A Survey & Strengthening of Barnette's Conjecture

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Abstract

Tait and Tutte made famous conjectures stating that all members of certain graph classes contain Hamiltonian Cycles. Although the Tait and Tutte conjectures were disproved, Barnette continued this tradition by conjecturing that all planar, cubic, 3-connected, bipartite graphs are Hamiltonian, a problem that has remained open since its formulation in the late 1960s. This paper has a twofold purpose. The first is to provide a survey of the literature surrounding the conjecture. The second is to prove a new strengthened form of Barnette's Conjecture by showing that it holds if and only if for any arbitrary path P of length 3 that lies on a face in a planar, cubic, 3-connected, bipartite graph, there is a Hamiltonian Cycle which passes through the middle edge in P, and avoids both its leading and trailing edges. When combined with previous results, this has implications which further strengthen the conjecture.

Terminology

For the purpose of this paper, we assume that the reader is familiar with standard graph theoretic terminology. All graphs are undirected, with no self loops nor any multiple edges. **Barnette's Conjecture** states that all planar, cubic, 3-connected, bipartite graphs are Hamiltonian. For the sake of not having to constantly repeat these adjectives, these graphs will henceforth be referred to as **Barnette graphs**. A **planar** graph is one that can be drawn on a two-dimensional plane such that no two edges cross. A **cubic** graph is one in which all vertices have degree 3. A k**-connected** graph is one that cannot be disconnected by the removal of fewer than k vertices. A graph whose vertices can be coloured using exactly two different colours such that no two adjacent vertices have the same colour is called **bipartite**. A **Hamiltonian Cycle** is a simple cycle which passes through every vertex in a graph exactly once. A graph that contains a Hamiltonian Cycle is said to be **Hamiltonian**. A **Hamiltonian Path** between two vertices u and v is a path which contains every vertex in a graph exactly once and has endpoints u and v. The number of **sides** of a face refers to the number of edges surrounding it. A **marked edge** is one that has been marked as having to be included in every Hamiltonian Cycle in a graph, even if its inclusion is not necessary. A **forced edge** is one which must necessarily be included in every Hamiltonian Cycle.

Introduction

Barnette's Conjecture, first announced in [6] and later in [21], is part of a series of conjectures stating that all members of certain graph classes contain Hamiltonian Cycles. In [33] Tait conjectured that all planar, cubic, 3-connected graphs are Hamiltonian. Had this been true, it would have supported a short, elegant proof of the Four-Colour Theorem. Tait's Conjecture stood for more than 60 years, but was

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disproved by Tutte, who in [35] cleverly constructed a counterexample. In [37] Tutte conjectured that all cubic, 3-connected, bipartite graphs are Hamiltonian. Tutte's Conjecture fell due to a counterexample by Horton, published in [8]. Barnette's Conjecture subsumes the other two conjectures in that any counterexample would have to be a simultaneous counterexample to both the Tait and Tutte Conjectures.

Despite considerable effort to prove otherwise, Barnette's Conjecture has stood since its publication almost 40 years ago. Many partial and related results have been obtained and are described below, but it still remains unclear whether the conjecture is true or false. The first part of this paper is dedicated to providing a survey of these results. The second part of this paper contains a new result which strengthens Barnette's Conjecture, the hope being that a stronger conjecture may be easier to settle. Specifically, we prove that the conjecture is true if and only if for any arbitrary path P of length 3 that lies on a face in a Barnette graph, there is a Hamiltonian Cycle which passes through the middle edge in P, and avoids both its leading and trailing edges. When combined with a previous result, this theorem yields a number of interesting corollaries which further strengthen Barnette's Conjecture.

Survey of Partial & Related Results

Some Useful Facts

Although the following are too obvious and simplistic to be called theorems, they are nevertheless important facts pertaining to Barnette graphs:

- A plane graph is bipartite if and only if each of its faces has an even number of sides [22].
- The minimum size of any face in a Barnette graph is 4.
- Every Barnette graph has an even number of vertices.
- The bipartitions of every Barnette graph have equal size. This follows directly from the fact that Barnette graphs are cubic, so if one had unequal partitions, the number of edges leaving one partition would not equal the number of edges entering the other one, a clear contradiction. The relevance of this fact is that any effort to construct a non-Hamiltonian Barnette graph by using the obvious method of making it unevenly bipartite will fail.
- Every Barnette graph is 2-colourable.

