Problem 1 Solution

(a) Let $P_{off}^{(1)}$ be the probability that at time $k = 1$ the transceiver is in state $S_{off}$. We have, $P_{off}^{(1)} = \frac{2}{3}$.

(b) Let $P_{off}^{(k)}$ ($P_{on}^{(k)}$) be the probability that the transceiver is in state $S_{off}$ ($S_{on}$) at time $k$ for $k = 0, 1, 2, \ldots$.

From the initial conditions, we have that $P_{off}^{(0)} = 1$ and $P_{on}^{(0)} = 0$. For $k = 1, 2, 3, \ldots$, we have that

$$P_{off}^{(k)} = \frac{2}{3} P_{off}^{(k-1)} + \frac{1}{3} P_{on}^{(k-1)},$$

and

$$P_{on}^{(k)} = \frac{1}{3} P_{off}^{(k-1)} + \frac{2}{3} P_{on}^{(k-1)}.$$

We then obtain

\[
\begin{align*}
P_{off}^{(1)} &= \frac{2}{3}, & P_{on}^{(1)} &= \frac{1}{3} \\
P_{off}^{(2)} &= \frac{2}{3} + \frac{1}{3} = \frac{5}{9}, & P_{on}^{(2)} &= \frac{1}{3} + \frac{2}{3} = \frac{4}{9} \\
P_{off}^{(3)} &= \frac{1}{3} + \frac{2}{3} = \frac{14}{27}, & P_{on}^{(3)} &= \frac{1}{27} + \frac{4}{27} = \frac{13}{27} \\
P_{off}^{(4)} &= \frac{1}{81} + \frac{13}{27} = \frac{41}{81}, & P_{on}^{(4)} &= \frac{1}{27} + \frac{40}{81} = \frac{40}{81}
\end{align*}
\]

(c) See question (b):

$$P_{off}^{(1)} = \frac{2}{3} P_{off} + \frac{1}{3} P_{on},$$

$$P_{on}^{(1)} = \frac{1}{3} P_{off} + \frac{2}{3} P_{on}.$$

(d) From (b) and (c), we need that

$$P_{off} = \frac{2}{3} P_{off} + \frac{1}{3} P_{on},$$

and

$$P_{on} = \frac{1}{3} P_{off} + \frac{2}{3} P_{on}.$$

Any solution for this system of equations has the property that

$$P_{off} = P_{on}.$$

As $P_{off}$ and $P_{on}$ are probabilities, we need that

$$P_{off} + P_{on} = 1.$$

Combining the two conditions above, we obtain that

$$P_{off} = P_{on} = \frac{1}{2}.$$
(e) We have
\[
\lim_{k \to \infty} P_{\text{off}}^{(k)} = \frac{1}{2},
\]
and
\[
\lim_{k \to \infty} P_{\text{on}}^{(k)} = \frac{1}{2}.
\]

**Problem 2 Solution**

(a) You can utilize \[ \sum_{k=1}^{\infty} kx(1 - x)^{k-1} = \frac{1}{x}. \]
\[
E[L] = \sum_{l=1}^{\infty} l\alpha(1 - \alpha)^{l-1} = \frac{1}{\alpha}.
\]

(b) For \( l > l_0 \), we have that
\[
P(L = l|L > l_0) = \frac{P(L = l \text{ and } L > l_0)}{P(L > l_0)} = \frac{P(L = l)}{P(L > l_0)}.
\]
Note that
\[
P(L = l) = \alpha(1 - \alpha)^{l-1},
\]
and
\[
P(L > l_0) = \sum_{k=l_0+1}^{\infty} \alpha(1 - \alpha)^{(k-1)}
\]
\[
= \sum_{u=l_0}^{\infty} \alpha(1 - \alpha)^{u}
\]
\[
= \sum_{l=0}^{\infty} \alpha(1 - \alpha)^{(l_0+l)}
\]
\[
= (1 - \alpha)^{l_0} \sum_{l=0}^{\infty} \alpha(1 - \alpha)^{l}
\]
\[
= (1 - \alpha)^{l_0} \alpha \sum_{l=0}^{\infty} (1 - \alpha)^{l}...(i)
\]
\[
= (1 - \alpha)^{l_0} \alpha \frac{1}{1 - (1 - \alpha)}
\]
\[
= (1 - \alpha)^{l_0},
\]
where we used the identity \[ \sum_{k=0}^{\infty} x^n = \frac{1}{1-x} \text{ for } x < 1 \] in (i).
It then follows that for $l > l_0$, we have

$$P(L = l | L > l_0) = \frac{P(L = l)}{P(L > l_0)} = \frac{\alpha(1 - \alpha)^{l-1}}{(1 - \alpha)^{l_0}} = \alpha(1 - \alpha)^{l-l_0-1}.$$  

Note that this implies that the geometric distribution is "memoryless".

(c) Using part (b), we have that

$$E[L | L > l_0] = \sum_{l=l_0+1}^{\infty} l \alpha(1 - \alpha)^{l-l_0-1}$$

$$= \sum_{l=l_0+1}^{\infty} (l - l_0 + l_0) \alpha(1 - \alpha)^{l-l_0-1} \ldots (i)$$

$$= \sum_{x=1}^{\infty} (x + l_0) \alpha(1 - \alpha)^{x-1} \ldots (ii)$$

$$= \sum_{x=1}^{\infty} x \alpha(1 - \alpha)^{x-1} + \sum_{x=1}^{\infty} l_0 \alpha(1 - \alpha)^{x-1} \ldots (iii)$$

$$= \sum_{x=1}^{\infty} x \alpha(1 - \alpha)^{x-1} + l_0 \sum_{x=1}^{\infty} \alpha(1 - \alpha)^{x-1} \ldots (iv)$$

$$= \sum_{x=1}^{\infty} x \alpha(1 - \alpha)^{x-1} + l_0 \ldots (v)$$

$$= \sum_{x=1}^{\infty} x \alpha(1 - \alpha)^{x-1} + l_0 \ldots (vi)$$

$$= \frac{1}{\alpha} + l_0.$$  

In (i): we added and subtracted $l_0$.

In (ii): we used change of variables: $x = l - l_0$.

In (iii): we split the summation of $x$ and $l_0$.

In (iv): we moved $l_0$ outside the second summation.

In (v): we used the results of Problem 2(b) - (i) where we found that

$$\sum_{k=1}^{\infty} \alpha(1 - \alpha)^{k-1} = \sum_{u=0}^{\infty} \alpha(1 - \alpha)^{u} = 1$$

In (vi): we used $\sum_{k=1}^{\infty} kx(1 - x)^{k-1} = \frac{1}{x}$.