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Exclusive OR Operation That Leads to the Narrowest Intervals

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Formulation of the problem. Experts may be uncertain about their statements. To express this uncertainty, most expert systems assign numbers to expert statements, numbers that range from 0 (completely false) to 1 (completely true). These numbers are called *degrees of belief*, or *degrees of certainty*. To handle logical combinations of expert's statements, we must, therefore, extend logical connectives that are normally defined for the values 0 (= "false") and 1 (= "true") to arbitrary values from the interval [0, 1].

There are many different ways to extend a function from the 2-valued set {0, 1} to the interval [0, 1], and different choices may drastically change the quality of the resulting expert system. One of the possible methods of choosing the operations is to take into consideration that the numbers expressing the experts' degrees of belief can only be *approximately* determined, and therefore, a meaningful choice of a logical operation should be the one that is the least affected by a small change in these numerical values.

In [3], [4], we have formalized this requirement and used this formalization to find the least sensitive "and", "or", and "not" operations. It turns out that the least sensitive "and" operation is $f_{\&}(a, b) = \min(a, b)$, the least sensitive "or" operation is $f_{\vee}(a, b) = \max(a, b)$, and the least sensitive "not" operation is $f_{\neg}(a) = 1 - a$. Back then, however, we were unable to analyze another important logical connective that is often useful to describe the expert statements: exclusive "or" $\oplus$.

In this paper, we describe the least sensitive exclusive "or" operation. To describe our result, let us start with the definitions (mainly borrowed from [3]).

Definitions.

- By an *interpolation problem*, we mean the tuple

$$\mathcal{P} = (n, U, \mathcal{F}, N, \vec{x}^{(1)}, \dots, \vec{x}^{(N)}, y^{(1)}, \dots, y^{(N)}),$$

  where:
  - n is a positive integer;
  - U is a subset of R^n;
  - $\mathcal{F}$ is a set of functions from U to R;
  - N is a positive integer;
  - $\vec{x}^{(k)}$ $(1 \le k \le N)$ are elements of U;
  - $y^{(k)}$ $(1 \le k \le N)$ are real numbers.
- We say that a function $f \in \mathcal{F}$ is a *possible solution* to the interpolation problem if $f(\vec{x}^{(k)}) - y^{(k)}$ for all k.
- Let f be a possible solution to an an interpolation problem $\mathcal{P}$, and let $\delta > 0$ be a positive real number.
  - We say that a *δ-input uncertainty leads to a $\le$ α-output error*, if for every $\vec{x} \in U$ and $\vec{x}' \in U$, for which $x_i' \in [x_i - \delta, x_i + \delta]$ for all i, we have $f(\vec{x}') \in [f(\vec{x}) - \alpha, f(\vec{x}) + \alpha]$.
  - By a *δ-sensitivity* of a function $f(\vec{x})$ we mean the smallest of real numbers α, for which a δ-input uncertainty leads to a $\le$ α-output error. The δ-sensitivity of a function $f(\vec{x})$ will be denoted by $s_f(\delta)$.
- We say that a function $f(\vec{x})$ is *asymptotically less sensitive* than a function $g(\vec{x})$, if there exists a $\Delta > 0$ such that for every $\delta < \Delta$, $s_f(\delta) < s_g(\delta)$.
- We say that a function $f(\vec{x})$ is *the least asymptotically sensitive solution* to an interpolation problem $\mathcal{P}$ if f is a possible solution, and f is asymptotically less sensitive than any other possible solution.

Description of the problem in formal terms. Let $n = 2$ and $U = [0, 1] \times [0, 1]$. Let us define the following interpolation problem $\mathcal{P}_{\oplus}$:

- $\mathcal{F}$ = the set of all continuous functions $f : U \to [0, 1]$;
- $N = 4$, $f(0, 0) = f(1, 1) = 0$, $f(0, 1) = f(1, 0) = 1$.

THEOREM. *The only asymptotically least sensitive solution to the problem $\mathcal{P}_{\oplus}$ is* $f_{\oplus}(a, b) = \min(\max(a, b), 1 - \min(a, b))$.

Comments.

- Although the expression for $f(a, b)$ may appear clumsy, this solution is, in effect, very natural: it can be obtained if we take a formula from classical logic $a \oplus b = (a \vee b) \ \& \ \neg(a \ \& \ b)$ that expressed exclusive "or" in terms of "and", "or", and "not", and replace each of the logical operations &, $\vee$, and $\neg$ by the corresponding least sensitive operation.