The Tait & Tutte Conjectures in More Detail

Tait's Conjecture

As already mentioned, Barnette's Conjecture is similar to Tait and Tutte's conjectures, both of which have been settled. The Tait conjecture [33] stated that all planar, cubic, 3-connected graphs are Hamiltonian, and was disproved by Tutte [35], who constructed the non-Hamiltonian counterexample shown below in Figure 1.

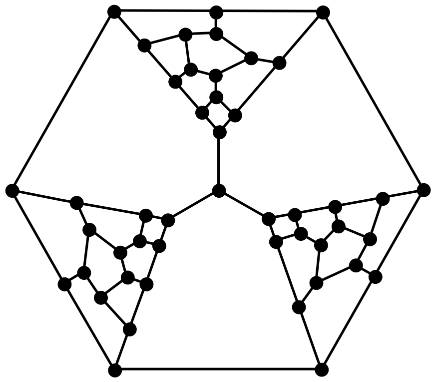

Figure 1: Tutte's Counterexample To Tait's Conjecture

Once this graph was discovered, many others followed; the literature contains dozens of interesting counterexamples to Tait's Conjecture. One of the more noteworthy ones was discovered by Grinberg, who proved a necessary condition for Hamiltonicity, and applied it to show that his graph is non-Hamiltonian. His necessary condition applies to planar graphs and can be stated as follows:

Theorem 1 (Grinberg's Theorem [8]). *Every planar graph $G = (V, E)$ containing a Hamiltonian Cycle C satisfies the following equation:*

$$\sum_{i=3}^{n}(i-2)(f_i' - f_i'') = 0, \tag{1}$$

where f_i' represents the number of faces of size i on the interior of C, and f_i'' represents the number of faces of size i on the exterior of C.

Grinberg's Graph, shown below in Figure 2, does not satisfy its associated equation, and is therefore not Hamiltonian.

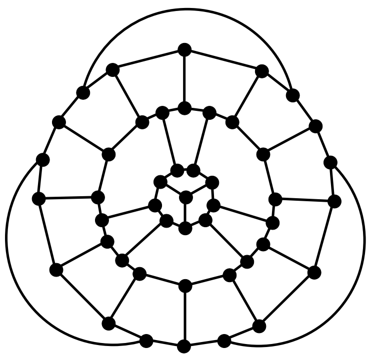

Figure 2: The Grinberg Graph

Using Grinberg's theorem to find a counterexample to Barnette's Conjecture seems to be difficult; since all faces in Barnette graphs have an even number of sides, constructing a graph with a contradictory associated equation may require some clever trick, if it is at all possible.

With Tait's conjecture so well studied, the only remaining problem was to find the smallest counterexample. With 46 vertices, both the Tutte and Grinberg graphs seem quite large. According to [23], Lederberg, Barnette, and Bosák together contributed 6 different counterexamples containing 38 vertices each. The papers [3] and [25] together showed that these are the only such graphs on 38 vertices, and furthermore that they are the smallest counterexamples to Tait's conjecture. Since Tait's Conjecture contains Barnette graphs as a proper subset, it follows that all Barnette graphs on 36 or fewer vertices are Hamiltonian.

The literature contains a considerable amount of work concerning cyclically connected planar graphs, both Hamiltonian and non-Hamiltonian. This research is peripherally related to Barnette's Conjecture, but too expansive to be included here. An interested reader should consult [4, 23].

Tutte's Conjecture

As already mentioned, Tutte's Conjecture [37], which stated that all cubic, 3-connected, bipartite graphs are Hamiltonian was disproved by Horton [8], who constructed the non-Hamiltonian counterexample shown below in Figure 3.

