Induction

Principle of Complete Induction: Let A be a set that satisfies the properties

1. 0 is an element of A

2. for any $i \in \mathbb{N}$, if every natural number less than i is an element of A then i is also in A

Then A is a superset of $\mathbb{N}$.

Similar to the principle of simple induction, we will not prove the principle of complete induction. Rather, we assume that is an *obvious truth*.

Remark. *The principle of complete induction is valid even if we remove the first conditions! [see page 18 of the textbook]*

The principle of complete induction allows us to define the "complete induction" (strong induction in some textbooks) proof technique. Complete induction proves a predicate $P(n)$ for all natural numbers in two steps:

1. *Base case:* Prove P(0) is true, i.e., the predicate $P(n)$ holds for $n = 0$

2. *Induction step:* For $i \in \mathbb{N}$, if $P(j)$ holds for all natural numbers $j < i$ (*Induction Hypothesis*), prove that $P(i)$ also holds

Remark. *Intuitively, the complete induction can be concluded from simple induction. In simple induction, the chain of arguments that allows us to say $P(i)$ is true starts from $P(0)$ and continues as:*

$$P(0) \Rightarrow P(1) \to P(1) \Rightarrow P(2) \to \cdots \to P(i-1) \Rightarrow P(i)$$

Therefore, by the time that we want to conclude the truth of $P(i)$, we have already established the truth of $P(0), P(1), \cdots, P(i-1)$.

Example 1. *You have a chocolate bar that consists of $n > 0$ squares arranged in a rectangular pattern. You can only break chocolate bars along the line between the squares. What is the minimum number of breaks that you have to do to split the chocolate bar into n squares?*

Ans: Discovery Phase: Let's explore different chocolate patterns

$$1 \times 1(0 \text{ breaks}), 1 \times 2(1 \text{ breaks}), \cdots, 1 \times n(n-1 \text{ breaks}), 2 \times 2(3 \text{ breaks}), 2 \times 3(5 \text{ breaks}) \cdots$$

Conjecture: For all $n \geq 1$, every chocolate bar with n squares can be split up (into individual squares) with $n-1$ breaks.
Let's prove the conjecture using complete induction.
$P(n)$: All chocolate bars with n squares can be split up with $n-1$ breaks.
Base case: A chocolate bar with only one square is already split up and needs $0 = 1 - 1$ breaks. So $P(1)$ holds.
Induction step: Assuming $i \geq 2$ and for all $1 \leq j < i$, $P(j)$ is true (IH), we want to prove that $P(i)$ is also true.
The first break splits the chocolate bar in two pieces A and B, with a and b squares, respectively. Note that $1 \leq a < n$, $1 \leq b < n$, and $n = a + b$. By the IH, A can be split up with $a - 1$ breaks and B can be split up with $b - 1$ breaks. Hence, the chocolate bar can be entirely split up with $1 + a - 1 + b - 1 = a + b - 1 = n - 1$ breaks. So, $P(i)$ holds.
Conclusion: By complete induction, $\forall n \geq 1, P(n)$ (i.e., for all $n \geq 1$, every chocolate bar with n squares can be split up with $n - 1$ breaks). □

Remark. *Similar to simple induction, complete induction can have a non-zero base case. You will see an example of this in tutorial.*

Well-Ordering

Principle of Well-Ordering: Every non-empty subset of $\mathbb{N}$ contains a smallest element.

Remark. *The well-ordering principle applies to* all *non-empty subsets, even infinite ones.*

Remark. *Note that well-ordering is a property of $\mathbb{N}$ – it does not apply to other sets such as $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$.*

Well-ordering principle is often used in proofs by contradiction to show $\forall n \in \mathbb{N}, P(n)$. A general template of such proof is as follows:

For a contradiction, suppose $\exists n, \neg P(n)$. Then set $S = \{n \in \mathbb{N} : \neg P(n)\}$ is not empty and hence by well ordering, S contains a smallest element k. At this point we know:

   i $\neg P(k)$ (because $k \in S$)

   ii $P(i), \forall i \in \{0, \cdots, k-1\}$ (because k is the smallest in S).

We can now use both facts to derive a contradiction. Two standard techniques to derive the contradiction is to either prove $(\forall i < k, P(i)) \Rightarrow P(k)$ to contradict the first fact (similar to proof by complete induction) or to prove $\neg P(k) \Rightarrow \exists i < k, \neg P(i)$, to contradict the second fact (equivalent to first technique but sometimes easier to do). Hence, by contradiction, $\forall n \in \mathbb{N}, P(n)$.

