Induction

Principle of Simple Induction: Let A be a set that satisfies the properties

1. 0 is an element of A
2. for any $i \in \mathbb{N}$, if i is in A then $i+1$ is also in A

Then A is a superset of $\mathbb{N}$.

Despite an initial mind struggle, our intuition suggests that this argument should hold (domino chain example). Therefore, we will accpet the principle as an "obvious truth" (hence the attribue principle), and will never attempt to prove this.

The aforementioned abstract principle leads to the definition of a proof technique which is called "simple induction". Simple induction proves a predicate $P(n)$ for all natural numbers in two steps:

1. *Base case:* Prove P(0) is true, i.e., the predicate $P(n)$ holds for $n = 0$
2. *Induction step:* For $i \in \mathbb{N}$, if $P(i)$ is true (*Induction Hypothesis*), prove that $P(i+1)$ is also true

Question. *Why does simple induction proves $P(n)$ for all natural numbers? [see page 21-22 of the textbook]*

Example 1. *Prove* $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Ans: The definition of predicate $P(n)$ is straightforward.

$$P(n) : \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Base case: $P(0) : \sum_{i=0}^{0} i^2 = \frac{0\times(0+1)\times(2\times0+1)}{6}$. But the truth of $P(0)$ is trivial as

$$\sum_{i=0}^{0} i^2 = 0^2 = 0 = \frac{0 \times (0+1) \times (2 \times 0 + 1)}{6}$$

Induction step: The induction hypothesis (IH) assumes that for $n \in \mathbb{N}$, $P(n)$ is true. Now we have to prove that $P(n+1)$ is also true, i.e., $\sum_{i=0}^{n+1} i^2 = \frac{(n+1)(n+1+1)(2(n+1)+1)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$.

$$\begin{aligned}
\sum_{i=0}^{n+1} i^2 &= \sum_{i=0}^{n} i^2 + (n+1)^2 \\
&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \qquad \text{(by the IH)} \\
&= (n+1) \times \Big(\frac{6(n+1) + (n \times (2n+1))}{6}\Big) \\
&= (n+1) \times \Big(\frac{2n^2 + 7n + 6}{6}\Big) \\
&= (n+1) \times \frac{(n+2)(2n+3)}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6}
\end{aligned}$$

and hence the $P(n+1)$ is true. $\square$

Example 2. *The ATM machine in 25Bank can only pay customers with \$2 coins and \$5 bills. The machine rejects a customers withdrawal request that cannot be paid in \$2 coins and \$5 bills. What \$ amounts can customers request to withdraw from this machine?*

Ans: Discovery Phase: Starting from 1, try various amounts

$$\cancel{\$1}, \$2, \cancel{\$3}, \$4, \$5, \$6, \$7, ...$$

Conjecture: In addition to \$2, every amount after \$4 can be withdrawn from the machine. For now, we will ignore \$2, and try to focus on amounts $\geq$ \$4.
Idea: Assuming \$4 and \$5 can be withdrawn, the machine can process other amounts by adding \$2 to an amount that it knows it can process.
Proof: We want to use simple induction on amounts of \$4 or more. But dollar amounts start from \$4 rather than \$0 which allows for a convenient definition of $P(n)$. So, what should $P(n)$ be?

$$P(n): \ \exists a, \exists b, 2a + 5b = n + 4$$

This will allow us to start n from 0 and easily follow the steps of simple induction.
Base case: Let $a = 2, b = 0$ then

$$2a + 5b = 2 \times 2 + 5 \times 0 = 4 = 0 + 4$$

Hence, $P(0)$ is true.
Induction Step: Let a', b' be such that $2a' + 5b' = n + 4$ (IH). Then $\exists a'', \exists b'', n + 1 + 4 = a'' + b''$. To prove this we need to look at the following subcases.

subcase A: If $b' > 0$, then

$$\begin{aligned} n + 1 + 4 &= 2a' + 5b' + 1 \\ &= 2a' + 5(b' - 1) + 5 + 1 \\ &= 2a' + 5(b' - 1) + 6 \\ &= 2(a' + 3) + 5(b' - 1) \end{aligned}$$

so we can pick $a'' = a' + 3$ and $b'' = b' - 1$.

subcase B: If $b' = 0$, then $n + 4 = 2a'$. Therefore, $a' \geq 2$ and

$$\begin{aligned} n + 1 + 4 &= 2a' + 1 \\ &= 2(a' - 2) + 4 + 1 \\ &= 2(a' - 2) + 5 \times 1 \end{aligned}$$

so we can pick $a'' = a' - 2$ and $b'' = 1$.

In both cases, $P(n + 1)$ holds and by induction $P(n)$ holds for all natural numbers, which means the ATM machine can pay any dollar amount ≥ 4. $\square$

Look back at the example. If instead of defining $P(n)$, we would have defined

$$P'(n): \ \exists a, \exists b, 2a + 5b = n$$

which is what comes into mind first, we couldn't prove that $P'(n)$ holds for all natural numbers. But if in the induction we start our base from $n = 4$, we can prove $P'(4)$ holds and a similar argument to the proof of example 2 will prove the induction step. In general, a statement of the form

$$\forall n \in \mathbb{N} \text{ such that } n \geq c, \ P(n) \text{ is true}$$

can be proved using the following variant of simple induction.

1. *Base case:* Prove $P(c)$ is true

2. *Induction step:* For $i \in \mathbb{N}$, if $P(i)$ is true (*Induction Hypothesis*), prove that $P(i+1)$ is also true

Considering the fact that we can rephrase the statement as

$$\forall n \in \mathbb{N}\ P(n+c) \text{ is true}$$

it is trivial, that this variant of simple induction can also be proved using the principle of simple induction.

The proof of example 2 does not match our intuition that we can start from \$4 and \$5 and try to prove that any greater amount can be paid by adding \$2s to another amount that we know is payable.

Next week, we will discuss a prove technique that can be used to provide a proof that follows our intuition.