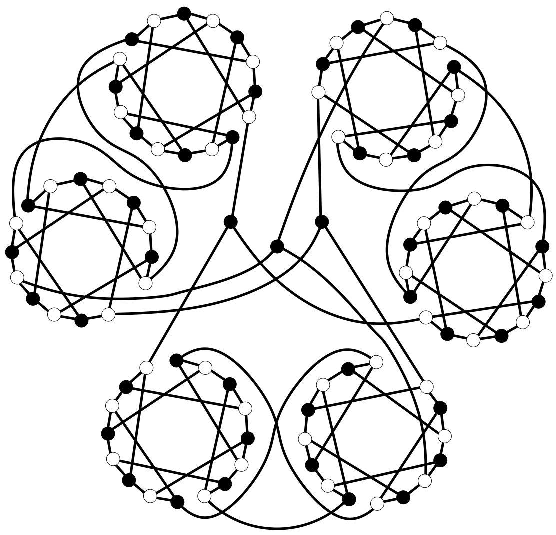

Figure 3: The Horton Graph

Much like Tait's conjecture, the Tutte conjecture has also been well-studied. The definitive survey on this topic is [20]. In it Gropp describes the search for ever smaller counterexamples to Tutte's Conjecture. With 96 vertices, Horton's graph is quite large. In 1982, Horton discovered a smaller one having 92 vertices [26]. In the same year, Ellingham discovered an infinite family of counterexamples, the smallest having 78 vertices. The following year, Ellingham, Horton, and Culberson discovered a counterexample with 54 vertices [13]. Finally, in 1989 Georges discovered a graph with 50 vertices [17]. This is the smallest counterexample currently known. Working from the other direction, in 1981 Ellingham proved that all cubic, 3-connected, bipartite graphs on 22 or fewer vertices are Hamiltonian [11]. This result includes Barnette graphs as a proper subset, but is subsumed by the lower bound of 36 vertices given by the work on Tait's Conjecture.

It turns out that there is a slight twist to this timeline. In fact, all of the above results pertaining to Tutte's Conjecture had first been proved by Soviet mathematicians, but remained unknown to researchers in the West. This was due mainly to the fact that most of the Soviet results were published in Russian, and appeared only in obscure journals which did not circulate outside of the Iron Curtain. The political climate in which other forms of information did not flow freely only added to the problem. The most important results worth mentioning belong to Kelmans, who published two papers. The first, published in 1986 [28] is rich with many results and contains a series of counterexamples to Tutte's conjecture, the smallest of which has 50 vertices. In a second paper [29], Kelmans constructs 4 non-isomorphic 50-vertex counterexamples, one of which is isomorphic to that of Georges. Kelmans' result therefore not only came before that of Georges, but also subsumes it, giving him clear precedence. Working from the other direction, Baraev and Faradzhev [5] in 1978 proved exactly the same result that Ellingham would later prove in [11]. According to [20], Kelmans and Lomonosov proved without the aid of a computer that Tutte's Conjecture holds for up to and including 30 vertices. Unfortunately, no reference is given. If this is true, then it is the strongest result known. Whether or not the graphs on 50 vertices are the smallest counterexamples to Tutte's Conjecture remains an open problem.

Other Related Results

Complexity results closely related to Barnette's Conjecture are also known; in [1] Akiyama et al. proved that the Hamiltonian Cycle problem remains $\mathcal{NP}$-Complete even when the input is restricted to planar, cubic, 2-connected, bipartite graphs. Nobody has been able to show that it remains $\mathcal{NP}$-Complete when restricted to Barnette graphs. Such a result would clearly disprove the conjecture because the reduction would allow us to take an unsatisfiable formula and convert it to a non-Hamiltonian Barnette graph.

There are two final related results worth mentioning. In [36], Tutte proved that all planar 4-connected graphs are Hamiltonian, and in [34] Thomassen extends this result by showing that every planar 4-connected graph is Hamiltonian connected, that is, for any pair of vertices, there is a Hamiltonian Path with those vertices as endpoints.

Results Directly Concerning Barnette's Conjecture

Support For The Conjecture

From the results on Tait's Conjecture, we know that a minimal counterexample to Barnette's conjecture must have at least 36 vertices. In fact, we have a much better lower bound; in [24], Holton et al. confirmed through a combination of clever analysis and computer search that all Barnette graphs with up to and including 64 vertices are Hamiltonian. In an announcement [14, 2], McKay et al. used computer search to extend this result to 84 vertices, inclusive. This implies that if Barnette's Conjecture is indeed false, then a minimal counterexample must contain at least 86 vertices, and is therefore considerably larger than the minimal counterexamples to Tait and Tutte's conjectures. This is not all we know about a possible counterexample; another interesting result is that of Fowler, who in an unpublished manuscript [15] provides a list of subgraphs that cannot appear in any minimal counterexample to Barnette's Conjecture.

