CSC 473 - Advanced Algorithms Design	Winter 2017

König's Theorem

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A classical result in graph theory states the following:

Theorem 1 (König's Theorem)**.** *Let $G(V, E)$ be a bipartite graph. The size of a maximum matching in G equals the size of a minimum vertex cover of G.*

In this lecture, we formulate maximum matching and minimum vertex cover as LPs, as ILPs in fact (integer linear programs) and we'll use strong duality as well as properties of TU matrices to prove König's theorem.

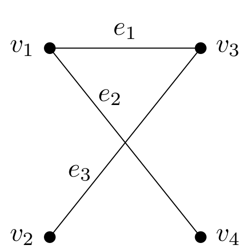
We recall some definitions first.

- A graph $G(V = X \cup Y, E)$ is a bipartite graph if X and Y are both independent sets and every edge $e \in E$ is of the form $e = (x, y), x \in X, y \in Y$. We call X and Y the bi-partitions of G.
- A vertex u is incident to an edge e if u is an endpoint of e, i.e. $e = (u, v)$ for some $v \in V$. We write $u \sim e$.
- A matching in a graph is a subset of edges $M \subseteq E$ such that for all $e_1, e_2 \in M$, e_1, e_2 have no common end points.
- A vertex cover in G is a subset of vertices $S \subseteq V$ such that every edge $e \in E$ is incident to at least one vertex in S (i.e. e is covered).
- An incidence matrix A of a graph G is an $n \times m$ binary matrix whose rows are the vertices of G and columns the edges of G. An entry is A is defined as

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is incident to } e_j \\ 0, & \text{otherwise.} \end{cases}$$

  it's not hard to see that every column of A has exactly two 1s, since every edge is incident to exactly two vertices.

We illustrate the above definitions in the following example. Consider the bipartite graph below with bi-partitions $X = \{v_1, v_2\}, Y = \{v_3, v_4\}$.



	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{e_3}$
$\mathbf{v_1}$	1	1	0
$\mathbf{v_2}$	0	0	1
$\mathbf{v_3}$	1	0	1
$\mathbf{v_4}$	0	1	0

Figure 1: A bipartite graph and its incidence matrix.

The set $M = \{e_1\}$ is a maximal matching of G, and $M' = \{e_2, e_3\}$ a maximum matching. The set $S = \{v_1, v_2\}$ is a vertex cover of G.

Lemma 1. *The incidence matrix of a bipartite graph is totally unimodular.*

Proof. Left as an exercise. □

Maximum Bipartite Matching: Let $G(V, E)$ be a bipartite graph. Consider the following linear program:

$$\begin{array}{ll} \text{maximize} & \sum_{j=1}^{m} x_j \\ \text{subj. to} & A\mathbf{x} \leq \mathbf{1} \\ & \mathbf{x} \geq \mathbf{0} \\ & \mathbf{x} \in \mathbb{Z}^{\mathbf{m}} \end{array}$$

Where A is the incidence matrix of a bipartite graph. Notice that the objective function can be rewritten as the maximization of $\sum_{j=1}^{m} \mathbf{1}^T\mathbf{x}$, where $\mathbf{1}$ is the all one vector.

For every edge $e_j \in E$, we introduce a new variable $x_j \in \mathbb{Z}_{\geq 0}$. Every row i of A corresponds to a vertex $v_i \in V$, and every vertex v_i induces the constraint $\sum\limits_{\substack{j \\ v_i \sim e_j}} x_j \leq 1$. For instance, using the graph in Figure 1, vertex v_3 induces the constraint

$$1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 \leq 1$$

In other words, we either set x_1 or x_3 to 1 but not both. Thus either edge e_1 or e_3 can be in a matching but not both. Since $x \geq 0$ and $x \in \mathbb{Z}^m$, this implies $x_j \in \{0, 1\}$ for $j \in [m]$.

Therefore, in an optimal solution $\mathbf{x}^*$ of the LP above, the edges e_j with $x_j^* = 1$ form a matching. Thus the LP is a formulation of the maximum matching in G.

Minimum Vertex Cover: Consider now the following LP:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} y_i \\ \text{subj. to} & A^T\mathbf{y} \geq \mathbf{1} \\ & \mathbf{y} \geq \mathbf{0} \\ & \mathbf{y} \in \mathbb{Z}^{\mathbf{n}} \end{array}$$

Where again A is the incidence matrix of a bipartite graph, and the objective function can be rewritten as the minimization of $\sum_{i=1}^{n} \mathbf{1}^T\mathbf{y}$. The rows of A^T correspond the the edges of G. For every vertex v_i, we introduce a variable $y_i \in \mathbb{Z}_{\geq 0}$. Every row j of A^T corresponds to an edge $e_j \in E$, and every edge e_j induces the constraint $\sum\limits_{\substack{i \\ v_i \sim e_j}} y_i \geq 1$. For instance, using again the same graph in Figure 1, edge e_3 induces the constraint

$$0 \cdot y_1 + 1 \cdot y_2 + 1 \cdot y_3 + 0 \cdot y_4 \geq 1$$

In other words, at least one of v_2, v_3 is set to 1. In fact, v_i can indeed be greater than 1, but since this is a minimization problem, reducing v_i to 1 - but not below - produces a better feasible solution. Therefore, in an optimal solution $\mathbf{y}^*$ of this LP, $y_i^* \in \{0, 1\}$ for $i \in [n]$. And thus, the vertices v_i with $y_i^* = 1$ form a minimum vertex cover of G.

We are now ready to prove König's Theorem.

Proof of Theorem 1. Since A is TU (Lemma 1), and so is A^T (since $det(A) = det(A^T)$), we can drop the integrality constraints of the LPs above and still get optimal integral solutions by Theorem 9 (Lecture Notes: Week 5). Notice that the LPs above are duals to each other, and thus by strong duality, their optimal values are equal. Therefore the size of a maximum matching in a bipartite graph equals the size of a minimum vertex cover. □