Remark. *Notice the similarity between the first method of deriving the contradiction and the proofs by induction. As we discuss in short, there is indeed a strong connection between principles of induction and the principle of well-ordering.*

Example 2 (Example 1.5 in textbook)**.** *Let m, n be natural numbers such that $n \neq 0$. Then, there exist natural numbers q and r such that $m = q \times n + r$ and $r < n$.*

Ans: Suppose $m, n \in \mathbb{N}$ and $n > 0$. Consider the set $R = \{r \in \mathbb{N} : \exists q, \ m = q \times n + r\}$. R is not empty because $m \in R (m = 0 \times n + m)$. Therefore, by well ordering, R contains a smallest element r, i.e.,

   i $\exists q, m = q \times n + r$

   ii $\forall r' \in \{0, \cdots, r-1\}, \forall q, m \neq q \times n + r'$

Now, we show $r < n$ to complete the proof.

For a contradiction, suppose $r \geq n$ and let $q \in \mathbb{N}$ such that $m = q \times n + r$. Then

$$m = q \times n + n + r - n = (q+1)n + (r-n)$$

so $r - n \in R$ (because $r - n \geq 0$). But $r - n < r$ (because $n > 0$) which contradicts the assumption ii above. Hence, $\exists r, q, r < n$ such that $m = q \times n + r$. $\square$
Compare this proof to the proof by induction provided in the textbook on pages 27–28.

Theorem 1. *The principles of well-ordering, simple induction, and complete induction are equivalent.*

Proof. We prove this theorem with a cyle of implications. In other words, we show well-ordering implies simple induction, simple induction implies complete induction, and complete induction implies well-ordering.

Well-ordering $\Rightarrow$ Simple induction:
Suppose well-ordering holds, i.e., every non-empty subset of $\mathbb{N}$ contains a smallest element. We prove the principle of simple induction, for arbitrary predicate P. i.e.,

$$P(0) \wedge (\forall i, P(i) \Rightarrow P(i+1)) \Rightarrow \forall n, P(n)$$

Suppose by contradiction that $\neg\forall n, P(n)$. Then $A = \{n \in \mathbb{N} : \neg P(n)\}$ is a non-empty subset of N and by well-ordering, it contains a smallest element s. Because $P(0)$, $s > 0$. Moreover, s is smallest element such that $\neg P(s)$ and $s > 0$, so $P(s-1)$ should hold. But by IH, $\forall n, P(i) \Rightarrow P(i+1)$. In particular, $P(s-1) \Rightarrow P(s)$. This contradicts $\neg P(s)$. Hence, simple induction holds.

Simple induction $\Rightarrow$ Complete induction:

Suppose simple induction holds, i.e., for arbitrary predicates P,

$$\Big(P(0) \wedge \forall i, P(i) \Rightarrow P(i+1) \Big) \Rightarrow \forall n, P(n)$$

Now, we have to prove complete induction also holds for P, i.e.,

$$\Big(P(0) \wedge \forall i, (\forall j < i, P(j)) \Rightarrow P(i) \Big) \Rightarrow \forall n, P(n) \tag{1}$$

Let $P'(n) \equiv \forall k \leq n, P(k)$. It is easy to see that $\forall n, P(n) \Leftrightarrow \forall n, P'(n)$. Now, instead of proving that $\forall n, p(n)$, we choose to prove $\forall n, P'(n)$ using simple induction.

1. *Base Case:* $P'(0) = \forall k \leq 0, P(k) = P(0)$. By left hand side of (1), $P(0)$ is true which means $P'(0)$ holds.

2. *Induction step:*   Let $i \in \mathbb{N}, P'(i)$. We want to prove $P'(i+1)$ holds, i.e., $\forall k \leq i+1, P(k)$.

$$\forall k <= i, P(k) \Rightarrow \forall k < i+1, P(k) \qquad \text{(By IH)}$$
$$\Rightarrow P(i+1) \qquad \text{(By 2nd argument on the left-hand side of (1))}$$
$$\Rightarrow P(i+1) \wedge \forall k < i+1, P(k)$$
$$\Rightarrow \forall k <= i+1, P(k) = P'(i+1)$$

Therefore by simple induction, $\forall n, \forall k <= n, P(k)$ which means $\forall n, P(n)$. So complete induction holds.

Complete induction $\Rightarrow$ Well-ordering:

Suppose complete induction holds, i.e., for arbitrary predicate P, (1) holds.

Let A be an arbitrary subset of $\mathbb{N}$. We show, by indirect proof, that if A is non-empty, then A contains a smallest element.

Suppose that A does not contain a smallest element, i.e.,

$$\forall n \in \mathbb{N}, n \in A \Rightarrow \exists k < n, k \in A$$

The contrapositive of this statement is

$$\forall n, (\forall k < n, k \notin A) \Rightarrow n \notin A$$

Let $P(n) \equiv n \notin A$. Then, $\forall n, (\forall k < n, P(k)) \Rightarrow P(n)$ (by reasoning above), so by complete induction, $\forall n, P(n)$, i.e., $\forall n, n \notin A$, meaning A is empty. Hence, well-ordering holds.

$\square$

Remark. *The proof technique that we used above is a standard technique to prove that a set of statements are equivalent. In this proof technique, we will order the statements and prove that each statement implies the subsequent statement. Finally, we prove that the last statement implies the first statement to complete the cycle of implications.*