Also supporting Barnette's Conjecture, Goodey in [18] considered proper subsets of the Barnette graphs and proved the following:

Theorem 2. *Every Barnette graph which has faces consisting exclusively of quadrilaterals and hexagons is Hamiltonian, and furthermore in all such graphs, any edge that is common to both a quadrilateral and a hexagon is part of some Hamiltonian Cycle.*

Theorem 3. *Every Barnette graph which has faces consisting of 7 quadrilaterals, 1 octagon, and any number of hexagons is Hamiltonian, and furthermore in all such graphs, any edge that is common to both a quadrilateral and an octagon is part of some Hamiltonian Cycle.*

The proof of Goodey's theorems proceeds by induction on the number of faces in the respective graph classes. These results tell us a little about what any counterexamples to Barnette's Conjecture must look like. For example, we know that it must contain at least one face that has 8 or more sides.

Barnette's Conjecture & Graph Colouring

In [27], Jensen and Toft report that Barnette's Conjecture is equivalent to two other open problems:

Theorem 4. *Barnette's Conjecture is true if and only if for every Barnette graph G, it is possible to partition its vertex set into two subsets so that each induces an acyclic subgraph of G.*

[This theorem is probably not correct as stated, or else it trivially settles Barnette's conjecture, since for any bipartite graph, you can simply delete one partition, and the remainder is acyclic, making the conjecture true.]

Theorem 5. *Barnette's Conjecture is false if and only if there exists a planar graph G satisfying $\chi(G) = \varrho(G) = 3$, where $\chi(G)$ is G's chromatic number, and $\varrho(G)$ is G's point arboricity, that is, the minimum number of colours used in a coloring of the vertices of G such that each colour class induces an acyclic subgraph.*

The second result is particularly interesting because it provides a connection between Barnette's Conjecture and the seemingly unrelated area of graph colouring. In [14], Feder and Subi proved another result relating Barnette's Conjecture to graph colouring:

Theorem 6. *For any Barnette graph, if its faces can be 3-coloured such that two of the three colour classes contain only squares and hexagons, and the third colour class contains faces surrounded by an even number of squares, then that graph is Hamiltonian.*

In addition, at least one relevant connection between edge colouring and Barnette's conjecture is known. In a 1916 paper [31], König proved the following:

Theorem 7. *The edges of any bipartite graph G can be coloured with Δ colours, where Δ is the maximum degree of any vertex in G.*

Since Barnette graphs are bipartite and cubic, this implies that all Barnette graphs are 3-edge-colourable. This has an immediate corollary pertaining to cycles:

Corollary 1. *Every Barnette graph contains at least 3 distinct 2-factors.*

One strategy for proving Barnette's Conjecture might be to show that for every Barnette graph, it is always possible to take some 2-facto and amalgamate its cycles to form a Hamiltonian Cycle.

Strengthening Barnette's Conjecture

A different direction of research into Barnette's Conjecture parallels Tutte's search for a counterexample to Tait's Conjecture. In [38], Tutte describes a series of results which repeatedly strengthened Tait's Conjecture until a counterexample finally became apparent. Researchers working on Barnette's Conjecture have adopted the same strategy. In [22], this author proved the following strengthening result, which immediately yields an interesting corollary:

Theorem 8. *Barnette's Conjecture holds if and only if any arbitrary edge in a Barnette graph is part of some Hamiltonian Cycle.*

Corollary 2. *Barnette's conjecture holds if and only if for every pair of adjacent vertices in a Barnette graph, there is a Hamiltonian Path with those vertices as endpoints.*

This corollary is interesting because it is similar to Thomassen's result pertaining to Hamltonian connectivity.

Shortly afterwards, Braverman [9] further strengthened Barnette's Conjecture with the following result, which subsumes the previous result and again yields a useful corollary:

Theorem 9. *Barnette's Conjecture holds if and only if for any arbitrary face in a Barnette graph, there is a Hamiltonian Cycle which passes through any two arbitrary edges on that face.*

Corollary 3. *Barnette's Conjecture holds if and only if for any arbitrary edge in a Barnette graph, there is a Hamiltonian Cycle which avoids that edge.*

Proof: $\Rightarrow$ Suppose that Barnette's Conjecture holds. Then from Theorem 9, we know that there is a Hamiltonian Cycle which passes through any two adjacent edges of any Barnette graph. This means that the remaining edge is avoided. $\Leftarrow$ Trivial. $\square$

