Principles of Computer Networks
Tutorial 1

Problem 1 Solution

(a) Let $P_{off}^{(1)}$ be the probability that at time $k = 1$ the transceiver is in state S_{off}. We have, $P_{off}^{(1)} = \frac{2}{3}$.

(b) Let $P_{off}^{(k)}$ ($P_{on}^{(k)}$) be the probability that the transceiver is in state S_{off} (S_{on}) at time k for $k = 0, 1, 2, ...$ From the initial conditions, we have that $P_{off}^{(0)} = 1$ and $P_{on}^{(0)} = 0$. For $k = 1, 2, 3, ...$, we have that

$$P_{off}^{(k)} = \frac{2}{3} P_{off}^{(k-1)} + \frac{1}{3} P_{on}^{(k-1)},$$

and

$$P_{on}^{(k)} = \frac{1}{3} P_{off}^{(k-1)} + \frac{2}{3} P_{on}^{(k-1)}.$$

We then obtain

$$\begin{array}{ll}
P_{off}^{(1)} = \frac{2}{3}, & P_{on}^{(1)} = \frac{1}{3} \\
P_{off}^{(2)} = \frac{2}{3}\frac{2}{3} + \frac{1}{3}\frac{1}{3} = \frac{5}{9}, & P_{on}^{(2)} = \frac{1}{3}\frac{2}{3} + \frac{2}{3}\frac{1}{3} = \frac{4}{9} \\
P_{off}^{(3)} = \frac{2}{3}\frac{5}{9} + \frac{1}{3}\frac{4}{9} = \frac{14}{27}, & P_{on}^{(3)} = \frac{1}{3}\frac{5}{9} + \frac{2}{3}\frac{4}{9} = \frac{13}{27} \\
P_{off}^{(4)} = \frac{2}{3}\frac{14}{27} + \frac{1}{3}\frac{13}{27} = \frac{41}{81}, & P_{on}^{(4)} = \frac{1}{3}\frac{14}{27} + \frac{2}{3}\frac{13}{27} = \frac{40}{81}
\end{array}$$

(c) See question (b):

$$P_{off}^{(1)} = \frac{2}{3} P_{off} + \frac{1}{3} P_{on},$$

$$P_{on}^{(1)} = \frac{1}{3} P_{off} + \frac{2}{3} P_{on}.$$

(d) From (b) and (c), we need that

$$P_{off} = \frac{2}{3} P_{off} + \frac{1}{3} P_{on},$$

and

$$P_{on} = \frac{1}{3} P_{off} + \frac{2}{3} P_{on}.$$

Any solution for this system of equations has the property that

$$P_{off} = P_{on}.$$

As P_{off} and P_{on} are probabilities, we need that

$$P_{off} + P_{on} = 1.$$

Combining the two conditions above, we obtain that

$$P_{off} = P_{on} = \frac{1}{2}.$$

(e) We have

$$\lim_{k\to\infty} P_{off}^{(k)} = \frac{1}{2},$$

and

$$\lim_{k\to\infty} P_{on}^{(k)} = \frac{1}{2}.$$

Problem 2 Solution

(a) You can utilize $\sum\limits_{k=1}^{\infty} kx(1-x)^{k-1} = \frac{1}{x}$.

$$\begin{aligned} E[L] &= \sum_{l=1}^{\infty} l\alpha(1-\alpha)^{l-1} \\ &= \frac{1}{\alpha}. \end{aligned}$$

(b) For $l > l_0$, we have that

$$P(L=l|L>l_0) = \frac{P(L=l \text{ and } L>l_0)}{P(L>l_0)} = \frac{P(L=l)}{P(L>l_0)}.$$

Note that

$$P(L=l) = \alpha(1-\alpha)^{l-1},$$

and

$$\begin{aligned} P(L>l_0) &= \sum_{k=l_0+1}^{\infty} \alpha(1-\alpha)^{(k-1)} \\ &= \sum_{u=l_0}^{\infty} \alpha(1-\alpha)^{u} \\ &= \sum_{l=0}^{\infty} \alpha(1-\alpha)^{(l_0+l)} \\ &= (1-\alpha)^{l_0} \sum_{l=0}^{\infty} \alpha(1-\alpha)^{l} \\ &= (1-\alpha)^{l_0} \alpha \sum_{l=0}^{\infty} (1-\alpha)^{l} ...(i) \\ &= (1-\alpha)^{l_0} \alpha \frac{1}{1-(1-\alpha)} \\ &= (1-\alpha)^{l_0}, \end{aligned}$$

where we used the identity $\sum_{k=0}^{\infty} x^n = \frac{1}{1-x}$ for $x < 1$ in (i).

It then follows that for $l > l_0$, we have

$$P(L = l | L > l_0) = \frac{P(L = l)}{P(L > l_0)} = \frac{\alpha(1-\alpha)^{l-1}}{(1-\alpha)^{l_0}} = \alpha(1-\alpha)^{l-l_0-1}.$$

Note that this implies that the geometric distribution is "memoryless".

(c) Using part (b), we have that

$$\begin{aligned}
E[L \mid L > l_0] &= \sum_{l=l_0+1}^{\infty} l\alpha(1-\alpha)^{l-l_0-1} \\
&= \sum_{l=l_0+1}^{\infty} (l - l_0 + l_0)\alpha(1-\alpha)^{l-l_0-1} \ldots .(i) \\
&= \sum_{x=1}^{\infty} (x + l_0)\alpha(1-\alpha)^{x-1} \ldots .(ii) \\
&= \sum_{x=1}^{\infty} x\alpha(1-\alpha)^{x-1} + \sum_{x=1}^{\infty} l_0\alpha(1-\alpha)^{x-1} \ldots (iii) \\
&= \sum_{x=1}^{\infty} x\alpha(1-\alpha)^{x-1} + l_0 \sum_{x=1}^{\infty} \alpha(1-\alpha)^{x-1} \ldots (iv) \\
&= \sum_{x=1}^{\infty} x\alpha(1-\alpha)^{x-1} + l_0 \ldots (v) \\
&= \sum_{x=1}^{\infty} x\alpha(1-\alpha)^{x-1} + l_0 \ldots (vi) \\
&= \frac{1}{\alpha} + l_0.
\end{aligned}$$

In (i): we added and subtracted l_0.

In (ii): we used change of variables: $x = l - l_0$.

In (iii): we split the summation of x and l_0.

In (iv): we moved l_0 outside the second summation.

In (v): we used the results of Problem 2(b) - (i) where we found that $\sum_{k=1}^{\infty} \alpha(1-\alpha)^{k-1} = \sum_{u=0}^{\infty} \alpha(1-\alpha)^{u} = 1$

In (vi): we used $\sum_{k=1}^{\infty} kx(1-x)^{k-1} = \frac{1}{x}$.