CSC358: Tutorial 3

Principles of Computer Networks

Problem 1

a)

\[ P_n = \binom{N}{n}(\lambda \Delta t)^n (1 - \lambda \Delta t)^{N-n} \quad n=0, 1, \ldots, N \]

b)

\[ P_n = \binom{N}{n}(\lambda \Delta t)^n (1 - \lambda \Delta t)^{N-n} \]

For \( T \) fixed, as \( N \to \infty \), we have \( \Delta t = T/N \to 0 \)

- \[ \lim_{\Delta t \to 0} (1 - \lambda \Delta t)^N = \lim_{\Delta t \to 0} (1 - \lambda \Delta t)^{T/\Delta t} = e^{-\lambda T} \]

- \[ \lim_{\Delta t \to 0} (1 - \lambda \Delta t)^{-n} = 1 \]

- \[ \lim_{N \to \infty} \frac{N(N-1)\ldots(N-(n-1))}{n^n} = \lim_{N \to \infty} \frac{1}{n} \left( 1 - \frac{1}{n} \right) \left( 1 - \frac{2}{n} \right) \ldots \left( 1 - \frac{n-1}{n} \right) = 1 \]

We then obtain that

- \[ P_n = \frac{(\lambda T)^n}{n!} e^{-\lambda T} \quad n=0, 1, \ldots, N \]

which is the Poisson distribution.

c) Let \( T \) be the inter-arrival time. Then using (b), we have that

\[ P(T \leq t) = P(\text{at least one arrival in the interval } [0,t]) \]

\[ = 1 - e^{-\lambda t}, \quad t > 0 \]

i.e. the inter-arrival time is given by an exponential distribution.
Problem 2

a) \[ \frac{\lambda_1}{\lambda_2 + \lambda_1} \]

b) \[ \frac{\lambda_2}{\lambda_2 + \lambda_1} \]

c) Define \( \lambda = \lambda_2 + \lambda_1 \). The state transition diagram is shown below

![State Transition Diagram]

d) Define \( \rho_1 = \frac{\lambda_1}{\mu} \) and \( \rho_2 = \frac{\lambda_2}{2\mu} \). It follows that

\[
P_1 = (1 - \mu \delta)P_1 + \lambda_1 \delta P_0
\]

\[
P_2 = (1 - 2\mu \delta)P_2 + \lambda_2 \delta P_0
\]

Which leads to,

\[
P_1 = \frac{\lambda_1}{\mu}P_0
\]

\[
P_2 = \frac{\lambda_2}{2\mu}P_0
\]

Using the condition that \( P_0 + P_1 + P_2 = 1 \), we obtain that

\[
P_0 \left(1 + \frac{\lambda_1}{\mu} + \frac{\lambda_2}{2\mu}\right) = 1
\]

Define \( \bar{\rho} = \frac{2\lambda_1 + \lambda_2}{2\mu} \), then we get

\[
P_0 = \frac{1}{1 + \bar{\rho}}
\]

\[
P_1 = \bar{\rho_1} \frac{1}{1 + \bar{\rho}}
\]

\[
P_2 = \bar{\rho_2} \frac{1}{1 + \bar{\rho}}
\]
Problem 3

(a) \[ P = \lambda_1 \Delta t \]

(b) \[ P = \lambda_2 \Delta t \]

(c) \[ P = \lambda_1 \Delta t (1 - \lambda_2 \Delta t) \approx \lambda_1 \Delta t \]

(d) \[ P = \lambda_2 \Delta t (1 - \lambda_1 \Delta t) \approx \lambda_2 \Delta t \]

(e) \[ P = (1 - \lambda_1 \Delta t)(1 - \lambda_2 \Delta t) = 1 - (\lambda_1 + \lambda_2)\Delta t + \lambda_1 \lambda_2 (\Delta t)^2 \approx 1 - (\lambda_1 + \lambda_2)\Delta t \]

(f) \[ P = \lambda_1 \Delta t (1 - \lambda_2 \Delta t) + \lambda_2 \Delta t (1 - \lambda_1 \Delta t) = (\lambda_1 + \lambda_2)\Delta t - 2\lambda_1 \lambda_2 (\Delta t)^2 \approx (\lambda_1 + \lambda_2)\Delta t \]

(g) \[ P = \lambda_1 \lambda_2 (\Delta t)^2 \approx 0 \]

(h) \[ P = \frac{\lambda_1 \Delta t (1 - \lambda_2 \Delta t)}{\lambda_1 \Delta t (1 - \lambda_2 \Delta t) + \lambda_2 \Delta t (1 - \lambda_1 \Delta t)} \approx \frac{\lambda_1}{\lambda_1 + \lambda_2} \]

(i) \[ P = \frac{\lambda_2 \Delta t (1 - \lambda_1 \Delta t)}{\lambda_1 \Delta t (1 - \lambda_2 \Delta t) + \lambda_2 \Delta t (1 - \lambda_1 \Delta t)} \approx \frac{\lambda_2}{\lambda_1 + \lambda_2} \]

(j) Let \( E_1 \) be the event that the link \( \text{OUT}_1 \) transmits a packet in a time-slot of length \( \Delta t \), and let \( A \) be the event that we have an arrival at the switch. Then we have that

\[ P(E_1) = P(E_1 \mid A)P(A) = p(\lambda_1 + \lambda_2)\Delta t \]

(k) Let \( E_2 \) be the event that the link \( \text{OUT}_2 \) transmits a packet in a time-slot of length \( \Delta t \), and let \( A \) be the event that we have an arrival at the switch. Then we have that

\[ P(E_2) = P(E_2 \mid A)P(A) = (1 - p)(\lambda_1 + \lambda_2)\Delta t \]