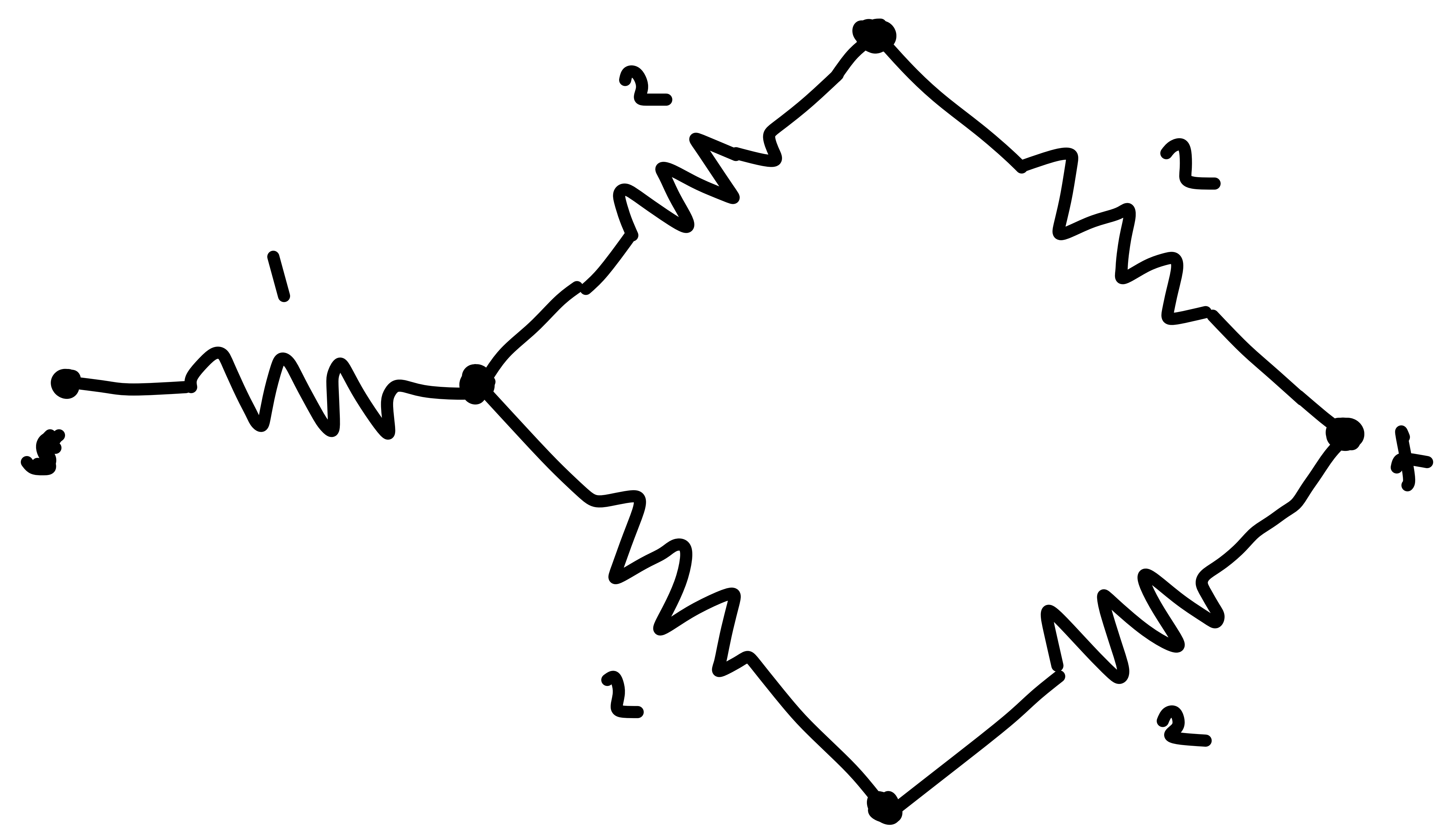
Electrical Flows and Max Flow

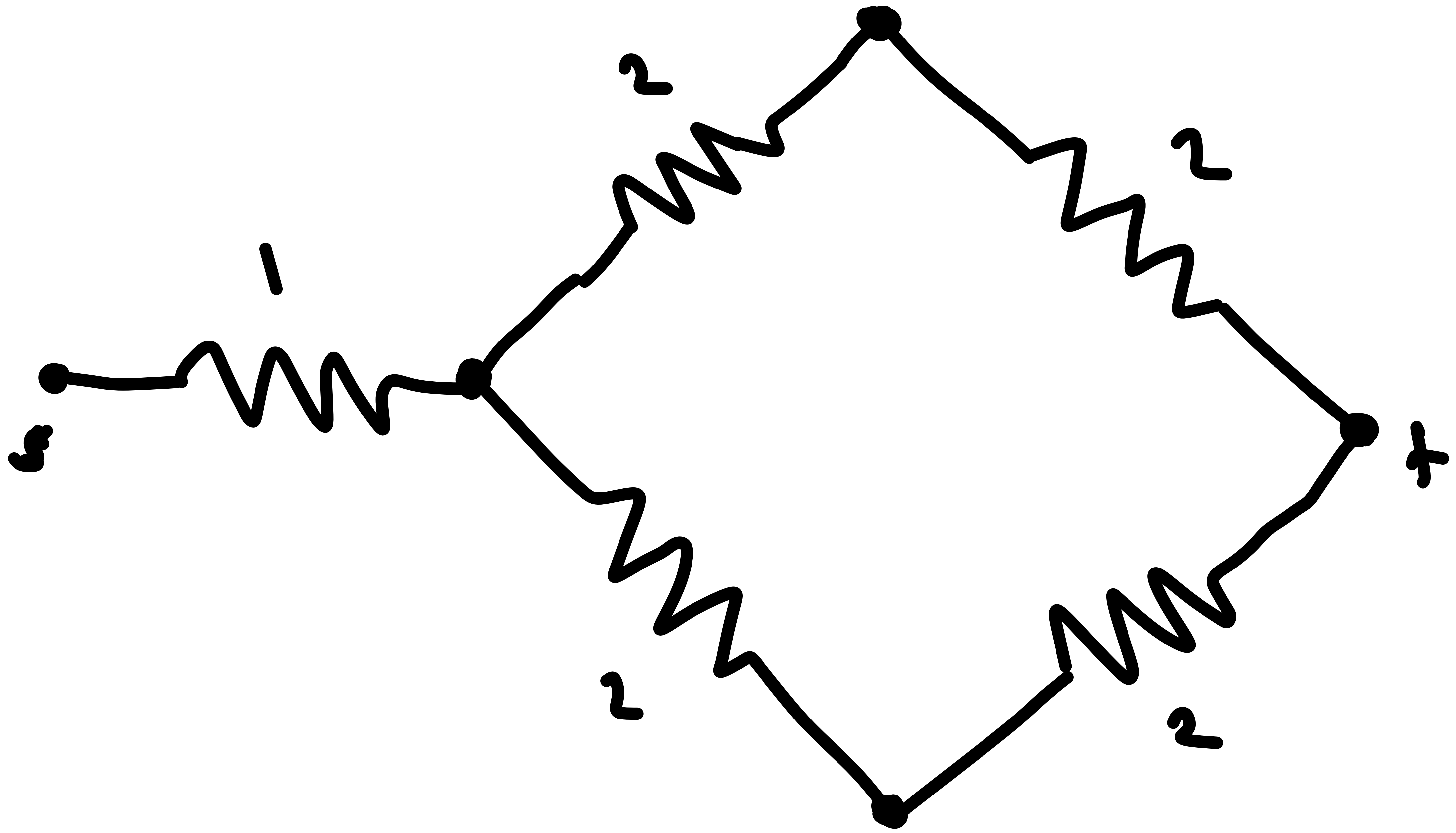
Theory Student Seminar 23/03/22

What are Electrical Flows?



What are Electrical Flows?

