1. A “prime factorization” of an integer $n$ is a sequence of prime numbers whose product equals $n$, e.g., $84 = 2 \times 2 \times 3 \times 7$. Prove all integers $n \geq 2$ have a prime factorization.

**Ans:** The precise definition of predicate is

$$P(n) : \exists \text{ prime numbers } p_1, p_2, \cdots, p_k, n = p_1 \times p_2 \times \cdots \times p_k$$

**Base case:** $2$ is a prime number. The prime factorization of a prime number is the number itself. So $P(2)$ holds.

**Induction step:** Assume for $i \geq 2, \forall 1 < k \leq i$, $P(k)$ (IH). There are two cases for $i + 1$

**sub-case A:** $i + 1$ is a prime number. In this case, $P(i+1)$ holds as every prime number is the prime factorization of itself.

**sub-case B:** $i + 1$ is a composite number. Then there exist numbers $a > 1$ and $b > 1$ such that $i + 1 = a \times b$. By definition, $a < n$ and $b < n$. Hence by IH there exist prime sequences $p_1, p_2, \cdots, p_k$ and $q_1, q_2, \cdots, q_s$ such that $a = p_1 \times p_2 \times p_k$ and $b = q_1 \times q_2 \times q_s$. Now, we can define the sequence $p_1, p_2, \cdots, p_k, q_1, \cdots, q_s$ such that

$$i + 1 = a \times b = p_1 \times \cdots \times p_k \times q_1 \times \cdots \times q_s$$

which proves $P(i + 1)$ holds. Therefore, by complete induction $\forall n, P(n)$.

2. What amounts of postages can be made exactly using only $3c$ and $5c$ stamps? Prove your claim by complete induction.

**Ans:** Discovery Phase:

$$\{\text{3c, 5c, 6c, 8c, 9c, 10c, 11c} \cdots\}$$

**Conjecture:** $3c, 5c, 6c$ and every postage amount greater than or equal $8c$ can be made.

**Idea:** Because the three consecutive amounts $8c, 9c, 10c$ can all be made, we can just keep adding $3c$ stamps to get everything thereafter.

**Proof:** It is easy to see how we can make $3c, 5c,$ and $6c$ postages. So we just prove by complete induction that $\forall n \geq 8$ the $nc$ postage can be made using $3c$ and $5c$ stamps.

Define

$$P(n) : \exists a, \exists b, 3a + 5b = n$$

**Induction step:** Assume $i \geq 8, \forall 8 \leq k < i$, $P(k)$ (IH). We want to prove $P(i)$.

Either $i = 8$ or $i = 9$ or $i = 10$ or $i \geq 11$.

**Case 1:** If $i = 8$, then $i = 8 = 1 \times 3 + 1 \times 5$, so $P(8)$ holds (This covers the base case in the format of complete induction that we covered in class).

**Case 2:** If $i = 9$, then $i = 9 = 3 \times 3 + 0 \times 5$, so $P(9)$ holds.

**Case 3:** If $i = 10$, then $i = 10 = 0 \times 3 + 2 \times 5$, so $P(10)$ holds.

**Case 4:** If $i \geq 11$, then $i - 3 \geq 8$ so there exist $a$ and $b$ such that $i - 3 = 3a + 5b$ (by IH). Then,

$$i = 3 + (i - 3)$$

$$= 3 + 3a + 5b$$

$$= 3(a + 1) + 5b$$

Hence, there exist $a' = a + 1$ and $b' = b$ such that $i = 3a' + 5b'$ and so $P(i)$ holds.

3. Define

$$P(n) : \text{In every set of } n \text{ kids, all kids have the same eye colour}$$

The following proof tries to use simple induction to show $\forall n \in \mathbb{N}, P(n)$. Can you explain why the proof is wrong?
Base case: \( P(0) \) is vacuously true: in every set of 0 kids, all kids have the same eye colour.

Induction step: Let \( i \in \mathbb{N} \) and suppose \( P(i) \) holds. Let \( S \) be a set of \( i + 1 \) kids, say \( S = \{h_1, h_2, ..., h_{i+1}\} \).
Consider \( \{h_1, h_2, ..., h_i\} \): this is a set of \( i \) kids, so by IH, all kids in that set have the same eye colour, say \( A \). Now, consider \( \{h_2, ..., h_i, h_{i+1}\} \): this is also a set of \( i \) kids, so by IH, all kids in that set have the same eye colour, say \( B \). Since kids \( h_2, \ldots, h_i \) belong to both sets and cannot have two different eye colours, it must be that \( A = B \). This means every kid in \( S \) has the same eye colour.

Ans: Base Case is OK. But Induction step makes an implicit assumption about \( i \). The reasoning only works if \( i > 1 \). If \( i = 1 \), reasoning fails for \( S = \{h_1, h_2\} \) because there is no kid in the range \( h_2, \ldots, h_n \). This could be fixed by considering two cases: \( i = 1 \) and \( i > 1 \). In the former case, \( P(1) \) is true because in every set of 1 kid, all kids have the same eye colour. For the latter case, we still need to prove \( P(2) \) so that IH can connect with a base case for which the truth is established. But \( P(2) \) cannot be proven. Hence, the proof fails. \( \square \)