History was to repeat itself. It turns out that the strengthening results above had already been proved by the same A.K. Kelmans that proved all of the results pertaining to Tutte's Conjecture results before they had been discovered in the West. Kelmans published his strengthening result of Barnette's Conjecture in [28], the same paper which contains his results on Tutte's Conjecture. In fact, Kelmans' strengthening result is strictly stronger than those above in the sense that it subsumes both results:

Theorem 10. *Barnette's Conjecture holds if and only if for any arbitrary face in a Barnette graph, and for any arbitrary edges e_1 and e_2 on that face, there is a Hamiltonian Cycle which passes through e_1 and avoids e_2.*

Kelmans' result has since been translated into English [30], but remains strangely obscure and is hardly ever referenced. Perhaps one of the goals of this paper should be to draw attention to Kelmans' publication, which contains many more interesting results. Not only would this give Kelmans the recognition that he deserves, but it would also make it less likely that future researchers overlook it.

One important note about the strengthening results is that instead of talking about Hamiltonian Cycles which "pass through" or "avoid" certain edges, it is often useful to word the theorems more prescriptively. For example, instead of saying that "a Hamiltonian Cycle passes through edge e_i", we might say "we may mark edge e_i as having to be included in any Hamiltonian Cycle found without destroying Hamiltonicity". Likewise, instead of saying "a Hamiltonian Cycle avoids edge e_j", we can say "we may delete edge e_j without destroying Hamiltonicity".

With that in mind, Kelmans' Theorem also yields an interesting Corollary:

Corollary 4. *Barnette's Conjecture holds if and only if in any Barnette graph, any path of length 3 which has two edges on a face F, and the final edge leaving that face is part of some Hamiltonian Cycle.*

Proof: $\Rightarrow$ Suppose that Barnette's Conjecture holds. Let P be any arbitrary path of length 3 that has two edges on a face F, and the final edge leaving F. By Theorem 10 we may delete any edge and mark any other edge on F without destroying Hamiltonicity; simply delete a relevant edge on F in order to force the portion of the path that has one edge on F and the other leaving it, and then mark the remaining edge on F in order to extend the path to length 3. Since P was arbitrary, this result generalizes to all such paths, as required. $\Leftarrow$ Trivial. $\square$

Barnette's Other Conjecture

Barnette's less-famous Hamiltonicity conjecture [32] states that all planar, cubic, 3-connected (bipartite omitted) graphs with all faces having 6 or fewer sides are Hamiltonian. This conjecture is of interest because this class of graphs contains the Fullerenes as a proper subset. They are planar, cubic, 3-connected graphs with exactly 12 faces of degree 5, and all other faces having degree 6, and are of interest to chemists because they represent possible molecular structures of pure carbon [23].

Partial results have also been achieved here. In [19], Goodey proved that the conjecture holds for a proper subset of graphs:

Theorem 11. *Every planar, cubic, 3-connected graph which has faces consisting exclusively of triangles and hexagons is Hamiltonian.*

In [4], Aldred et al. proved that Barnette's Second Conjecture holds for up to 176 vertices, inclusive. In [10], this result was extended by Brinkmann et al., who confirm that it holds for up to 250 vertices, inclusive.

A relevant complexity-related result is that of Garey, Johnson, and Tarjan, who in [16] proved that the Hamiltonian Cycle problem remains $\mathcal{NP}$-Complete when restricted to planar, cubic, 3-connected graphs with no face having fewer than 5 sides.

Commentary on Results

The above results show that Barnette graphs seem to exist on a knife-edge between Hamiltonicity and non-Hamiltonicity. As shown by Goodey [18], we can further restrict the set by placing constraints on the face sizes, and show that all of the resulting graphs are Hamiltonian. Similarly, if we omit some of Barnette's adjectives, and strengthen others, again we can prove that the resulting set is Hamiltonian; for example, as Tutte showed in [36], all planar 4-connected graphs are Hamiltonian.

On the other hand, if we remove any of the adjectives from Barnette's Conjecture, then the resulting set contains non-Hamiltonian graphs. The Tutte and Horton graphs show this to be true for the removal of bipartiteness and planarity, respectively. The Kirkman graph [7], shown below in Figure 4, is a non-Hamiltonian, planar, 3-connected, bipartite graph, showing that the set resulting from removing the cubic condition contains non-Hamiltonian graphs. This fact in and of itself is interesting, because the minimum degree in Kirkman's graph is 3, and it contains three degree-4 vertices. Since its average degree is higher than that of a Barnette graph, one would expect that if anything, Kirkman's graph would be more Hamiltonian. Instead, the opposite is true, showing that this area of research has some counterintuitive aspects.