* Resistors : r_{uv}
* Current : f_{uv}
* Voltage (Potential) : p_u



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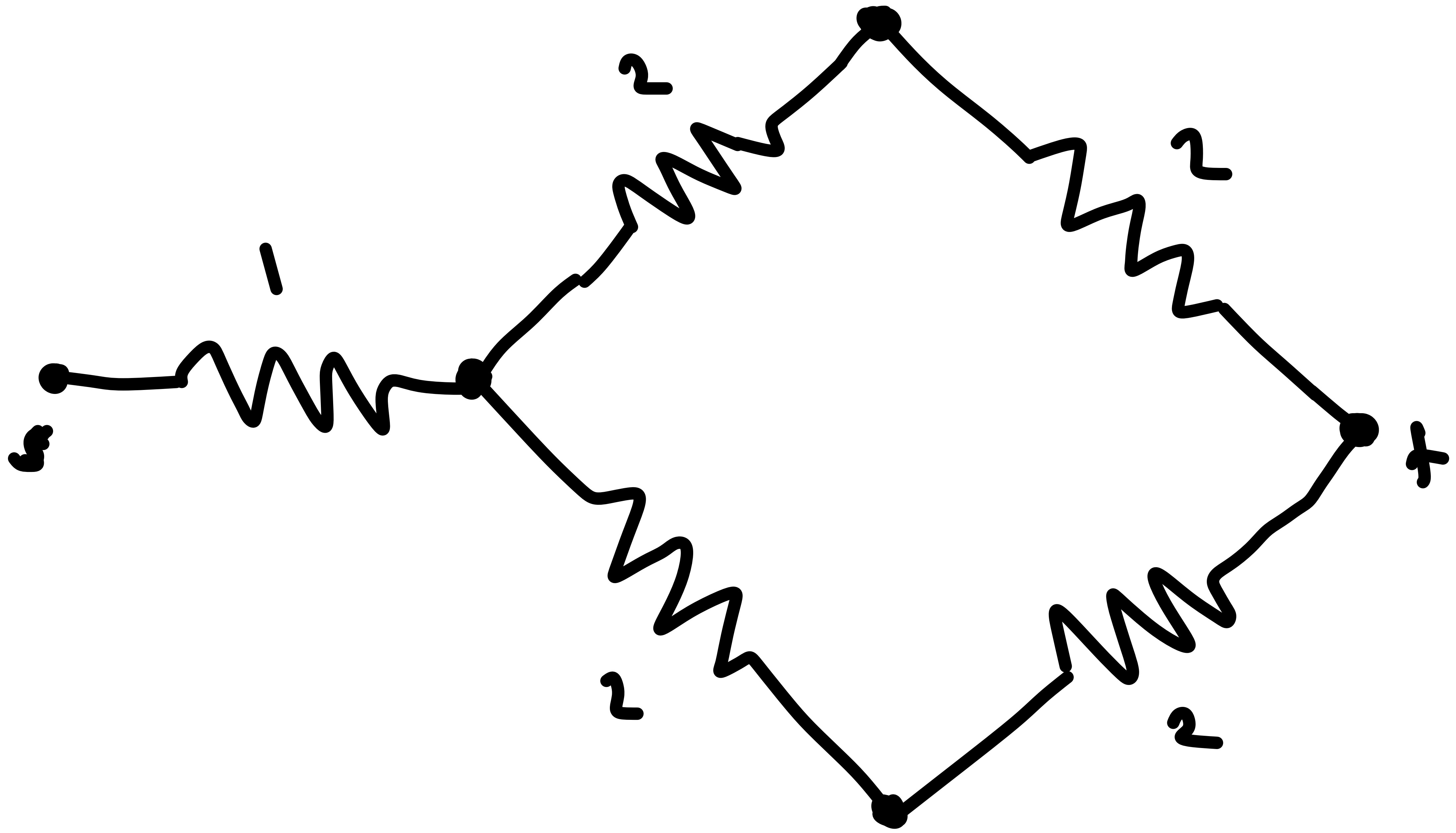

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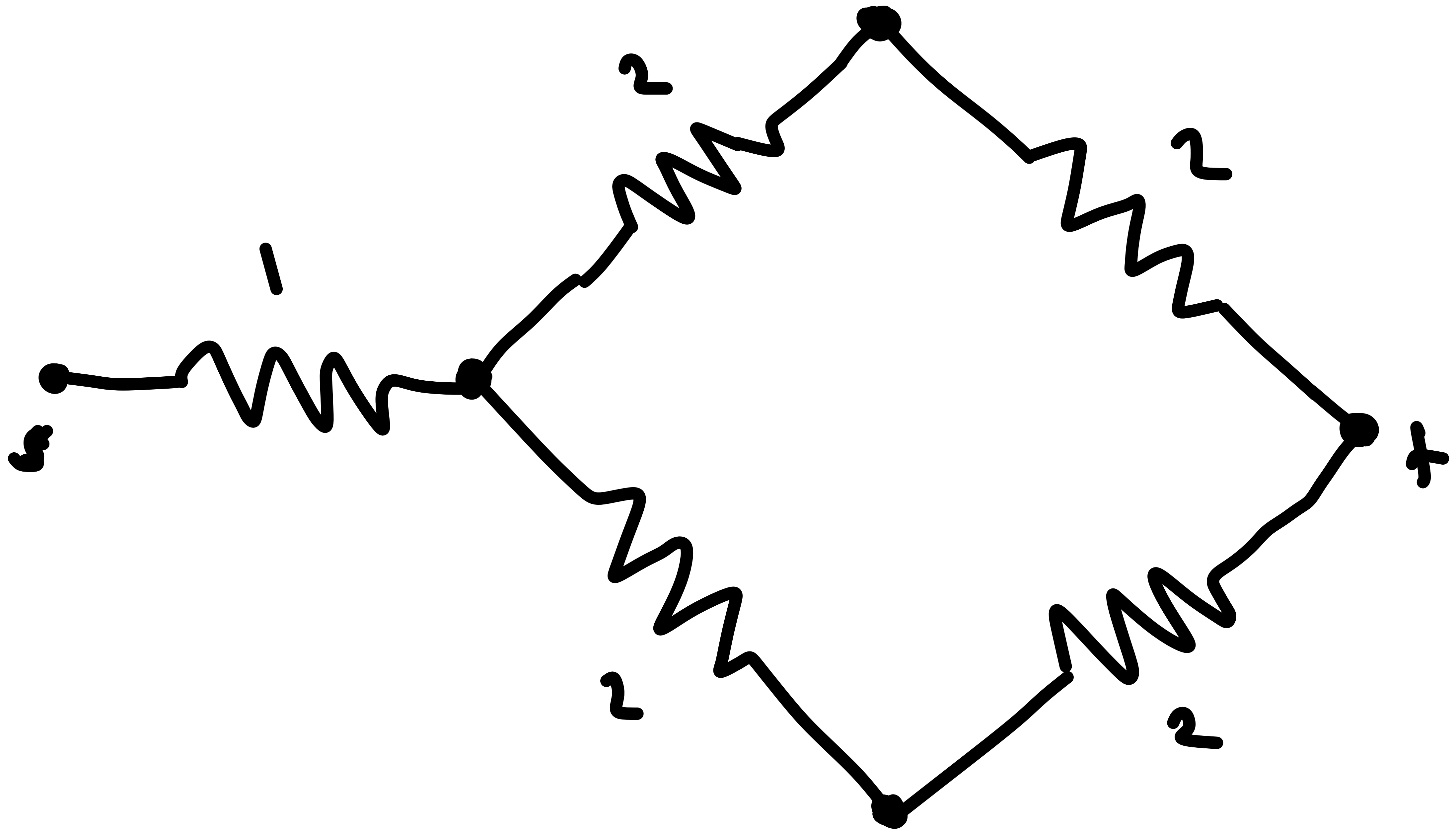

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$$\sum_{u: v \to u} f_{uv} - \sum_{v: w \to v} f_{wv} = 0$$

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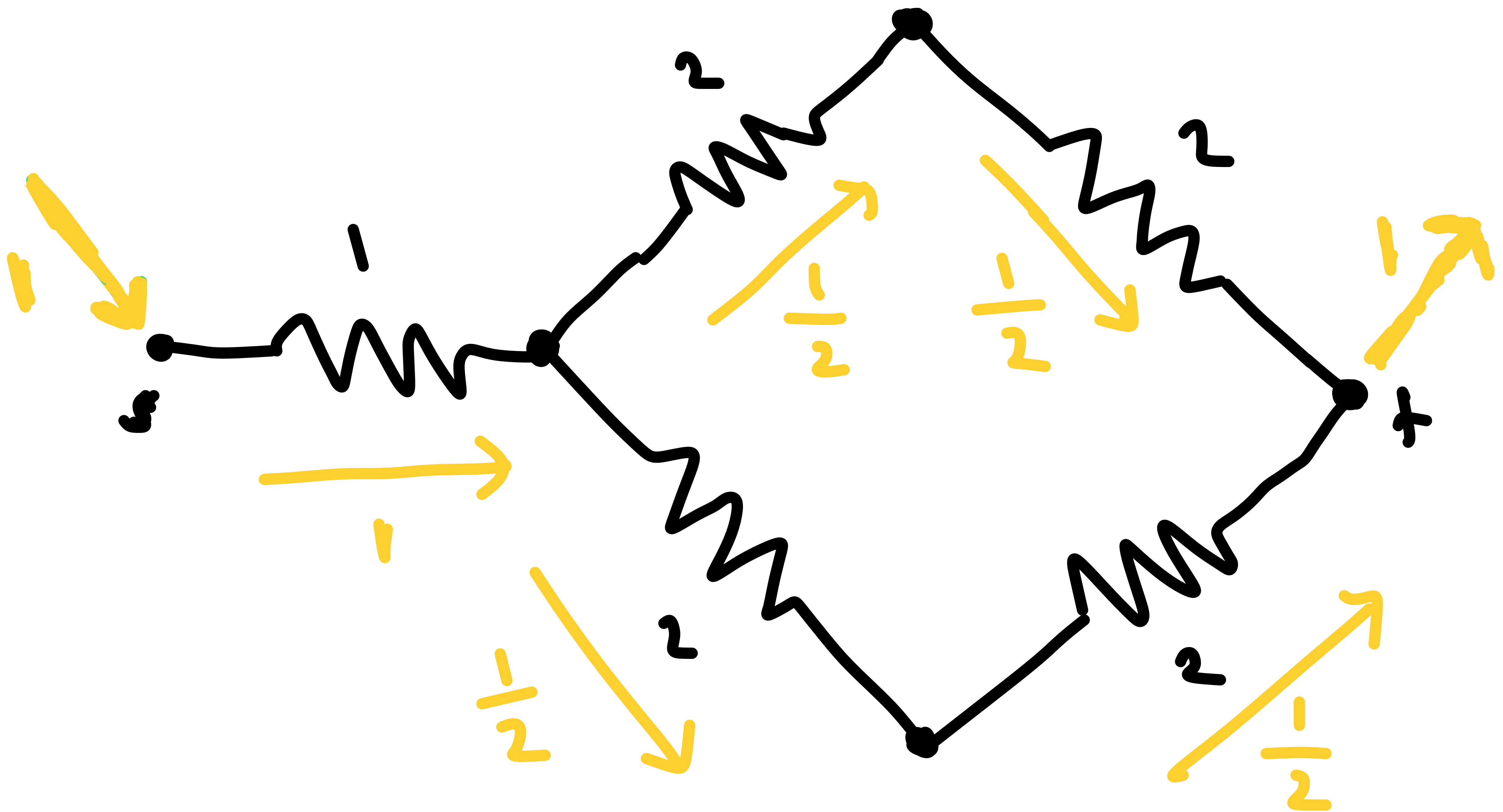

