Proposition 5.1 (Bollobás and Riordan)

In a preferential attachment model in which each newborn node forms \( m \geq 2 \) links, as \( n \) grows the resulting network consists of a single component with diameter proportional to \( \frac{\log(n)}{\log \log(n)} \) almost surely.

Does this give a bound for the diameter of a hybrid model?
Proposition 5.2 (Jackson and Rogers)

Consider a growing hybrid random-network formation process as described in section 5.3. Under the mean-field estimate, a node i’s degree is larger than a node j’s degree at time t after both are born if and only if i is older than j. In that case, if $\alpha > 0$, then the estimated distribution of i’s neighbours’ degrees strictly first-order stochastically dominates that of j’s at each time $t > j$ relative to younger nodes; that is, $F_i^t(d) < F_j^t(d)$ for all $d < d_j(t)$. 
Proof of Proposition 5.2

The degree of a node is:

\[ d_i(t) = \left( m + \frac{2\alpha m}{1-\alpha} \right) \left( \frac{t}{i} \right)^{(1-\alpha)/2} - \frac{2\alpha m}{1-\alpha} \]

This implies that if \( d_i(t) > d_j(t) \), then \( i < j < t \).

\[ F_i^t(d) = 1 - \frac{d_i(t^*(d, t))}{d_i(t)} \]

\( t^*(d, t) \) is the date of birth of a node with degree \( d \) at time \( t \). We want to show that for \( i < j < t' < t \):

\[ \frac{d_i(t')}{d_i(t)} > \frac{d_j(t')}{d_j(t)} \]

\[ \frac{d_i(t')}{d_i(t)} = \left( m + \frac{2\alpha m}{1-\alpha} \right) \left( \frac{t'}{i} \right)^{(1-\alpha)/2} - \left( m + \frac{2\alpha m}{1-\alpha} \right) \left( \frac{t}{i} \right)^{(1-\alpha)/2} \]
5.4.5 - Clustering

Transitive Triples:

\[ C^{TT}(g) = \frac{\sum_{i:j \neq i; k \neq j} g_{ij} g_{jk} g_{ik}}{\sum_{i:j \neq i; k \neq j} g_{ij} g_{jk}} \]

Proposition 5.3 (Jackson and Rogers)

Under a mean-field approximation the fraction of transitive triples, \( C^{TT} \), tends to

\[
\begin{align*}
\frac{1}{(r+1)m} & \quad r \geq 1 \\
\frac{(m-1)r}{m(m-1)(1+r)r-m(1-r)} & \quad r < 1
\end{align*}
\]

Where \( m = m_r + m_n \) and \( r = \frac{m_r}{m_n} \).
6.1 - Pairwise Stability

A network is pairwise stable if, where \( u_i(g) \) is the utility of a network \( g \) for node \( i \):

1. For all \( ij \in g \), \( u_i(g) \geq u_i(g - ij) \) and \( u_j(g) \geq u_j(g - ij) \)

2. For all \( ij \notin g \), if \( u_i(g + ij) > u_i(g) \) then \( u_j(g + ij) < u_j(g) \)
6.2 - Efficient Networks

Efficiency: $g$ is efficient relative to the utility functions $u_1, \ldots, u_n$ if
$$\sum_i u_i(g) \geq \sum_i u_i(g')$$
for all $g' \in G(N)$.

Pareto Efficiency: $g$ is Pareto efficient if there does not exist any $g' \in G$ such that $u_i(g') \geq u_i(g)$ for all $i$, with strict inequality for some $i$.

![Figure 6.1](image_url)