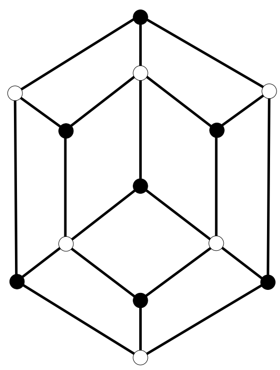

Figure 4: Kirkman's Planar, 3-Connected, Bipartite, Non-Hamiltonian Graph

Finally, if we relax the 3-connectivity condition, we again end up with a set containing non-Hamiltonian graphs; as already mentioned, from [1] we know that the Hamiltonian Cycle problem remains $\mathcal{NP}$-Complete even when the input is restricted to planar, cubic, 2-connected, bipartite graphs, so it is possible to take an unsatisfiable formula and map it to a non-Hamiltonian planar, cubic, 2-connected graph.

In effect, the surrounding results seem to give some insight into why Barnette's Conjecture has been so hard to settle; the Barnette graphs are very close to both Hamiltonicity and non-Hamiltonicity, and border cases are often the hardest ones.

New Strengthening Result

We now present our new result. It was inspired by a technique that Tutte used in the construction of his counterexample to Tait's Conjecture [38]. Tutte showed that it is possible to add some edges and vertices to any face in a planar, cubic, 3-connected graph without destroying any of those properties. He then applied one of his strengthened versions of Tait's Conjecture in order to prove another strengthened version. Our result follows this template. We show that it is possible to add edges to any face in a Barnette graph G in such a way that the result G^* will always be a Barnette graph. We then apply Theorem 10, which allows us to delete one edge, and mark another one on any face without destroying Hamiltonicity. The resulting Hamiltonian Cycle in G^* corresponds to a Hamiltonian Cycle in G which passes through the middle edge in a path P of length 3, and avoids its other two edges. Since the face that we chose was arbitrary, the result holds for any path P of length 3 on any face of a Barnette graph.

Theorem 12. *Barnette's Conjecture holds if and only if for any arbitrary path P of length 3 that lies on a face in a Barnette graph, there is a Hamiltonian Cycle which passes through the middle edge in P, and avoids both its leading and trailing edges.*

Proof: $\Rightarrow$ Suppose that Barnette's Conjecture is true. Let $P = (A, B, C)$ be any arbitrary path of length 3 lying on the face of a Barnette graph G, as shown below in Figure 5.1. P lies on the face F_2, which is separated from face F_1 by edge B. We know that every face in a Barnette graph has an even number of sides, and that the smallest possible face has 4 edges surrounding it, so the dotted arc on F_1 represents a path with an odd number of edges in it. We will show that there is a Hamiltonian Cycle in G which avoids edges A and C, and passes through B. The construction proceeds as follows: subdivide F_1 by adding vertices and edges as shown in figure 5.2 in order to create G^*. Note that every face created inside F_1 has an even number of sides, including the face bordered by the dotted arc, which contains an odd number of edges. In addition, we always add two vertices onto edges surrounding F_1, so the adjacent faces remain even. Since all faces affected or created by this construction are even and the construction also clearly preserves planarity, 3-regularity, and 3-connectivity, we know that G^* is a Barnette graph. Since we assumed that Barnette's Conjecture is true, Theorem 10 applies, so we may delete any edge and mark any other edge on any face in G^* without destroying Hamiltonicity. We will therefore delete edge x, and mark edge y. This allows us to force further edges, and delete others, as shown in Figure 5.3, based on the principles that any edges incident on degree-2 vertices is forced, and that any unmarked edges incident on degree-3 vertices which already have 2 marked edges incident on them may be deleted without destroying any Hamiltonian Cycles. The last edge to be deleted is z; the justification for its removal is that if it had been included, it would have created a small cycle. Since G^* with x removed and y marked was Hamiltonian, so is the resulting graph $G^{*'}$, shown in Figure 5.4. It is not hard to see that the original graph $G^{'}$, shown in Figure 5.5, therefore must also contain a Hamiltonian Cycle and furthermore that it avoids edges A and C while passing through B. Since P was arbitrary, the result generalizes to all paths of length 3 on faces of Barnette graphs, as required.