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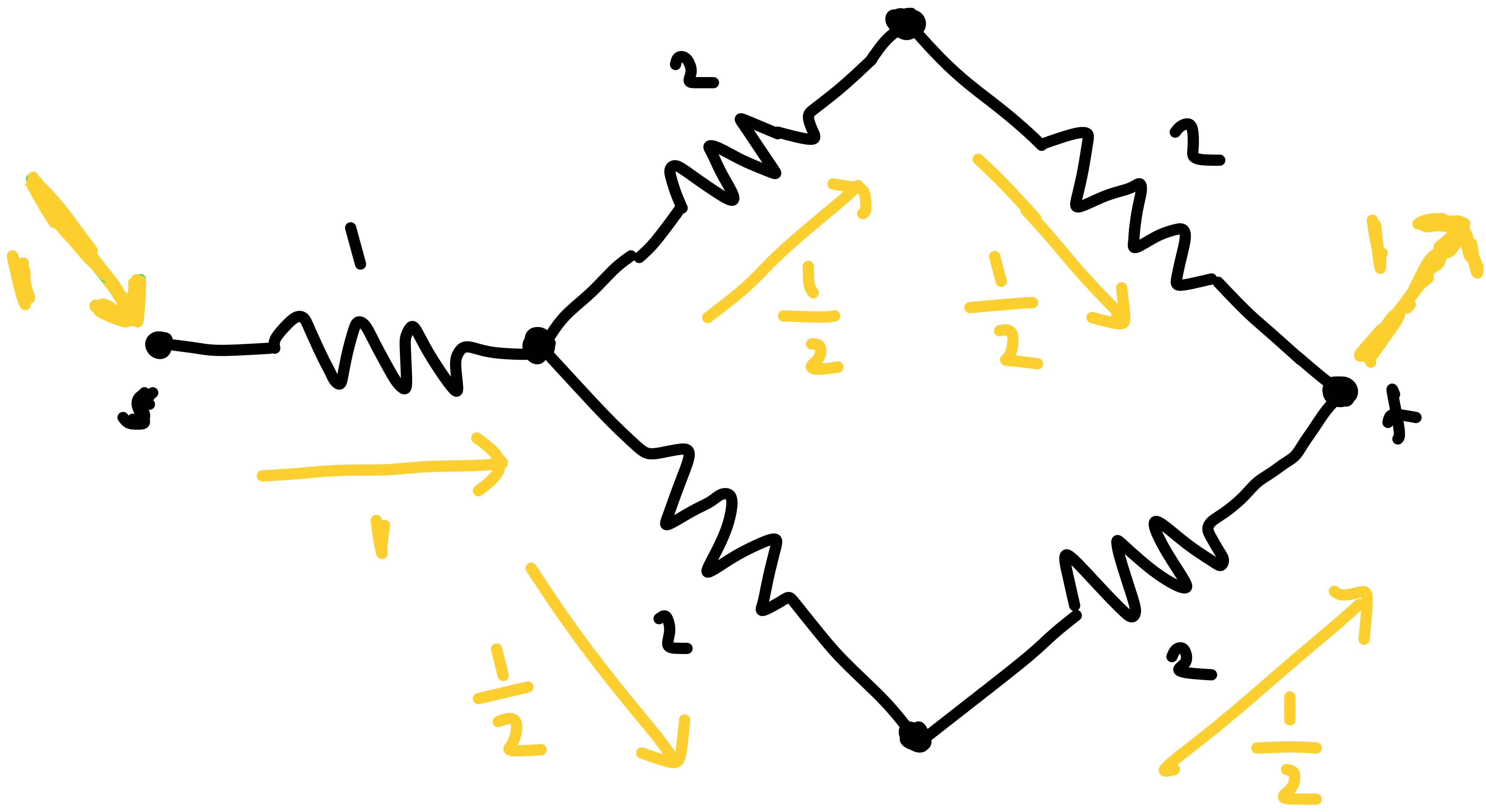

Electrical Flows minimise energy:

$$\text{Minimize} \sum_{e \in E} \frac{1}{2} r_e f_e^2$$

subject to $Bf = b$

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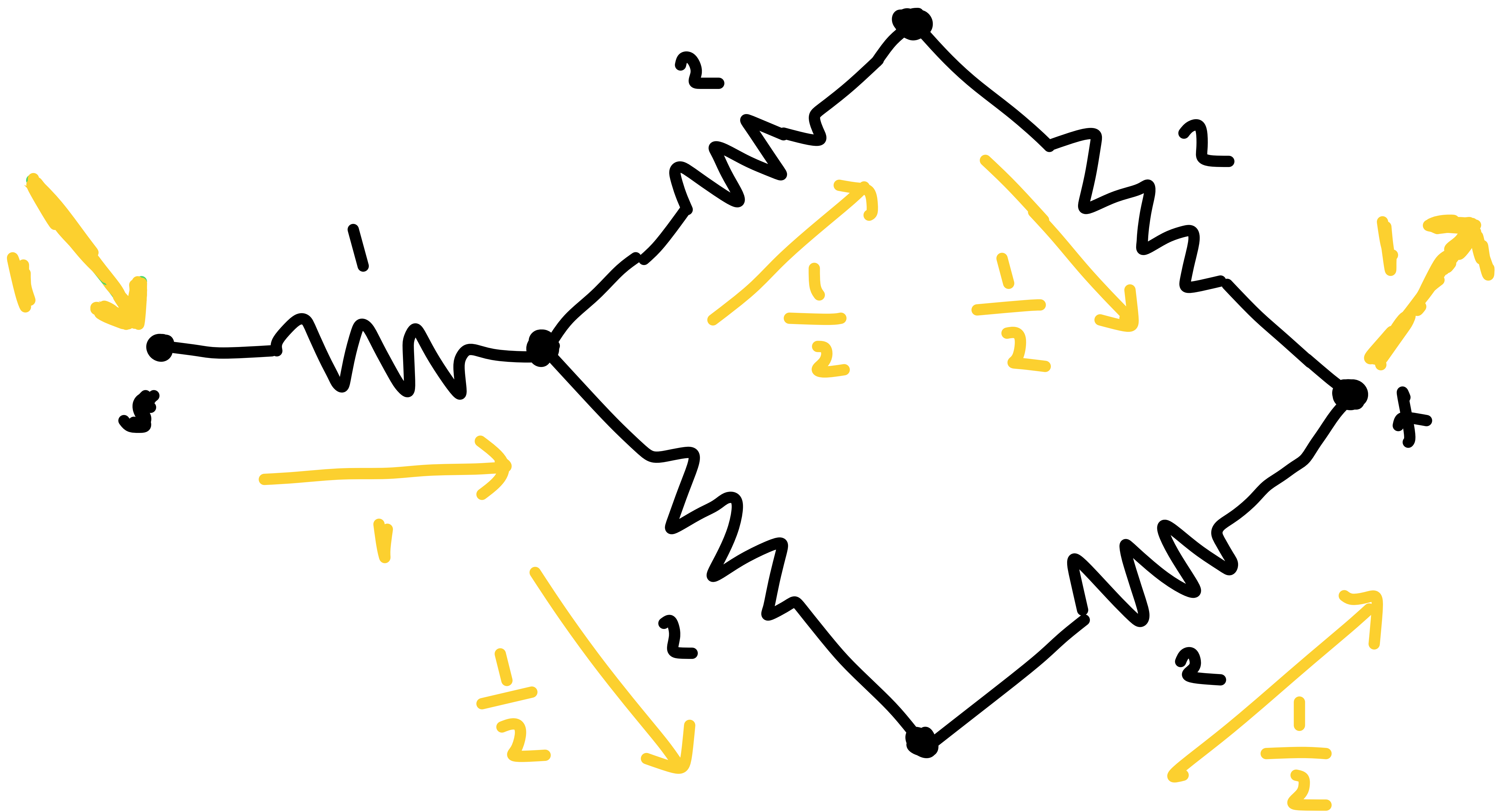

