1. Consider the function T(n) defined by the following recurrence:

$$T(n) = \begin{cases} 2 & \text{if } n = 1 \\ 3 + T(\lceil \frac{n}{2} \rceil) & \text{if } n > 1 \end{cases}$$

Prove that $T(n)$ is in $\Theta(\log n)$.

Ans: In order to prove $\Theta(\log n)$, we have to show $T(n) \in \Omega(\log n)$ and $T(n) \in O(\log n)$. We prove both by induction.
[$T(n) \in \Omega(\log n)$]: We need constants $B, c > 0$ such that $T(n) \geq c \log n$ for all $n \geq B$. But, how can we find these values? Imagine the final proof and see what we need for the inductive step and the base case. In the induction step, we will have

$$T(n) = 3 + T(\lceil \frac{n}{2} \rceil)$$

We should use the IH to say that $T(n) \geq 3 + c\log(\lceil \frac{n}{2} \rceil)$. Then, we have to work with this to simplify:

$$\begin{aligned} T(n) &\geq 3 + c\log(\tfrac{n}{2}) && \text{(because } \lceil \tfrac{n}{2} \rceil \geq \tfrac{n}{2} \text{ and } \log \text{ is an increasing function)} \\ &= 3 + c(\log n - \log 2) \\ &\geq 3 + c\log n - c\log 2 \end{aligned}$$

But our goal is to show $T(n) \geq c \log n$. So we have to able to set the following inequality:

$$c \log n + (3 - c\log 2) \geq c \log n$$

If we solve this inequality for c, we find that $c \leq \frac{3}{\log 2}$.
Note: We are not done yet. We should double-check the base case to be sure it works. In this example, we choose $B = 1$ as our base value.

$$T(1) = 2 >= 0 = \log(1) = \frac{3}{\log 2}\log(1)$$

so we're good. Now let's prove our statement assuming $c = \frac{3}{\log 2}$.

Proof. Define our predicte as $P(n) : T(n) \geq c\log n,\ c = \frac{3}{\log 2}$. We use complete induction to prove that $P(n), \forall n \geq 1$.
Base Case: $T(1) = 2 \geq 0 = c\log(1)$.
Induction step: Let $k > 1$ and suppose $T(j) \geq c\log j$ for $1 \leq j < k$ (IH). We know prove $T(k) \geq c\log k$.

$$\begin{aligned} T(k) &= 3 + T(\lceil \tfrac{k}{2} \rceil) && \text{(by recurrence for } T\text{, since } k > 1) \\ &\geq 3 + c\log(\lceil \tfrac{k}{2} \rceil) && \text{(by IH, since } 1 \leq \lceil \tfrac{k}{2} \rceil < k \text{ for } k > 1) \\ &\geq 3 + c(\log k - \log 2) && (\lceil x \rceil \geq x \Rightarrow \log\lceil x \rceil \geq \log x \text{ for all } x) \\ &\geq c\log k && \text{(by choice of } c) \end{aligned}$$

Hence, by induction, $T(n) \geq c\log n$ for all $n \geq 1$. □

$[T(n) \in O(\log n)]$: As above, we start with induction step to find constraints on c.

$$\begin{aligned} T(n) &= 3 + T(\lceil \tfrac{n}{2} \rceil) \\ &\leq 3 + c \log \lceil \tfrac{n}{2} \rceil \end{aligned}$$

But wait, there is a problem here, we cannot go any further as $\lceil \frac{n}{2} \rceil \geq \frac{n}{2}$ but it is ideal for us to have the reverse inequality. We can overcome this problem by trying $T(n) \leq c \log(n-1)$.

$$\begin{aligned} T(n) &= 3 + T(\lceil \tfrac{n}{2} \rceil) \\ &\leq 3 + c \log(\lceil \tfrac{n}{2} \rceil - 1) && \text{(by IH)} \\ &\leq 3 + c \log(\tfrac{n+1}{2} - 1) && (\lceil \tfrac{x}{2} \rceil \leq \tfrac{x+1}{2} \text{–needs proof as a lemma}) \\ &\leq 3 + c \log \tfrac{n-1}{2} \\ &\leq c \log(n-1) + (3 - c \log 2) \end{aligned}$$

As before, $T(n) \leq c \log(n-1)$ iff $\frac{3}{\log 2} \leq c$. So we can pick $c = \frac{3}{\log 2}$. Similar to the previous part, we have to check the base case to make sure the choice of c works. We choose $B = 2$ as our base case. Notice that if we instead choose $B = 1$, then the base case will be $T(1) = 2 \leq c \log(1-1)$. But $\log 0$ is undefined!

$$T(2) = 3 + T(\lceil \frac{2}{2} \rceil) = 3 + T(1) = 5 \not\leq c \log(2-1) = c \log 1 = 0$$

It seems like, the base case doesn't want to yield! But don't panic, we can slightly change our bound to fix this. Let's see if we can prove $T(n) \leq \log(n-1) + 5$ instead! For the Base case:

$$T(2) = 5 \leq c \log(2-1) + 5.$$

Yey! But wait, we should revisit the induction step.

$$\begin{aligned} T(n) &= 3 + T(\lceil \tfrac{n}{2} \rceil) \\ &\leq 3 + c \log(\lceil \tfrac{n}{2} \rceil - 1) + 5 \\ &\leq 3 + c \log(n-1) - c \log 2 + 5 \\ &= c \log(n-1) + 5 && \text{(by choice of } c\text{)} \end{aligned}$$

Proof. Define predicate as $P(n): \; T(n) \leq c \log(n-1) + 5, \forall n \geq 2$. We proof $\forall n \geq 2, P(n)$ by complete induction.
Base case:

$$T(2) = 3 + T(\lceil \frac{2}{2} \rceil) = 3 + 2 = 5 \leq 0 + 5 = c \log(1) + 5$$

Induction step: Let $i > 2$ and suppose $T(j) \leq c \log(j-1) + 5$ for all $2 \leq j < i$ (IH). We have

$$\begin{aligned} T(i) &= 3 + T(\lceil \tfrac{i}{2} \rceil) \\ &\leq 3 + c \log(\lceil \tfrac{i}{2} \rceil - 1) + 5 && \text{(by recurrence for } T\text{, since } i > 2) \\ &\leq 3 + c \log(\tfrac{i-1}{2}) + 5 && \text{(by IH and because } 2 \leq \lceil \tfrac{x}{2} \rceil < x \text{ for } x > 2) \\ &\leq 3 + c(\log(i-1) - \log(2)) + 5 && (\lceil \tfrac{x}{2} \rceil - 1 \leq \tfrac{x+1}{2} - 1 = \tfrac{x-1}{2}) \\ &\leq c \log(i-1) + 5 && \text{(by choice of } c\text{)} \end{aligned}$$

Hence, by induction, $T(n) \leq c\log(n-1)+5$ for all $n \geq 2$. Since $c\log(n-1) \leq c\log n$, then $T(n) \in O(\log n)$. $\square$

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