Consider the following algorithm.

1. function Mystery(A, s, f)
2.   # A is a list and s, f are indices such that 0 ≤ s ≤ f + 1 ≤ length(A)
3.   if s > f then
4.      return 0
5.   end if
6.   m = ⌊(f − s + 1)/4⌋
7.   res = Mystery(A, s, s + m − 1)
8.   # loop precondition goes here...
9.   for i = s + m, ⋯ , f − m do
10.      res = res + A[i]
11.   end for
12.   # loop postcondition goes here...
13.   res = res + Mystery(A, f − m + 1, f)
14.   return res
15. end function

1. State clear and precise preconditions for this algorithm.

Ans: If you look at the comment at the start of the algorithm, you can find a partial answer. Moreover line 10 suggests that elements of A must be numeric. We can express these formally as

1. A is a list of numbers
2. s, f are integers that satisfy 0 ≤ s ≤ f + 1 ≤ length(A)

2. State clear and precise postconditions for this algorithm.

Ans: The indices of lines 7, 9 and 13 are consecutive values and we do sum values in these lines. Hence, we intuitively expect the algorithm to add all the elements of A together. More formally:

1. Mystery(A, s, f) returns A[s] + ⋯ + A[f]

3. Prove the correctness of this algorithm.

Note: You may assume that the loop is correct without proof, as long as you state clear preconditions and postconditions specifically for the loop where indicated by comments.

Ans:

Loop precondition: \( res = A[s] + \cdots + A[s + m − 1] \) (\( res = 0 \) if \( m = 0 \))

Loop postcondition: \( res = A[s] + \cdots + A[f − m] \)

By induction on \( n = f − s + 1 \), we prove that Mystery(A, s, f) terminates and returns \( A[s] + \cdots + A[f] \) for all lists of numbers \( A \) and integers \( s, f \) such that \( 0 ≤ s ≤ f + 1 ≤ length(A) \).

Base Case: Suppose \( A \) is a list of numbers and \( s, f \) are integers such that \( 0 ≤ s = f + 1 ≤ length(A) \). Then, the call Mystery(A, s, f) executes only line 3 and returns 0, and 0 is the sum of empty list. Hence, Mystery(A, s, f) terminates and returns \( A[s] + \cdots + A[f] \).

Induction Step: Suppose \( n ≥ 1 \) and, for all integers \( s, f \) such that \( 0 ≤ f − s + 1 < n \), Mystery(A, s, f) terminates and returns \( A[s] + \cdots + A[f] \) for all lists of numbers \( A \) such that \( 0 ≤ s ≤ f + 1 ≤ length(A) \) (IH). Suppose \( f − s + 1 = n \) and \( A \) is a list of numbers such that \( 0 ≤ s ≤ f + 1 ≤ length(A) \). Then, Mystery(A, s, f) executes the test on line 1 and fails since \( n ≥ 1 \) implies \( f ≥ s \). Then it executes lines 6 and 7 and hence make recursive
call on an input that satisfies the preconditions $0 \leq s \leq s + m - 1 + 1 = s + m \leq \text{length}(A)$. The input size of the recursive call is exactly $s + m - 1 - s + 1 = m = \left\lceil \frac{n}{4} \right\rceil < n$ so by IH, we know that this call will terminate and return $A[s] + \cdots + A[s + m - 1]$. This means that the loop on line 9 is entered with its preconditions satisfied. Hence, when the loop terminates, the postconditions are also satisfied. In other words, $\text{res} = A[s] + \cdots + A[f - m]$ upon termination of the loop. When the recursive call on line 13 is made, the arguments meet the precondition $0 \leq f - m + 1 \leq f + 1 \leq \text{len}(A)$ (since $m \geq 0$), and the input size is again exactly $\left\lceil \frac{n}{4} \right\rceil$. By the IH, this call terminates and returns $A[s - m + 1] + \cdots + A[f]$. Therefore, the current call also terminates and returns


Hence, for all $s,f$ such that $f - s + 1 = n$, Mystery($A,s,f$) terminates and returns $A[s] + \cdots + A[f]$, for all lists of numbers $A$ such that $0 \leq s \leq f + 1 \leq \text{length}(A)$. In conclusion, by induction, Mystery($A,s,f$) terminates and returns $A[s] + \cdots + A[f]$ for all lists of numbers $A$ such that $0 \leq s \leq f + 1 \leq \text{length}(A)$. □