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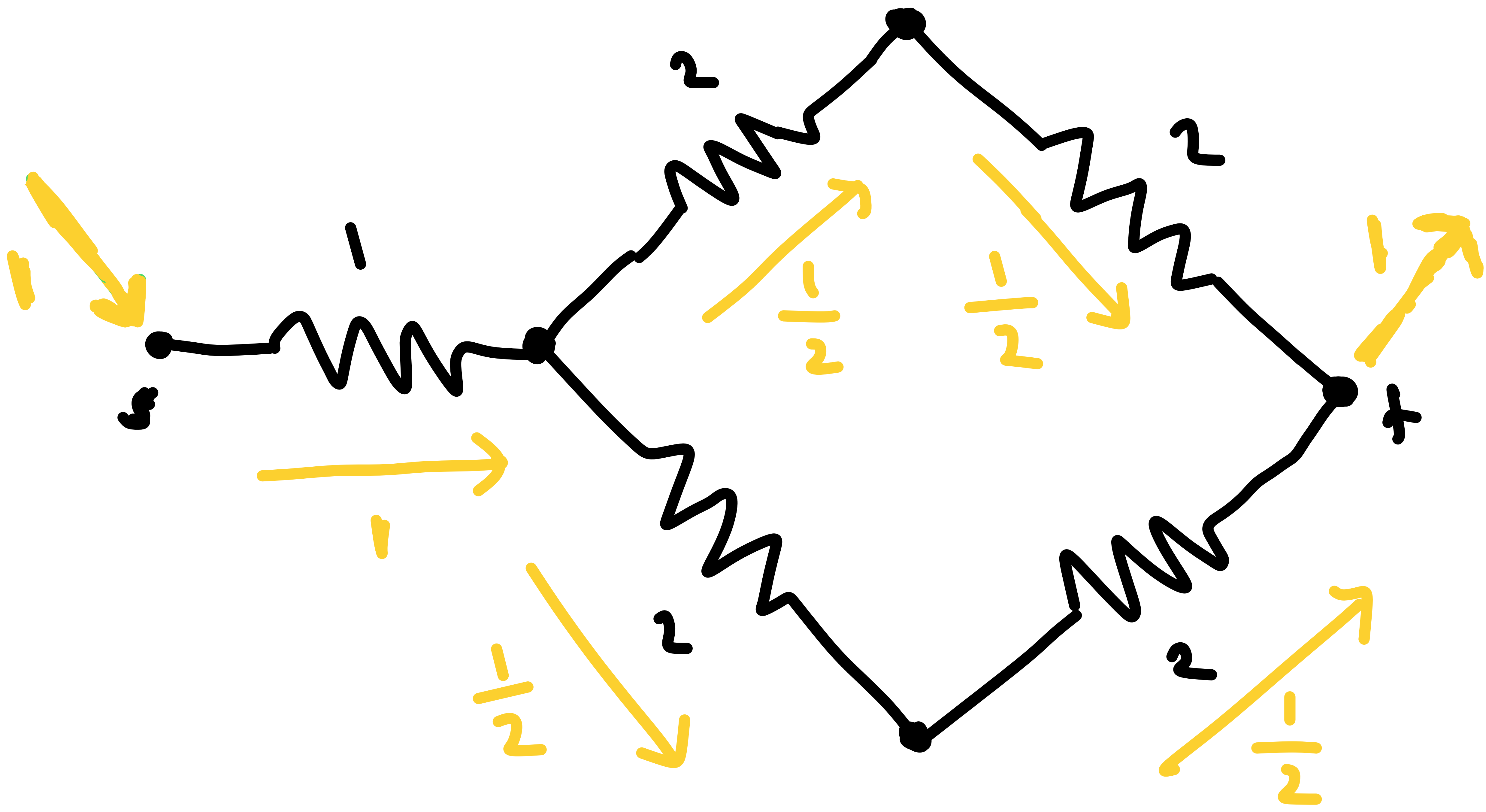

Electrical Flows minimise energy:

Minimize $\sum_{e \in E} \frac{1}{2} r_e f_e^2$

subject to $Bf = b^*$

Electrical Flows <u>ARE</u> random walks

f_{uv} = expected net passes through edge

$\frac{p_v}{p_s - p_t}$ = probability random walk hits s before t

Why Electrical Flows?

Efficiently computable ($\tilde{O}(m)$ time ST'04, KMP'10)

Turns out, for any given circuit, you can compute the induced flow in a single iteration of a Laplacian solver

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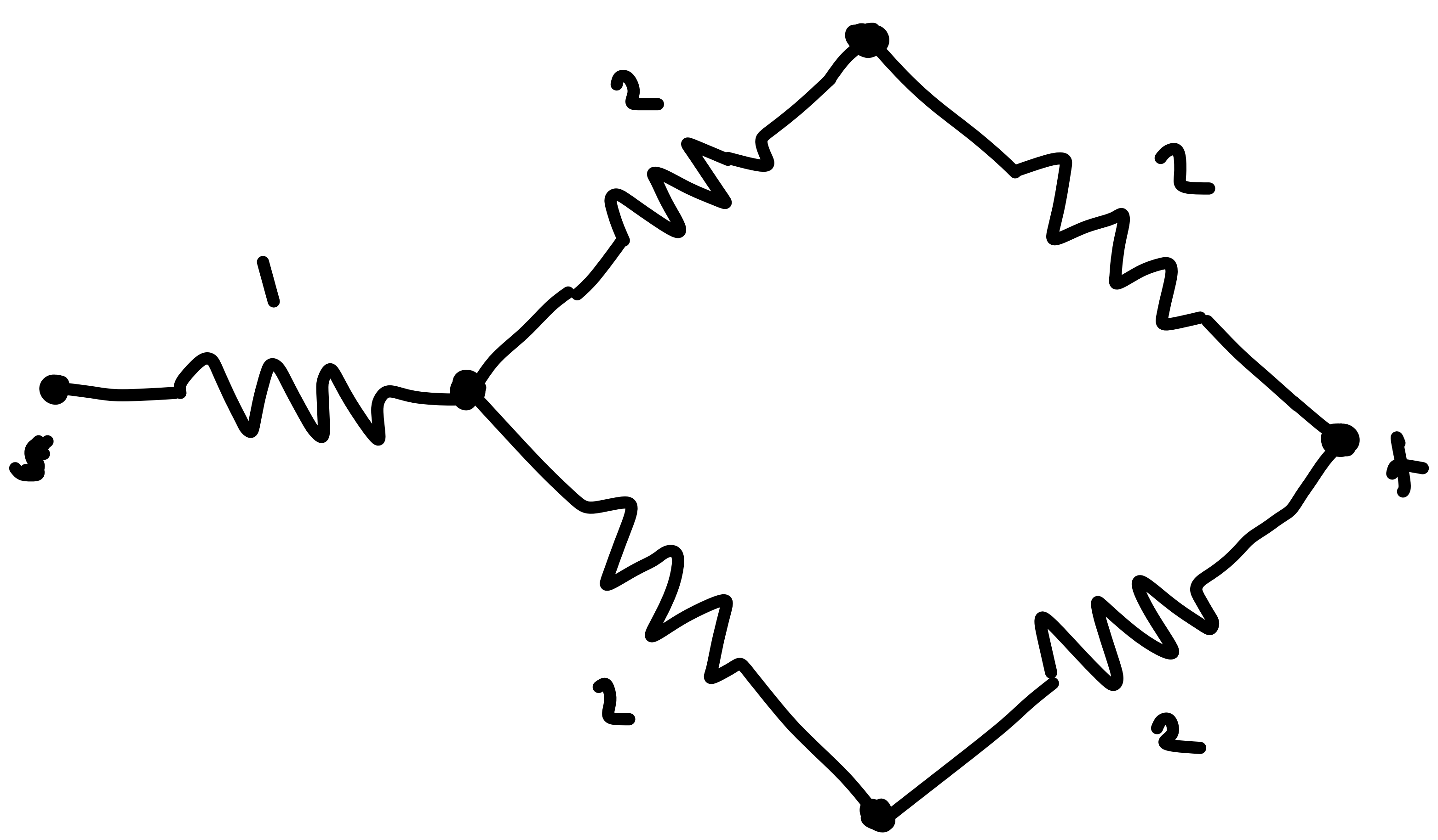
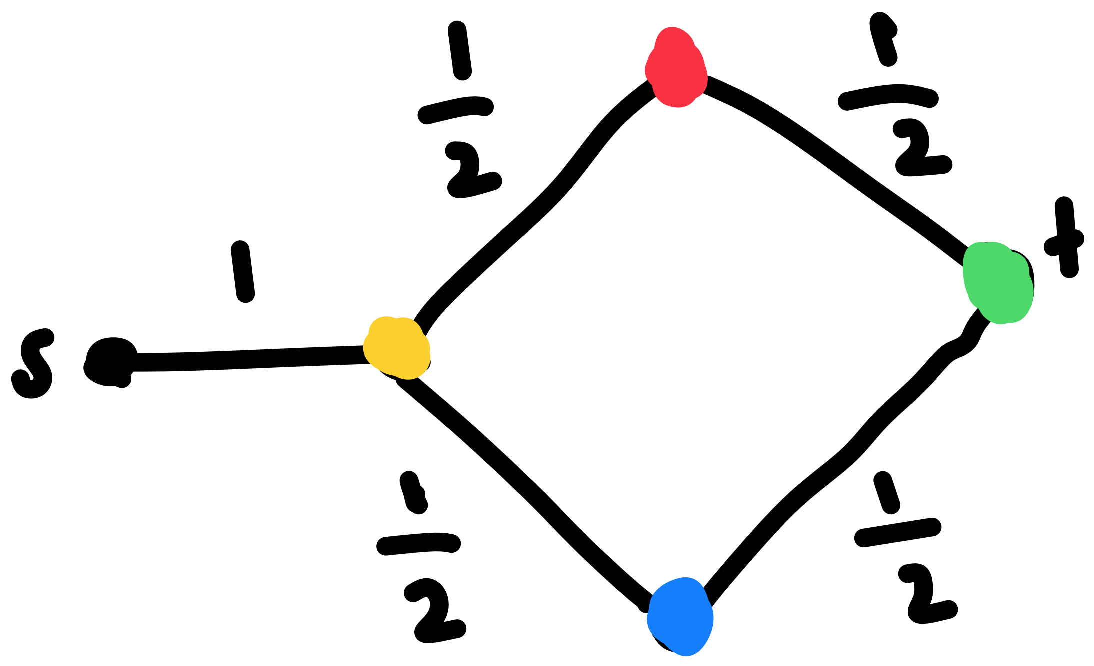
Circuit



