1. A complete binary tree is a binary tree in which every node has two children, except for the leaves (which have no child), and every leaf is at the same depth (same distance from the root).

a) Give a recursive definition for the set of all complete binary trees.

b) Use structural induction on the definition of part a) to prove that, for all complete binary trees T, if h is the height of T, then T contains exactly $2^{h+1} - 1$ nodes.

Ans: a) Let $\mathrm{T_c}$ denotes the set of complete binary tree. The following is a recursive definition of $\mathrm{T_c}$:

i A single node is a complete binary tree.

ii If T_1 and T_2 are complete binary trees of the same height such that $\mathbf{nodes}(T_1) \cap \mathbf{nodes}(T_2) = \emptyset$ and $r \notin \mathbf{nodes}(T_1) \cup \mathbf{nodes}(T_2)$, then the tree constructed by placing T_1 and T_2 under root node r is also a complete binary tree.

iii Nothing else is a complete binary tree.

b)Define $P(T):$ T contains $2^{h+1}-1$ nodes, where h is the height of T. Now we proceed with structural induction as follows:
Base Case: T is a single node. Then T has height 0 (easy to check from the defintion of height) and contains $1 = 2^{0+1} - 1$ nodes.

Induction step: Assume T_1 and T_2 are complete binary trees of height k and both contain $2^{k+1} - 1$ nodes (IH). Consider the complete binary tree T with root r, that contains T_1 and T_2 as children of r. This tree by definition of height is of height $h(T) = max(h(T_1), h(T_2)) + 1 = k + 1$ and it contains $(2^{h(T_1)+1} - 1) + 1 + (2^{h(T_2)+1} - 1) = 2^{h+1} + 2^{h+1} - 1 = 2^{h+2} - 1$ nodes. which proves that $P(T)$.

Hence, by structural induction, every complete binary tree T contains exactly $2^{h(T)+1} - 1$ nodes, where $h(T)$ is the height of the tree. $\square$

2. Let E be the set of well-formed algebraic expressions. i.e.,

i $x, y, z \in E$

ii If $e_1, e_2 \in E$, then $(e_1 + e_2)$, $(e_1 - e_2)$, $(e_1 * e_2)$, and $(e_1 \div e_2)$ also belong to E

iii E contains nothing else.

For any expression $e \in E$, let $vr(e)$ and $op(e)$ denote the number of variable and operator occurrences in e. Prove by structural induction over E that $\forall e \in E$, $vr(e) = op(e) + 1$.

Ans: Let's check an example to get some idea regarding the statement of problem. It is easy to check that $((x + x) + y) \in E$. Now, $vr(((x + x) + y)) = 3$ because the expression contains 2 occurrences of x and one occurrence of y. Moreover, $op(((x + x) + y)) = 2$ because the operation $+$ has occurred twice.

Now let's prove the statement. For $e \in E$ define $P(e):$ $vr(e) =$ $op(e) + 1$.
Base case: Let $e = x$ or $e = y$ or $e = z$. In each case, $vr(e) = 1 = 0 + 1 = op(e) + 1$, so $P(e)$ holds.

Induction step: Assume $e_1, e_2 \in E$ and $P(e_1)$ and $P(e_2)$ (IH). Let e be an element defined recursively from e_1 and e_2: either $e = (e_1 + e_2)$, or $e = (e_1 - e_2)$, or $e = (e_1 * e_2)$, or $e = (e_1 \div e_2)$. In each case, $vr(e) = vr(e_1) + vr(e_2)$ and $op(e) = op(e_1) + op(e_2) + 1$. Hence,

$$\begin{aligned}
vr(e) &= vr(e_1) + vr(e_2) \\
&= (op(e_1) + 1) + (op(e_2) + 1) \qquad \text{(by IH)} \\
&= (op(e_1) + op(e_2) + 1) + 1 \\
&= op(e) + 1
\end{aligned}$$

Therefore, by structural induction on E, $\forall e \in E$, $P(e)$. $\square$

3. Give a recursive definition for the set $\{x \in \mathbb{Z} : \ x \text{ is even}\}$ (the set of all even integers). Show that the aforementioned set is uniquely defined by the recursive definition.

Ans: Consider the following recursive definition:

i $0 \in S$

ii for all $x \in S$, $x + 2 \in S$ and $x - 2 \in S$

iii nothing else belongs to S

In order to show that this definition uniquely defines the set of all even integers, we have to prove two things:

1. ($\Rightarrow$) every element in S is an even integer (S is a subset of the set in problem)
2. ($\Leftarrow$) every even integer belongs to S (the set in problem is a subset of S)

$\Rightarrow$: We will prove this by structural induction on set S. Let $P(e)$: e is even.
Base case: 0 is an even integer and by basis of the defintion $0 \in S$. So $P(0)$ holds.
Induction step: Let x be an element of S and let $P(x)$. i.e., x is an even integer or $\exists k \in \mathbb{Z}$ such that $x = 2k$. Now $x + 2 = 2k + 2 = 2(k + 1)$ which means $x + 2$ is an even integer. $x - 2 = 2k - 2 = 2(k - 1)$ which means $x - 2$ is also an even integers. Hence $P(x + 2)$ and $P(x - 2)$ holds.

Hence by structural induction $\forall x \in S$, $P(x)$.

$\Leftarrow$: This part is more complicated than it seems. In order to prove something for every even integer, we need to use a form of induction. But in order to use induction, we need to have a recursive definition of the set of even integers. But that's what we're trying to show is correct!

The solution is to do induction on another set whose recursive structure we already know is correct, e.g., $\mathbb{N}$. Then find a relationship between our set and $\mathbb{N}$. In this case,

$$\begin{aligned}
\{x \in \mathbb{Z} : \ x \text{ is even}\} &= \{x \in \mathbb{Z} : \exists y \in \mathbb{Z}, x = 2y\} \\
&= \{x \in \mathbb{Z} : \exists y \in \mathbb{N}, x = 2y \vee x = -2y\}
\end{aligned}$$

Then, use simple induction to show that $\forall y \in \mathbb{N}, \ 2y \in S \wedge -2y \in S\}$.
Base case: $2 \times 0 = -2 \times 0 = 0 \in S$
Induction step: for any y, if $2y \in S$ and $-2y \in S$ (IH), then $2(y + 1) = 2y + 2 \in S$ (by IH) and $-2(y + 1) = -2y - 2 \in S$ (by IH). $\square$