1. Prove that for all $n \in \mathbb{N}$, every set of size n has exactly 2^n subsets.

Ans: The precise definition of predicate is

$$P(n) : \text{all sets of size } n \text{ have } 2^n \text{ subsets}$$

Base case: An empty set (set of size 0) has only one subset which is the empty set itself. So $P(0)$ is true.
Induction step: Assume for $n \in \mathbb{N}$, $P(n)$ is true (IH). In other words, assume all sets of size n have 2^n subsets. Suppose $S = \{a_1, a_2, \cdots, a_{n+1}\}$ is a set of size $n+1$. Subsets of S can be split into those that contain the element a_{n+1} and those that don't. Any subset in the former can be converted to a unique subset in the latter by removing the element a_{n+1} and any subset in the latter can be converted to a unique subset in the former by adding the element a_{n+1}. Therefore, there are an equal number of subsets in both categories. Now, each subset of S that does not contain a_{n+1} is a subset of $S' = S - \{a_{n+1}\} = \{a_1, a_2, \cdots, a_n\}$. Moreover, any subset of S' is in fact a subset of S that does not contain the element a_{n+1}. S' is a set of size n and by IH has 2^n subsets. Therefore, both categories of subsets of S have a size of 2^n and hence S has $2 \times 2^n = 2^{n+1}$ subsets.

Given the proofs for the base case and the induction step, we can conclude, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$. □

2. Prove that $\forall n \geq 13, n^2 > 12n + 5$.

Ans: This examples requires the use of simple induction with a non-zero base. Because of that, we simply define the predicate as

$$P(n) : n^2 > 12n + 5$$

and try to prove, as our base case, that $P(13)$ holds (since $n \geq 13$).
Base case:

$$\begin{aligned} 13^2 &= 169 \\ &> 161 = 156 + 5 = 12 \times 13 + 5 \end{aligned}$$

Therefore, $P(13)$ is true.
Induction step: Assuming that for $n \in \mathbb{N}$, $P(n)$ is true (IH). We will prove that $P(n+1)$ is also true.

$$\begin{aligned} (n+1)^2 &= n^2 + 2n + 1 \\ &> 12n + 5 + 2n + 1 && \text{(by the IH)} \\ &\geq 12n + 5 + 26 + 1 && (n \geq 13) \\ &> 12n + 12 + 5 = 12(n+1) + 5 \end{aligned}$$

Now, we have proved that $P(n+1)$ is true, so we can conclude by induction that $\forall n \geq 13, n^2 > 12n + 5$. □