Graph*



Laplacian

$$\begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & -\frac{1}{2} & -\frac{1}{2} & \\ & -\frac{1}{2} & 1 & & -\frac{1}{2} \\ & -\frac{1}{2} & & 1 & -\frac{1}{2} \\ & & -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}$$

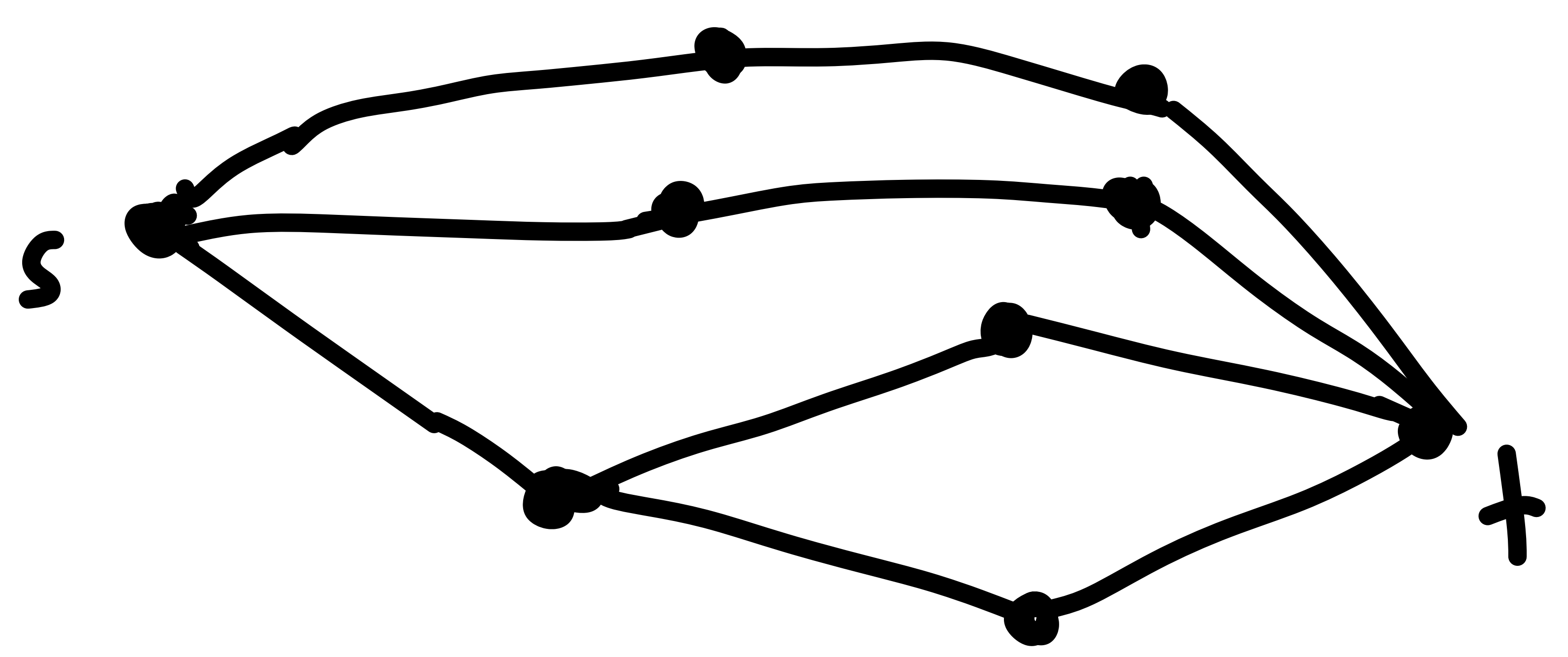
$$p = L^{+}(\mathbb{1}_s - \mathbb{1}_t)$$

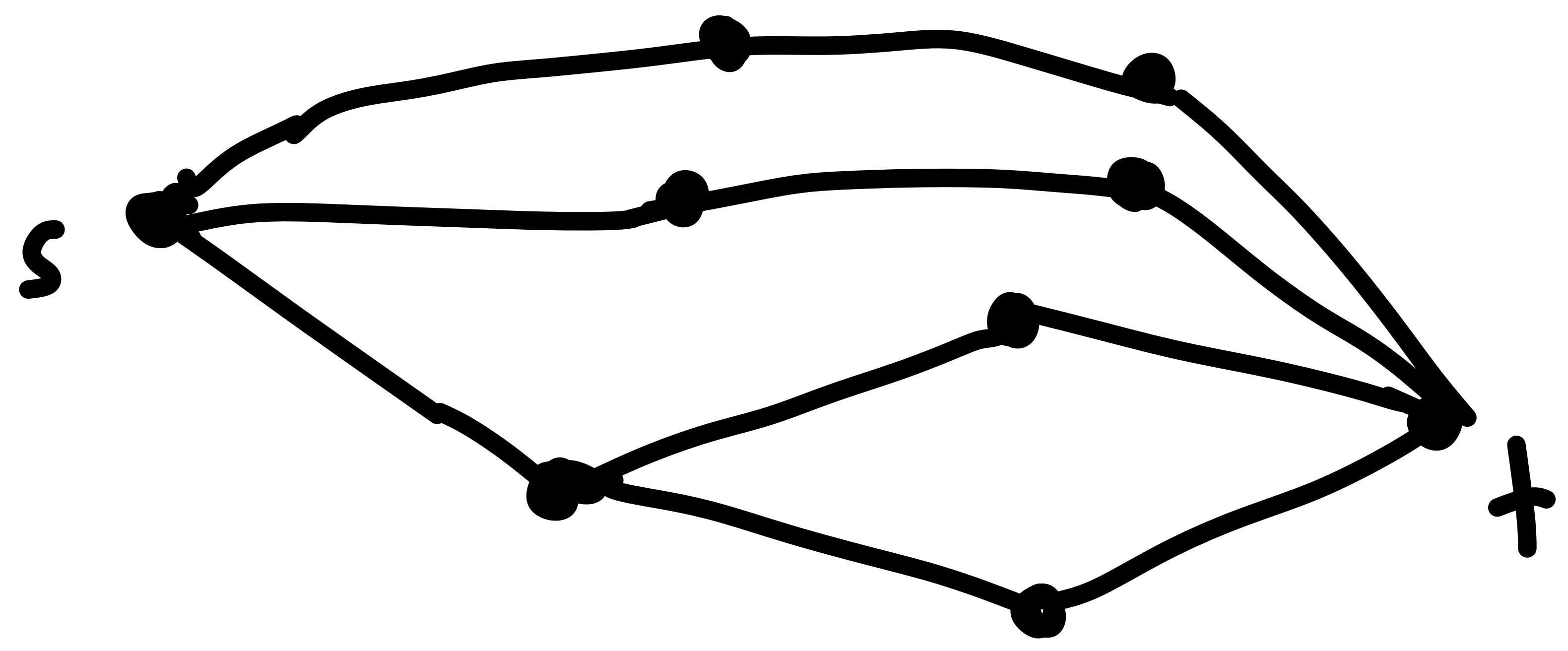
Electrical Flows to Approximate Undirected Max Flow

Focus: Unit Weight, Sparse

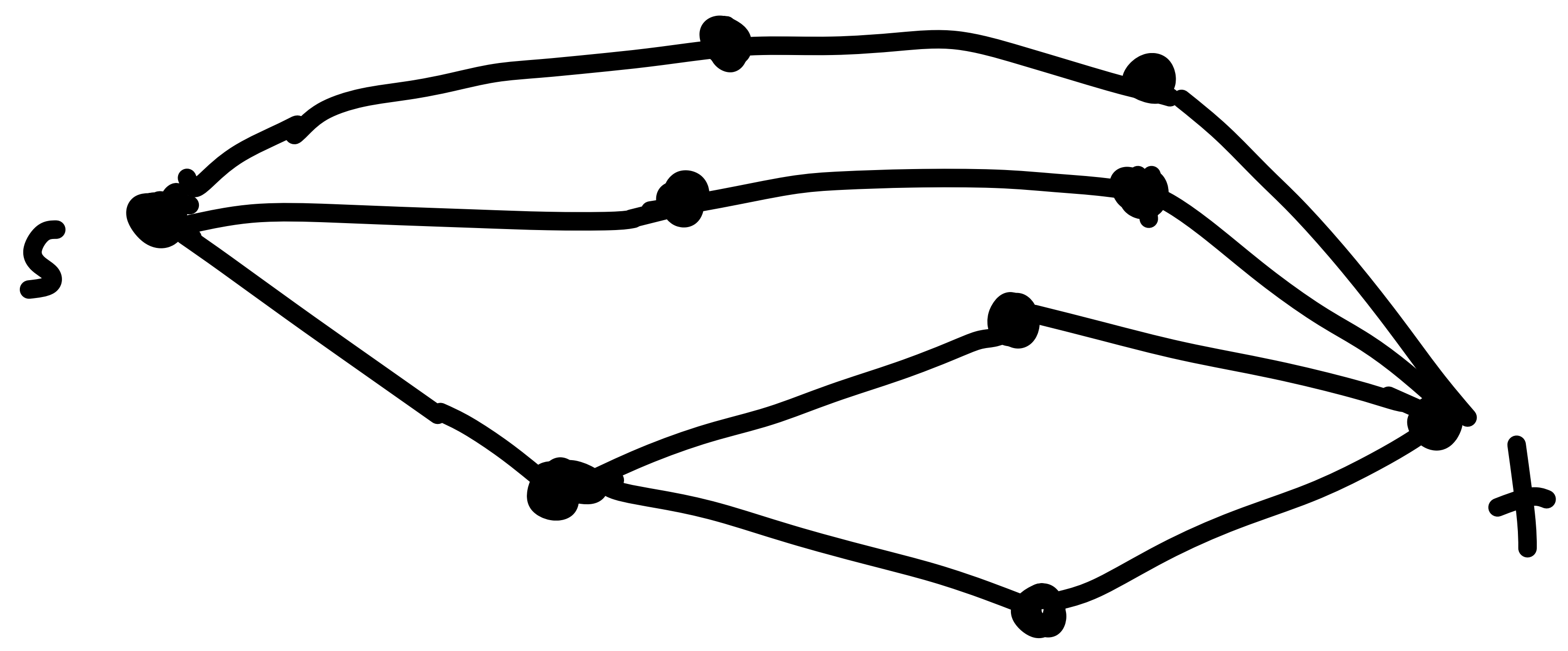
Max Flow	General	Unit Weight, sparse
[Ford Fulkerson '56]	$O(mnU)$	$O(n^2)$
[Dinitz '70]	$O(mn^2)$	$O(n^3)$
[Dinitz '73][Gabow '85]	$O(mn\log U)$	$\tilde{O}(n^2)$
[Lee Sidford '14]	$\tilde{O}(mn^{\frac{1}{2}}\log U)$	$\tilde{O}(n^{\frac{3}{2}})$
[CKMST '11]		$\tilde{O}(n^{\frac{4}{3}})$

