1. Dynkin systems

Definition 1 *A* **Dynkin system** *on a set* Ω *is a subset* $\mathcal{D}$ *of the power set* $\mathcal{P}(\Omega)$*, with the following properties:*

$$
\begin{array}{rl}
(i) & \Omega \in \mathcal{D} \\
(ii) & A, B \in \mathcal{D}, A \subseteq B \ \Rightarrow\ B \setminus A \in \mathcal{D} \\
(iii) & A_n \in \mathcal{D}, A_n \subseteq A_{n+1}, n \geq 1 \ \Rightarrow\ \bigcup_{n=1}^{+\infty} A_n \in \mathcal{D}
\end{array}
$$

Definition 2 *A* **σ-algebra** *on a set* Ω *is a subset* $\mathcal{F}$ *of the power set* $\mathcal{P}(\Omega)$ *with the following properties:*

$$
\begin{array}{rl}
(i) & \Omega \in \mathcal{F} \\
(ii) & A \in \mathcal{F} \ \Rightarrow\ A^c \stackrel{\triangle}{=} \Omega \setminus A \in \mathcal{F} \\
(iii) & A_n \in \mathcal{F}, n \geq 1 \ \Rightarrow\ \bigcup_{n=1}^{+\infty} A_n \in \mathcal{F}
\end{array}
$$

Exercise 1. Let $\mathcal{F}$ be a σ-algebra on Ω. Show that $\emptyset \in \mathcal{F}$, that if $A, B \in \mathcal{F}$ then $A \cup B \in \mathcal{F}$ and also $A \cap B \in \mathcal{F}$. Recall that $B \setminus A = B \cap A^c$ and conclude that $\mathcal{F}$ is also a Dynkin system on Ω.

Exercise 2. Let $(\mathcal{D}_i)_{i \in I}$ be an arbitrary family of Dynkin systems on Ω, with $I \neq \emptyset$. Show that $\mathcal{D} \stackrel{\triangle}{=} \cap_{i \in I} \mathcal{D}_i$ is also a Dynkin system on Ω.

Exercise 3. Let $(\mathcal{F}_i)_{i \in I}$ be an arbitrary family of σ-algebras on Ω, with $I \neq \emptyset$. Show that $\mathcal{F} \stackrel{\triangle}{=} \cap_{i \in I} \mathcal{F}_i$ is also a σ-algebra on Ω.

Exercise 4. Let $\mathcal{A}$ be a subset of the power set $\mathcal{P}(\Omega)$. Define:

$$D(\mathcal{A}) \stackrel{\triangle}{=} \{\mathcal{D} \text{ Dynkin system on } \Omega : \ \mathcal{A} \subseteq \mathcal{D}\}$$

Show that $\mathcal{P}(\Omega)$ is a Dynkin system on Ω, and that $D(\mathcal{A})$ is not empty. Define:

$$\mathcal{D}(\mathcal{A}) \stackrel{\triangle}{=} \bigcap_{\mathcal{D} \in D(\mathcal{A})} \mathcal{D}$$

Show that $\mathcal{D}(\mathcal{A})$ is a Dynkin system on Ω such that $\mathcal{A} \subseteq \mathcal{D}(\mathcal{A})$, and that it is the smallest Dynkin system on Ω with such property, (i.e. if $\mathcal{D}$ is a Dynkin system on Ω with $\mathcal{A} \subseteq \mathcal{D}$, then $\mathcal{D}(\mathcal{A}) \subseteq \mathcal{D}$).

Definition 3 *Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. We call* **Dynkin system generated** *by $\mathcal{A}$, the Dynkin system on Ω, denoted $\mathcal{D}(\mathcal{A})$, equal to the intersection of all Dynkin systems on Ω, which contain $\mathcal{A}$.*

EXERCISE 5. Do exactly as before, but replacing Dynkin systems by σ-algebras.

Definition 4 *Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. We call* **$\sigma$-algebra generated** *by $\mathcal{A}$, the σ-algebra on Ω, denoted $\sigma(\mathcal{A})$, equal to the intersection of all σ-algebras on Ω, which contain $\mathcal{A}$.*

Definition 5 *A subset $\mathcal{A}$ of the power set $\mathcal{P}(\Omega)$ is called a* **π-system** *on Ω, if and only if it is closed under finite intersection, i.e. if it has the property:*

$$A, B \in \mathcal{A} \ \Rightarrow \ A \cap B \in \mathcal{A}$$

Exercise 6. Let $\mathcal{A}$ be a π-system on Ω. For all $A \in \mathcal{D}(\mathcal{A})$, we define:

$$\Gamma(A) \stackrel{\triangle}{=} \{B \in \mathcal{D}(\mathcal{A}) : \ A \cap B \in \mathcal{D}(\mathcal{A})\}$$

1. If $A \in \mathcal{A}$, show that $\mathcal{A} \subseteq \Gamma(A)$
2. Show that for all $A \in \mathcal{D}(\mathcal{A})$, $\Gamma(A)$ is a Dynkin system on Ω.
3. Show that if $A \in \mathcal{A}$, then $\mathcal{D}(\mathcal{A}) \subseteq \Gamma(A)$.
4. Show that if $B \in \mathcal{D}(\mathcal{A})$, then $\mathcal{A} \subseteq \Gamma(B)$.
5. Show that for all $B \in \mathcal{D}(\mathcal{A})$, $\mathcal{D}(\mathcal{A}) \subseteq \Gamma(B)$.
6. Conclude that $\mathcal{D}(\mathcal{A})$ is also a π-system on Ω.

Exercise 7. Let $\mathcal{D}$ be a Dynkin system on Ω which is also a π-system.

1. Show that if $A, B \in \mathcal{D}$ then $A \cup B \in \mathcal{D}$.

2. Let $A_n \in \mathcal{D}, n \geq 1$. Consider $B_n \stackrel{\triangle}{=} \cup_{i=1}^{n} A_i$. Show that $\cup_{n=1}^{+\infty} A_n = \cup_{n=1}^{+\infty} B_n$.

3. Show that $\mathcal{D}$ is a σ-algebra on Ω.

EXERCISE 8. Let $\mathcal{A}$ be a π-system on Ω. Explain why $\mathcal{D}(\mathcal{A})$ is a σ-algebra on Ω, and $\sigma(\mathcal{A})$ is a Dynkin system on Ω. Conclude that $\mathcal{D}(\mathcal{A}) = \sigma(\mathcal{A})$. Prove the theorem:

Theorem 1 (Dynkin system) *Let $\mathcal{C}$ be a collection of subsets of Ω which is closed under pairwise intersection. If $\mathcal{D}$ is a Dynkin system containing $\mathcal{C}$ then $\mathcal{D}$ also contains the σ-algebra $\sigma(\mathcal{C})$ generated by $\mathcal{C}$.*