- Instead of requiring the smallest *worst-case* sensitivity as we did, we could require the smallest *average-case* sensitivity. The corresponding formalism is described in [2], [5], [6]. According to the results from [2], [5], [6], the operation with the least average sensitivity must satisfy the Laplace equation

  $$\frac{\partial^2 f}{\partial a^2} + \frac{\partial^2 f}{\partial b^2} = 0.$$

  So, if we fix the boundary conditions by requiring that $f_\oplus(a, 0) = f_\oplus(0, a) = a$ and $f_\oplus(a, 1) = f_\oplus(1, a) = 1 - a$, we conclude that $f_\oplus(a, b) = a + b - 2 \cdot a \cdot b$.

  The same operation follows from the general Maximum Entropy approach described in [1] (it actually follows as a general result from a theorem proved in [1] for arbitrary logical operations).

- A more general probabilistic approach leads to an *interval* of possible values: namely, if we know that the probability $P(A)$ of an event A is equal to a and that $P(B) = b$, then one can check that the probability $P(A \oplus B)$ can take any value from the interval $[|a - b|, \min(a + b, 2 - a - b)]$.

Proof of the theorem. One can easily check that for the chosen function, $s_f(\delta) = \delta$.

Let us show that this sensitivity function is indeed asymptotically the smallest, and that the operation described in the formulation of the theorem is indeed the only one for which this sensitivity is attained.

Our proof is based on two ideas. The first idea is the remark that if we restrict a function to a subset, its sensitivity can only decrease.

The second idea is Theorem 1 from [3], according to which when we interpolate a function of one variable between two known values, then linear interpolation is the only one that leads to the least asymptotically sensitive solution.

Therefore, if we consider, e.g., the function $f'(a) = f(a, 0)$ of one variable, that is known to be equal to 0 for $a = 0$ and to 1 for $a = 1$, then any non-linearity would make this function asymptotically less sensitive than our chosen our chosen $f_\oplus$ (for which $s_f(\delta) = \delta$)). We can, therefore, make two conclusions:

- First, that there is no way that a possible solution to the interpolation problem is asymptotically better than our choice of $f_\oplus$.
- Second, that unless we want the function to be asymptotically worse than $f_\oplus$, we must have linear extrapolation of $f(a, 0)$, i.e., we must have $f(a, 0) = a$.

Similarly, we can conclude that if a function is not linear on any other edge of the unit square U, this function is asymptotically more sensitive than our choice. Thus, we must have $f(a, 0) = f(0, a) = a$ and $f(a, 1) = f(1, a) = 1 - a$.

Let us now describe the values of the desired asymptotically least sensitive operation *inside* the square. The inside of the square can be subdivided into four triangles by the diagonals $a = b$ and $a = 1 - b$.

Without losing generality, let us consider a triangle in which $b \le a$ and $b \le 1-a$. Let (a_0, b_0) be an arbitrary point from this triangle. Let us consider the following function of one variable: $f'(a) = f(a, k(a))$, where:

$$k(a) = \frac{b_0}{a_0} \cdot a \quad \text{for } a \le a_0, \qquad \text{and} \qquad k(a) = \frac{b_0}{1-a_0} \cdot (1-a) \quad \text{for } a \ge a_0.$$

For $a = 0$ and $a = 1$, we have $k(0) = k(1) = 0$.

One can easily see that for every $a, a' \in [0, 1]$, $|k(a) - k(a')| \le |a - a'|$; therefore, the condition that $|a-a'| \le \delta$ and $|k(a)-k(a')| \le \delta$ is simply equivalent to $|a - a'| \le \delta$. Thus, the sensitivity of a function $f(a, b)$, as limited to the piece-wise linear set $S = \{(a, k(a)) \mid a \in [0, 1]\}$, is exactly equal to the sensitivity of the function $f'(a)$ of one variable. Hence, due to Theorem 1, if a function $f'(a)$ is not linear, then the sensitivity of the restriction of f to S is asymptotically worse than δ. In this case, the sensitivity of the un-restricted function f itself is also worse than δ.

Thus, if for a function $f(a, b)$, one of its restrictions $f'(a)$ is non-linear, then $s_f(\delta)$ is asymptotically worse than δ. The only possibility for the sensitivity to be asymptotically equal to δ is when *all* the functions $f'(a)$ are linear, i.e., if $f(a, k(a)) = a$ for all a. In particular, this means that $f(a_0, b_0) = a_0$ as long as $b_0 \le a_0$ and $b_0 \le 1 - a_0$. Similarly, we conclude that:

- $f(a, b) = b$ if $a \le b \le 1 - a$;
- $f(a, b) = 1 - a$ if $b \ge a$ and $b \ge 1 - a$;
- $f(a, b) = 1 - b$ if $1 - a \le b \le a$.

These four expression describe exactly our function $f_\oplus(a, b)$. The theorem is proven. □

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