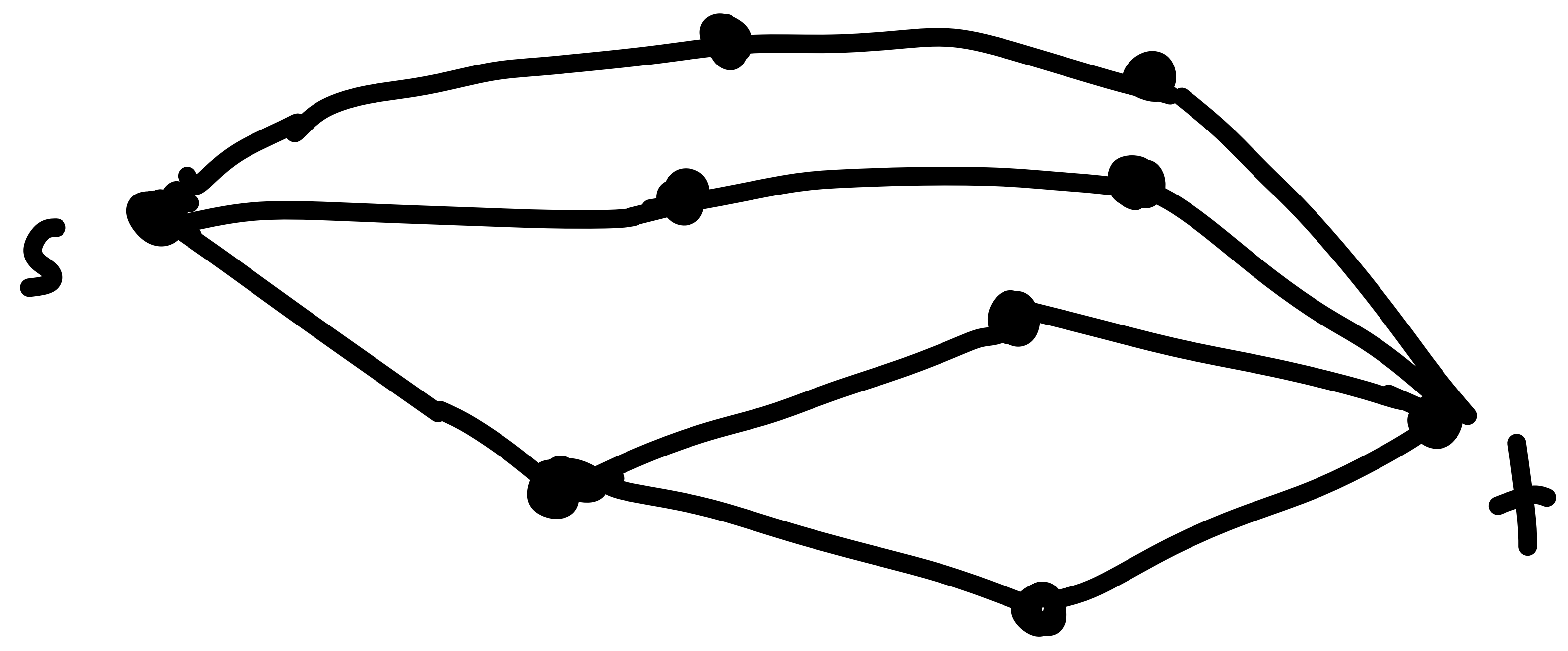
Max Flow vs Electrical Flow





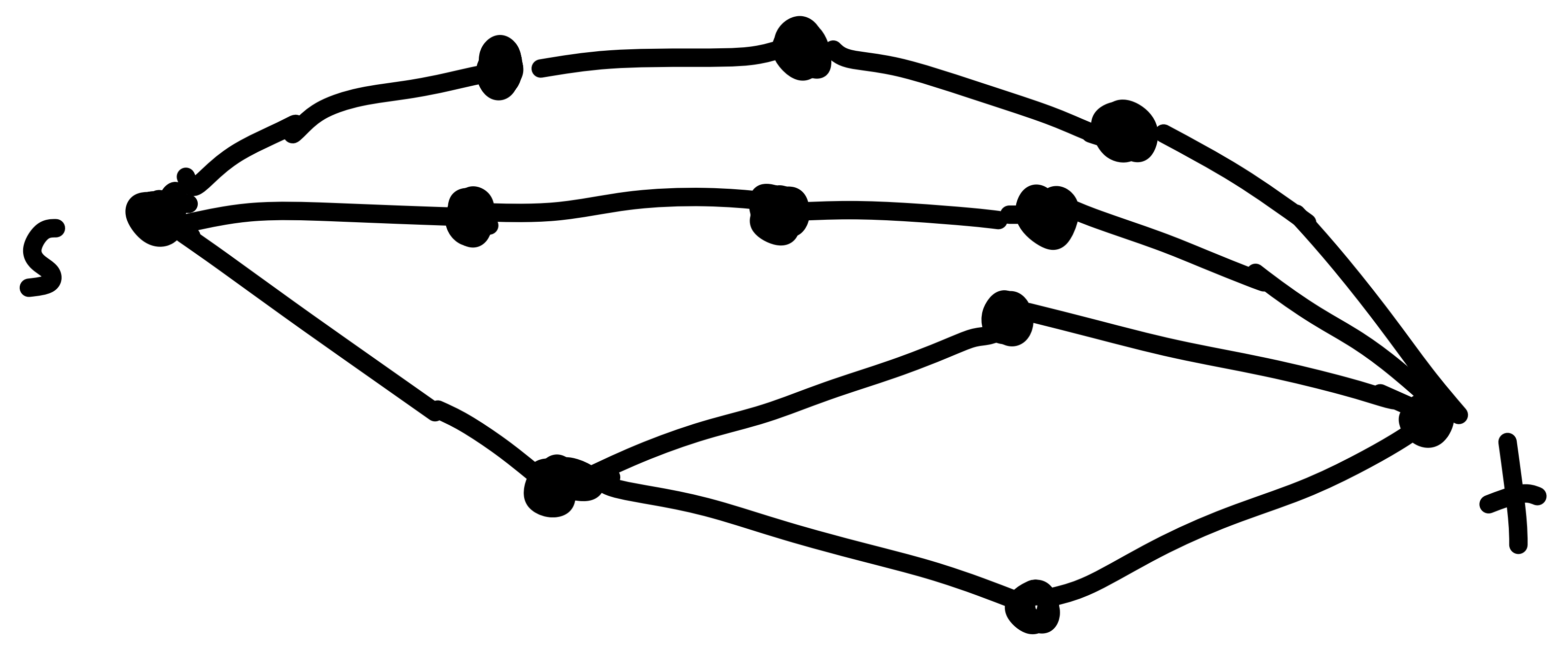
Max Flow vs Electrical Flow

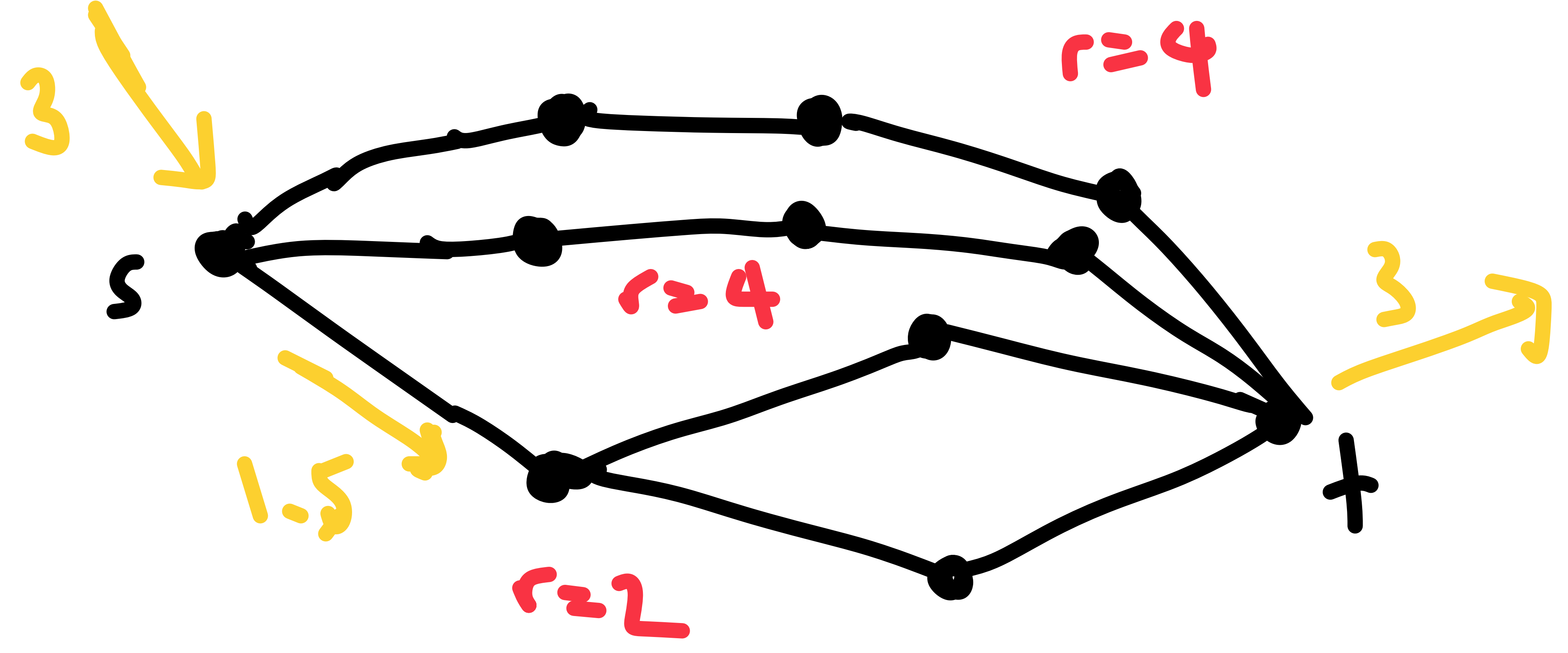




1) Guess Flow F^* (Binary Search)
2) Solve for electrical flow with F^* induced
   $\Rightarrow$ Some edges might be over capacity
3) Increase resistance on those edges

Max Flow vs Electrical Flow





1) Guess Flow F^* (Binary Search)
2) Solve for electrical flow with F^* induced] $O(n)$ time *
   ⇒ Some edges might be over capacity] Laplacian Solver
3) Increase resistance on those edges] $O(\sqrt{n})$ times

   Repeat. MWU

Feasibility on Average

We need to satisfy: $|f_e| \leq 1$ for all e

Given some flow f, we look at a weaker condition:

$$\sum_e w_e |f_e| \leq \sum w_e$$

Capacities are preserved on average.

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Fact about electrical circuits:

$$\sum_e r_e |f_e| \le \sum r_e$$

Multiplicative Weights Update

(AHK'05)

"Turning weak algorithms into strong ones"

Algorithm

1) Start with weights $w_e = 1$
