Chapter 4 of: Social and Economic Network / Matthew O. Jackson.

Properties of Random Networks
Thresholds and Phase Transitions

Erdös-Renyi model is completely specified by the link formation probability $p(n)$.

For a given monotone property $A$, we define a threshold function $t(n)$ as a function that satisfies:

$$ P(\text{property } A) \begin{cases} \rightarrow 0 & : \frac{p(n)}{t(n)} \rightarrow 0 \\ \rightarrow 1 & : \frac{p(n)}{t(n)} \rightarrow \infty \end{cases} $$

- What is a property?
- What is a monotone property?
Properties are generally specified as a set of networks for each \( n \), and then a property is satisfied if the realized network is in the set.

The property that a network has no isolated nodes:

\[
A(N) = \{ g : N_i(g) \neq \emptyset, \forall i \in N \}
\]
Monotone property are properties such that if a given network satisfies the property, then any supernetwork (in the sense of set inclusion) satisfies it.

\[ g \in A(N) \text{ and } g \subseteq g' \implies g' \in A(N) \]

- Example of a monotone property: being connected.
- Example of a non-monotone property: having an even number of links.
Recall:

For a given monotone property $A$, we define a threshold function $t(n)$ as a function that satisfies:

\[
P(\text{property } A) \begin{cases} 
  \to 0 & : \frac{p(n)}{t(n)} \to 0 \\
  \to 1 & : \frac{p(n)}{t(n)} \to \infty
\end{cases}
\]
Property: the network has some links.

\[ t(n) = \frac{1}{n^2} \]

\[(n = 50, \ p = 0.01) \ \text{threshold is 0.0004} \]
Property: the network has a cycle.

\[ t(n) = \frac{1}{n} \]

\((n = 50, \ p = 0.03)\) threshold is 0.02
(n = 50, p = 0.05) threshold is 0.02
Property: network is connected.

\[ t(n) = \frac{\log(n)}{n} \]

\((n = 50, \ p = 0.1)\) threshold is \(\sim 0.078\)
Connectedness (a special phase transition)

Theorem [Erdös-Renyi]

A threshold function for the connectedness of the Poisson random network is \( t(n) = \log(n)/n \).

Proof Sketch:

- If \( p(n)/t(n) \to 0 \) then there will be isolated nodes with probability 1.
- If \( p(n)/t(n) \to \infty \) then there will not be any component of size less than \( n/2 \).
- KEY IDEA: threshold for the isolated node is the same as threshold for small components.
proof sketch continued:

- We first show that for $E[d]$, expected degree, $\log(n)$ is the threshold above which we expect each node to have some links.
- Once nodes have many links, the chance of disconnected components vanishes.
- Let $E[d] = (1 - n)p(n) = f(n) + \log(n)$ for some function $f$.
- Probability that some node is isolated:

$$\left(1 - p(n)\right)^{n-1} = \left(1 - \left(f(n) + \log(n)\right) / (n - 1)\right)^{n-1}$$
proof sketch continued:

- Probability that a given node is completely isolated:

\[
(1 - p(n))^{n-1} \sim (1 - p(n))^n \sim \exp^{-np(n)} \text{ as } p(n) \to 0.
\]

as \( p(n) \to 0 \)

\[
e^x = \lim_{n \to \infty} (1+x/n)^n
\]

and similarly:

\[
(1 - p(n))^{n-1} \sim e^{-(f(n)+\log(n))} = e^{-f(n)}/n
\]
proof sketch continued:

Thus the expected number of isolated nodes is $e^{f(n)}$

- $f(n) \to \infty$: expected number of isolated nodes approaches 0.
- $f(n) \to -\infty$: expected number of isolated nodes becomes infinite.