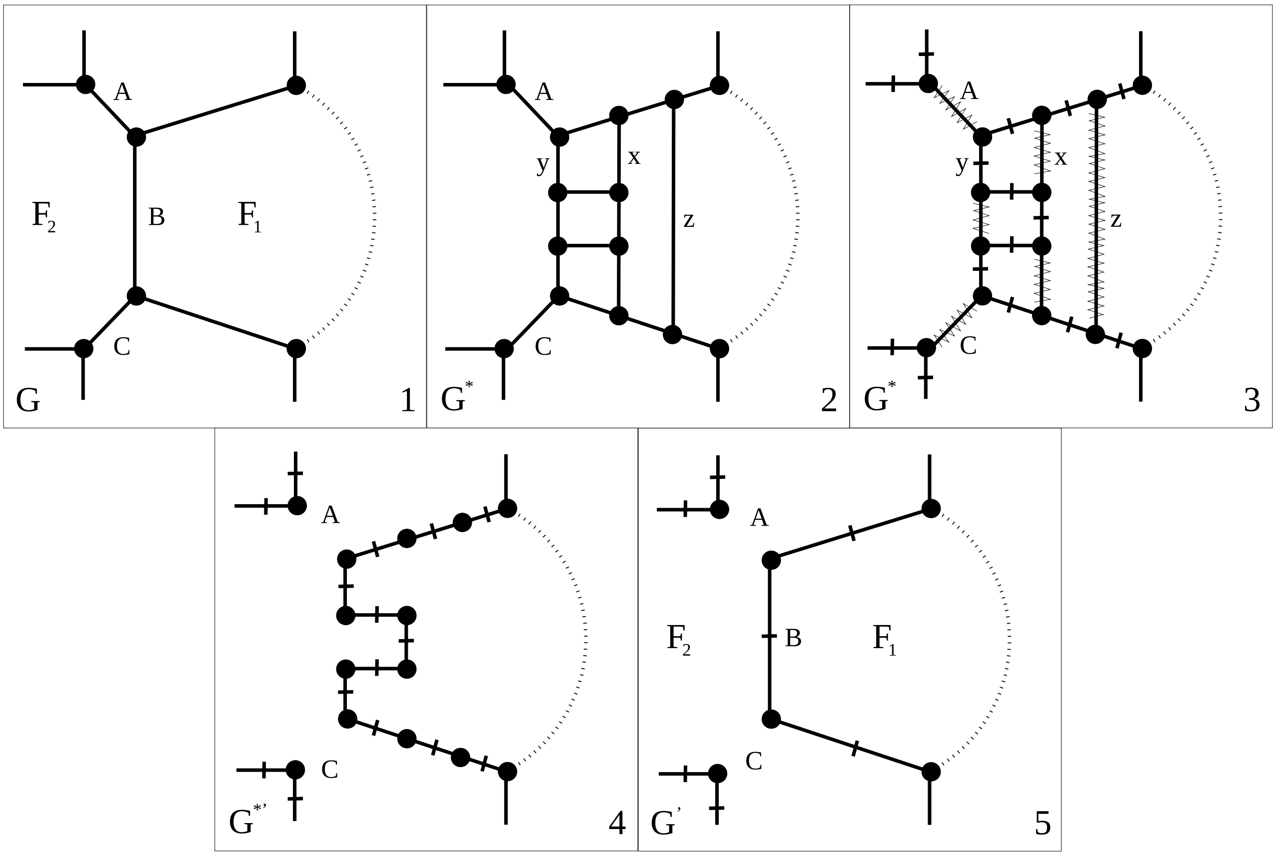

Figure 5: The Construction for the Proof of Theorem 12

$\Leftarrow$ Trivial. □

This theorem gives rise to some interesting corollaries:

Corollary 5. *Barnette's Conjecture holds if and only if in any Barnette graph, any arbitrary path of length 3 lying on a face is part of some Hamiltonian Cycle.*

Proof: $\Rightarrow$ This follows immediately from the construction shown in Figure 5 above. The result of deleting the two edges on face F_2 was that a path of length 3 was forced on F_1. Since we may control where we apply Theorem 12, we can use it to force any path of length 3 lying on any arbitrary face, as required. $\Leftarrow$ Trivial. □

Corollary 6. *Barnette's Conjecture holds if and only if in any Barnette graph, any arbitrary path of length 3 is part of some Hamiltonian Cycle.*

