1. A "prime factorization" of an integer n is a sequence of prime numbers whose product equals n, e.g., $84 = 2 \times 2 \times 3 \times 7$. Prove all integers $n \geq 2$ have a prime factorization.

Ans: The precise definition of predicate is

$$P(n) : \exists \text{ prime numbers } p_1, p_2, \cdots, p_k, n = p_1 \times p_2 \times \cdots \times p_k$$

Base case: 2 is a prime number. The prime factorization of a prime number is the number itself. So $P(2)$ holds.
Induction step: Assume for $i \geq 2, \forall 1 < k \leq i,\ P(k)$ (IH). There are two cases for $i+1$

sub-case A: $i+1$ is a prime number. In this case, $P(i+1)$ holds as every prime number is the prime factorization of itself.

sub-case B: $i+1$ is a composite number. Then there exist numbers $a > 1$ and $b > 1$ such that $i+1 = a \times b$. By definition, $a < n$ and $b < n$. Hence by IH there exist prime sequences $p_1, p_2, \cdots, p_k$ and $q_1, q_2, \cdots, q_s$ such that $a = p_1 \times p_2 \times p_k$ and $b = q_1 \times q_2 \times q_s$. Now, we can define the sequence $p_1, p_2, \cdots, p_k, q_1, \cdots, q_s$ such that

$$i + 1 = a \times b = p_1 \times \cdots \times p_k \times q_1 \times \cdots \times q_s$$

which proves $P(i+1)$ holds. Therefore, by complete induction $\forall n, P(n)$. □

2. What amounts of postages can be made exactly using only 3¢ and 5¢ stamps? Prove your claim by complete induction.

Ans: Discovery Phase:

~~1¢~~, ~~2¢~~, 3¢, ~~4¢~~, 5¢, 6¢, ~~7¢~~, 8¢, 9¢, 10¢, 11¢ ⋯

Conjecture: 3¢, 5¢, 6¢ and every postage amount greater than or equal 8¢ can be made.
Idea: Because the three consecutive amounts 8¢, 9¢, 10¢ can all be made, we can just keep adding 3c stamps to get everything thereafter.
Proof: It is easy to see how we can make 3¢, 5¢, and 6¢ postages. So we just prove by complete induction that $\forall n \geq 8$ the n¢ postage can be made using 3¢ and 5¢ stamps.

Define

$$P(n) :\ \exists a, \exists b, 3a + 5b = n$$

Induction step: Assume $i \geq 8, \forall 8 \leq k < i, P(k)$ (IH). We want to prove $P(i)$.
Either $i = 8$ or $i = 9$ or $i = 10$ or $i \geq 11$.

Case 1: If $i = 8$, then $i = 8 = 1 \times 3 + 1 \times 5$, so $P(8)$ holds (This covers the base case in the format of complete induction that we covered in class).

Case 2: If $i = 9$, then $i = 9 = 3 \times 3 + 0 \times 5$, so $P(9)$ holds.

Case 3: If $i = 10$, then $i = 10 = 0 \times 3 + 2 \times 5$, so $P(10)$ holds.

Case 4: If $i >= 11$, then $i - 3 \geq 8$ so there exist a and b be such that $i - 3 = 3a + 5b$ (by IH). Then,

$$\begin{aligned} i &= 3 + (i - 3) \\ &= 3 + 3a + 5b \\ &= 3(a + 1) + 5b \end{aligned}$$

Hence, there exist $a' = a + 1$ and $b' = b$ such that $i = 3a' + 5b'$ and so $P(i)$ holds. □

3. Define

$$P(n) :\ \text{In every set of } n \text{ kids, all kids have the same eye colour}$$

The following proof tries to use simple induction to show $\forall n \in \mathbb{N}, P(n)$. Can you explain why the proof is wrong?

Base case: $P(0)$ is vacuously true: in every set of 0 kids, all kids have the same eye colour.

Induction step: Let $i \in \mathbb{N}$ and suppose $P(i)$ holds. Let S be a set of $i+1$ kids, say $S = \{h_1, h_2, ..., h_{i+1}\}$. Consider $\{h_1, h_2, ..., h_i\}$: this is a set of i kids, so by IH, all kids in that set have the same eye colour, say A. Now, consider $\{h_2, ..., h_i, h_{i+1}\}$: this is also a set of i kids, so by IH, all kids in that set have the same eye colour, say B. Since kids $h_2, \cdots, h_i$ belong to both sets and cannot have two different eye colours, it must be that $A = B$. This means every kid in S has the same eye colour.

Ans: Base Case is OK. But Induction step makes an implicit assumption about i. The reasoning only works if $i > 1$. If $i = 1$, reasoning fails for $S = \{h_1, h_2\}$ because there is no kid in the range $h_2, \cdots, h_n$. This could be fixed by considering two cases: $i = 1$ and $i > 1$. In the former case, $P(1)$ is true because in every set of 1 kid, all kids have the same eye colour. For the latter case, we still need to prove $p(2)$ so that IH can connect with a base case for which the truth is established. But $P(2)$ cannot be proven. Hence, the proof fails. $\square$