CSC473: Advanced Algorithm Design Winter 2019

Linear Programming Duality

Aleksandar Nikolov

In class we saw that the linear program (LP) $\max\{c^\top x : Ax \le b, x \ge 0\}$ is dual to $\min\{b^\top y : A^\top y \ge c, y \ge 0\}$. This result can be used to derive duals for program which are not in standard form. Here are some examples:

- $\max\{c^\top x : Ax = b, x \ge 0\}$ is dual to $\min\{b^\top y : A^\top y \ge c\}$;
- $\min\{c^\top x : Ax = b, x \ge 0\}$ is dual to $\max\{b^\top y : A^\top y \le c\}$.

A general principle is:

- If a primal variable is always non-negative, then corresponding dual constraint is an inequality constraint.
- If a primal constraint is an equality constraint, then the corresponding dual variable is unrestricted, i.e. could be positive or negative.

Finally, here is a very general formulation. Consider the LP

$$\begin{aligned} \max\ & c^\top x + d^\top y && (1)\\ \text{s.t.}\ & Ax + By \le a && (2)\\ & Cx + Dy = b && (3)\\ & x \ge 0. && (4) \end{aligned}$$

Here the variables are given by the vectors x and y. One can get the example LPs above as special cases. For example $\max\{c^\top x : Cx = b, x \ge 0\}$ is the special case when $d = 0$, $a = 0$, $A = 0$, $B = 0$, $D = 0$. The dual of (1)–(4) is

$$\begin{aligned} \min\ & a^\top u + b^\top v\\ \text{s.t.}\ & A^\top u + C^\top v \ge c\\ & B^\top u + D^\top v = d\\ & u \ge 0. \end{aligned}$$

Here the dual variables are given by the vectors u and v. In the special case $\max\{c^\top x : Cx = b, x \ge 0\}$ above, we get that the dual is $\min\{b^\top v : C^\top v \ge c\}$.