CSC 265: Enriched Data Structures and Analysis
Review of Probability

A *sample space* S is a set of elementary events (which can be viewed as the possible outcomes of an experiment).
A *probability distribution function* f maps each point in S to a number in [0,1], so that $\sum_{x \in S} f(x) = 1$.
A *probability space* is a sample space plus a probability distribution function.

An event is a subset of the sample space.
If $A \subseteq S$ is an event, then $\text{Prob}[A] = \sum_{x \in A} f(x)$.

If A and B are events, then $\text{Prob}[A \cup B] = \text{Prob}[A] + \text{Prob}[B] - \text{Prob}[A \cap B]$.

A *random variable* is a function that assigns a value to each element of a probability space.

The *expected value* of a random variable $V : S \to R$ is $E[V] = \sum_{x \in S} V(x) \cdot f(x)$, where S is the sample space and f is the probability distribution function. An equivalent definition is $\sum_{r \in R} r \cdot \text{Prob}[V = r]$ where $\text{Prob}[V = r] = \sum \{f(x) \mid V(x) = r\}$.

Linearity of Expectation

$E[X + Y] = E[X] + E[Y]$ for any random variables X and Y
$E[aX] = aE[X]$ for any constant a

X and Y are *independent random variables* if

$$Prob[X = x \text{ and } Y = y] = \text{Prob}[X = x] \cdot \text{Prob}[Y = y]$$

for all values x of X and all values y of Y.
If X and Y are independent random variables, then $E[XY] = E[X]E[Y]$.

If range$(X) \subseteq N$ then

$$\begin{aligned} E[X] &= \sum_{i=0}^{\infty} i \text{Prob}[X = i] \\ &= \sum_{i=0}^{\infty} i(\text{Prob}[X \geq i] - \text{Prob}[X \geq i + 1]) \\ &= \sum_{i=1}^{\infty} \text{Prob}[X \geq i]. \end{aligned}$$

The last equality follows since each term $Prob[X \geq i]$ is added i times and subtracted $i - 1$ times.

Indicator Variables

A random variable with range $\{0, 1\}$ is called an indicator variable.

For example, if we consider the sample space of the rolls of a die, then the random variable $Odd : \{1, 2, 3, 4, 5, 6\} \to \{0, 1\}$ is an indicator variable, where

$$Odd(i) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases} .$$

For any indicator variable X,
$E[X] = 1 \cdot \mathrm{Prob}[X = 1] + 0 \cdot \mathrm{Prob}[X = 0] = \mathrm{Prob}[X = 1]$.

Conditional Probability

The conditional probability of an event A occurring given that another event B has occurred is $\mathrm{Prob}[A|B] = \mathrm{Prob}[A \cap B]/\mathrm{Prob}[B]$. Note that this is only defined if $\mathrm{Prob}(B) > 0$.

Bayes' Theorem $\mathrm{Prob}[A|B] = \mathrm{Prob}[B|A] \cdot \mathrm{Prob}[A]/\mathrm{Prob}[B]$.