Proof: $\Rightarrow$ Suppose that Barnette's Conjecture holds. There are only two possibilities for a path of length 3 in a Barnette graph; either all of its edges lie on one face, or two of its edges lie on a face, and one leaves it. The first case is handled by Corollary 5, which allows us to mark any path of length 3 that has all of its edges on a face without destroying Hamiltonicity. The second case is handled by Corollary 4, which allows us to mark any path of length 3 that has two edges on a face, and one leaving it. Therefore, in either case there is a Hamiltonian Cycle which includes the path of length 3, as required. $\Leftarrow$ Trivial. □

Corollary 7. *Barnette's Conjecture holds if and only if any Barnette graph which contains a quadrilateral face contains at least 6 distinct Hamiltonian Cycles.*

Proof: $\Rightarrow$ Suppose that Barnette's Conjecture holds. We may therefore apply both Corollary 5 and Theorem 12. Let G be any arbitrary Barnette graph that contains a quadrilateral face. There are exactly 6 ways in which a Hamiltonian Cycle may pass through the square face; either the cycle includes 3 edges on that face, or it only uses 2. Figure 6.1 below shows one way in which a Hamiltonian Cycle may use 3 edges. There are exactly 4 such ways. By applying Corollary 5, we may mark any path of length 3 on a face without destroying Hamiltonicity, and thereby cause any of these 4 ways to occur. Figure 6.2 shows one way in which a Hamiltonian Cycle may use 2 edges. There are exactly 2 such ways. By applying Theorem 12, we may take any path of length 3 on a face, and delete its leading and trailing edges without destroying Hamiltonicity. This lets us delete any two opposite edges on a quadrilateral face, allowing each of the 2 ways to occur. Therefore, all 6 ways in which a Hamiltonian Cycle may pass through a square face may occur. Since G was arbitrary, the result generalizes to all Barnette graphs containing quadrilateral faces, as required.

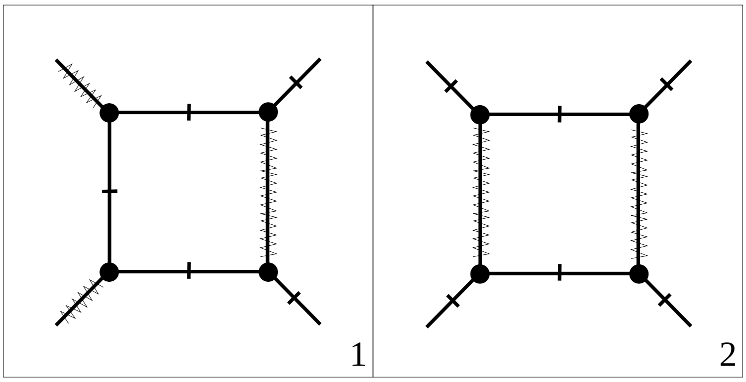

Figure 6: The Different Ways in Which a Hamiltonian Cycle May Pass Through a Quadrilateral Face

$\Leftarrow$ Trivial. $\square$

Future Research

It is difficult to say whether any of the techniques described above will aid in settling Barnette's Conjecture. Certainly many of them seem to be useful and worth extending. One strategy is to keep chipping away at it; if Barnette's Conjecture is true, then perhaps Goodey's results can be extended to show that successively larger and larger subsets of Barnette graphs are Hamiltonian.

If Barnette's Conjecture is false, then any one of the many strategies above may help to settle it. We know some of the characteristics that a minimal counterexample must have. For example, we know that it must contain at least one face of with 8 or more sides. We have a list of subgraphs which it cannot contain. In addition, we know that it must have at least 86 vertices. This last result was proved using computers, but the days of computers being very helpful with Barnette's Conjecture may be drawing to a close. Not only is the number of graphs to be investigated becoming enormous, but any potential candidates are becoming so large that even proving non-Hamiltonicity may be difficult in a computational sense. It stands to reason that future research will most likely involve more theoretical analysis and proportionally less computer power.

The strategy of strengthening Barnette's Conjecture seems to hold some promise. The known results are quite strong; if Barnette's Conjecture is false, then it probably won't take many more strengthening results before a counterexample is found. On the other hand, if the conjecture is true, then such robust characteristics in sparse, planar graphs may have some practical applications.

Then again, the key to settling Barnette's Conjecture may lie in a different area of graph theory altogether; perhaps we have better tools for tackling one of its equivalent problems.

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