2) Use Crude algorithm to get some flows $f^i(e)$ satisfying feasibility on average
3) $w_e^i \leftarrow w_e^{i-1}\left(1+\varepsilon \frac{|f^i(e)|}{\rho_i}\right)$

ρ_i = maximum congestion, $\max |f^i(e)|$

Repeat 2,3

4) Output <u>average</u> of all flows

Algorithm

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→ If congestion is large, then weight increases quickly.

→ Since feasible on average, sum $\sum_e w_e$ grows slowly

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$\tilde{O}(\rho\,\varepsilon^{-2})$ iterations

Algorithm

1) Start with weights ~~$w_e = 1$~~ resistances

2) Use ~~Crude algorithm~~ Laplacian solver to get some flows $f^i(e)$ satisfying feasibility on average

3) $\cancel{w}_e^i \leftarrow \cancel{w}_e^{i-1} \left(1 + \varepsilon \frac{|f^i(e)|}{\rho_i}\right)$

$c_e^i \quad c_e^{i-1}$

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4) Output <u>average</u> of all flows $\tilde{O}(\rho \varepsilon^{-2})$ iterations *

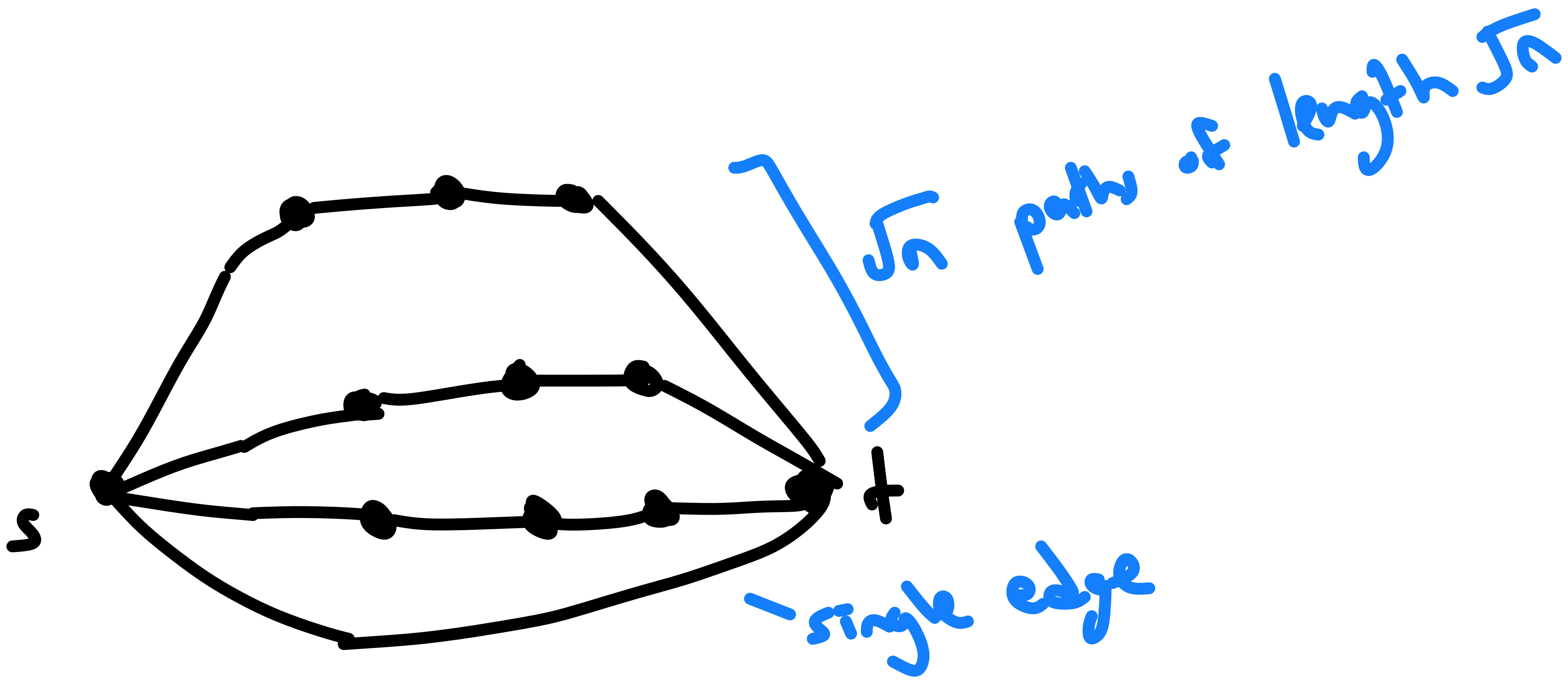
How large can p be?

Can show that if we regularize resistances by averaging:

$$r_e \leftarrow r_e + \varepsilon \frac{\sum r_e}{m}$$

then p is bounded by $O(n^{\frac{1}{2}} \varepsilon^{-1})$

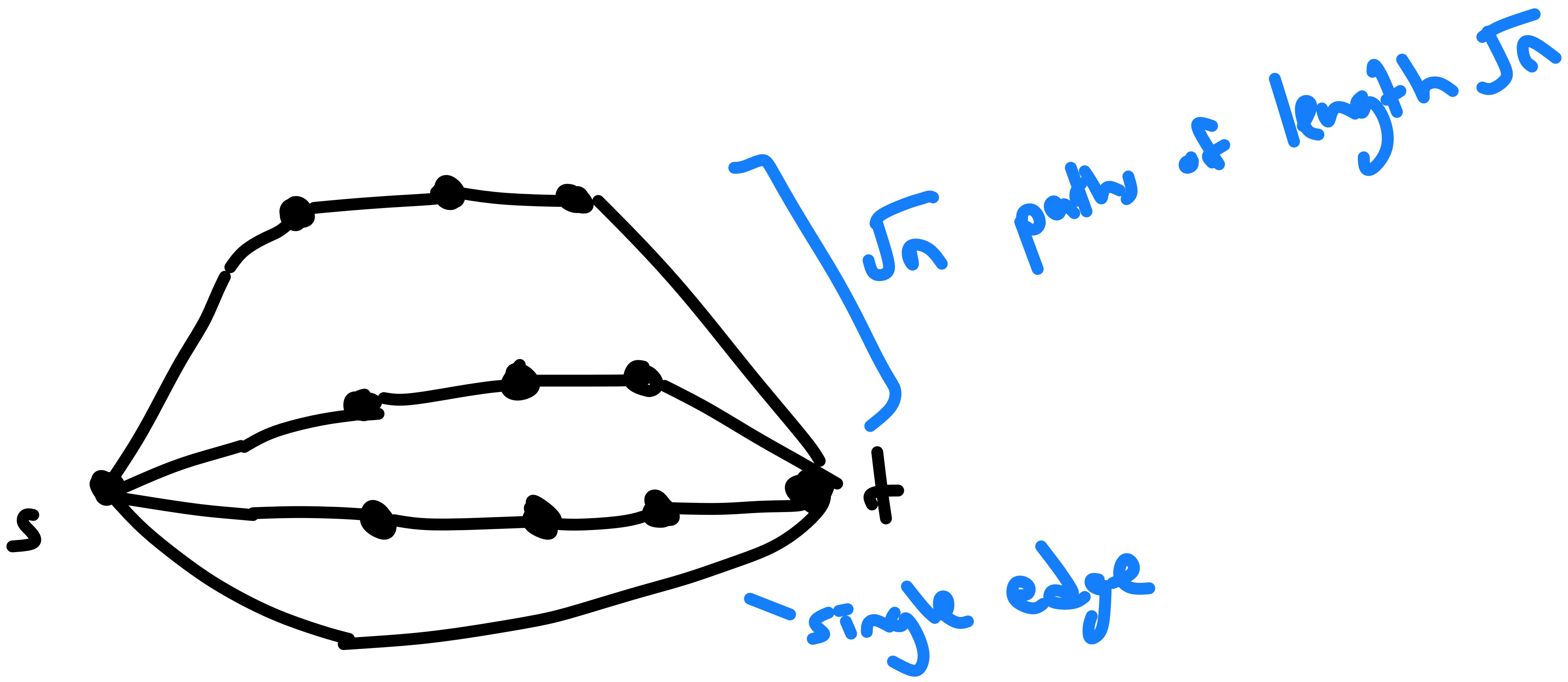
Maximal Congestion



Maximum flow $= \sqrt{n}$

congestion on single edge $= \frac{\sqrt{n}}{2}$

Maximal Congestion



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congestion on single edge $= \frac{\sqrt{n}}{2}$

How to do better? Notice graph is nicely behaved if single bad edge is removed.

$\Rightarrow$ If $\rho \geq n^{\frac{1}{3}}$ for an edge, just delete it.

The End

Key Takeaways

Max Flow as a linear programming problem:

Minimise $|f|_\infty$

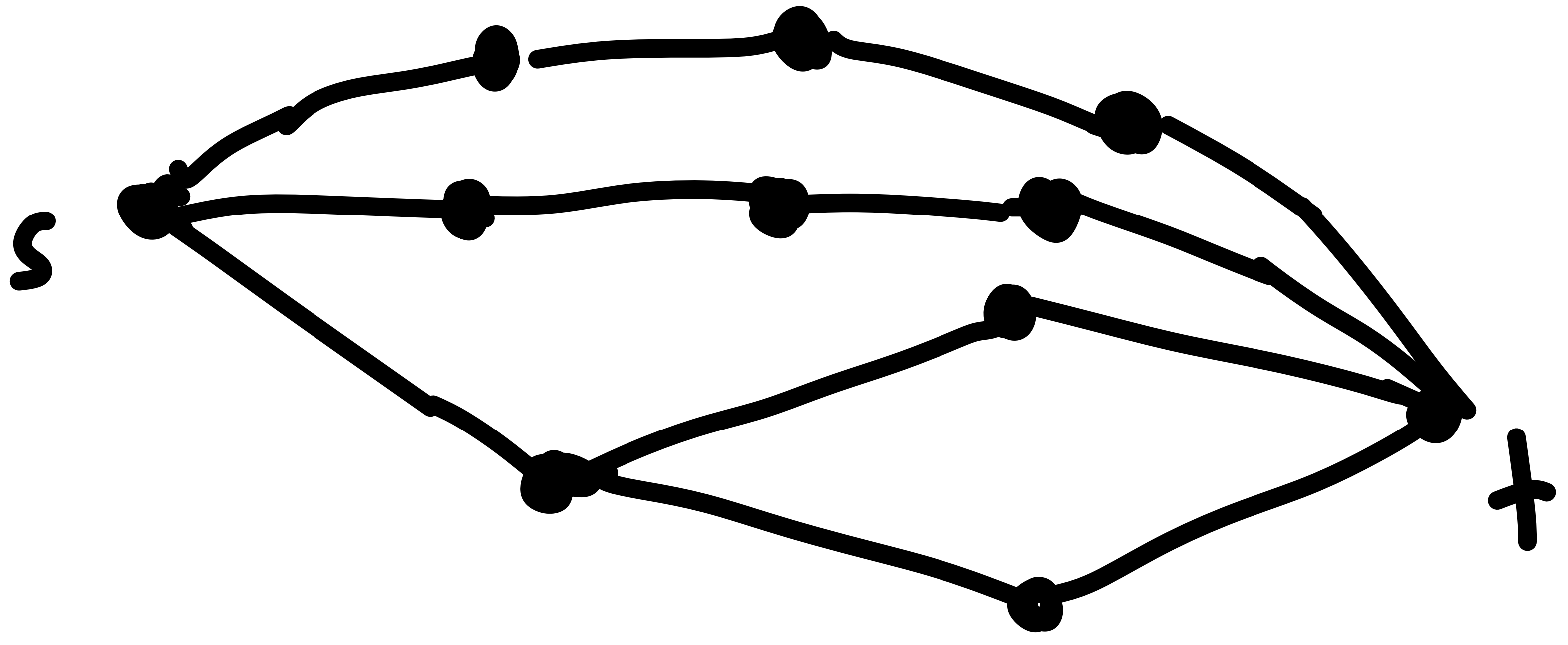
For each node v,

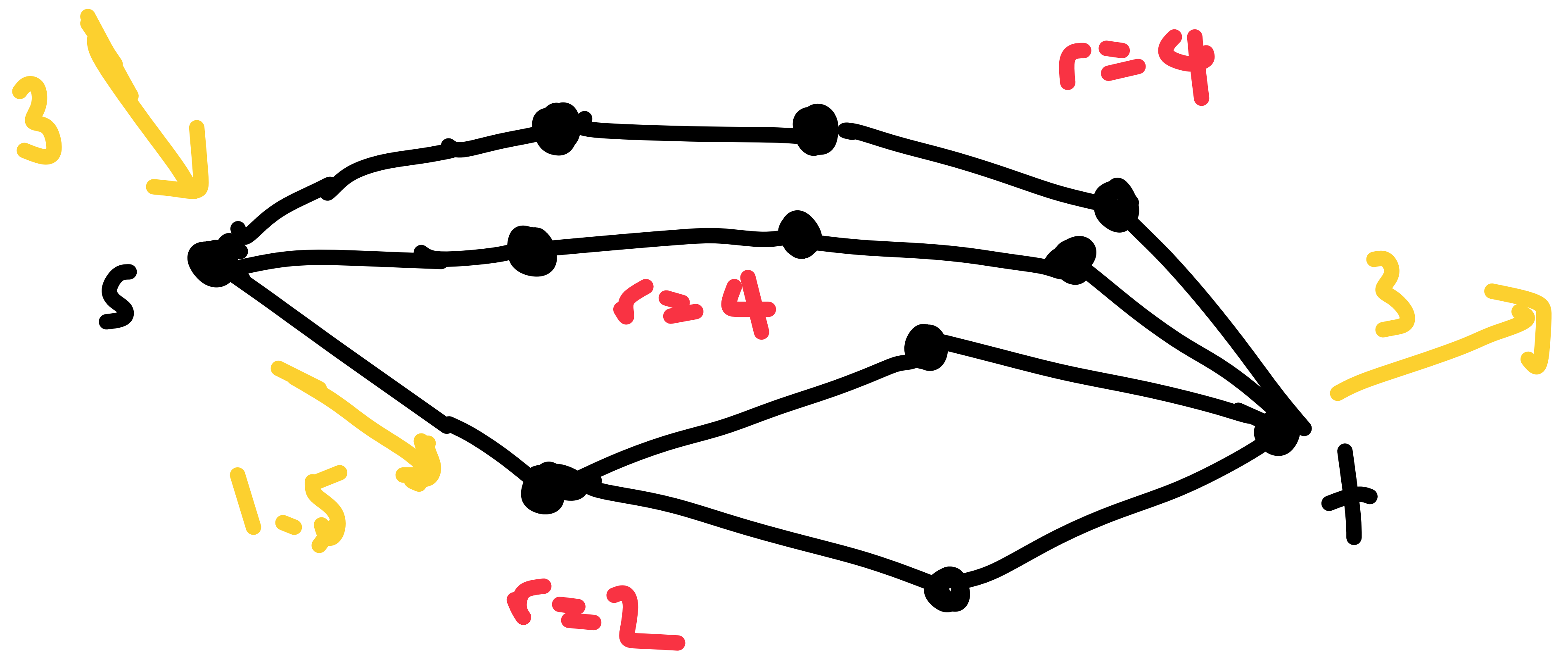
given:

$$\sum_{u: v \to u} f_{uv} - \sum_{v: w \to v} f_{wv} = 0$$

$$\sum_{u: v \to u} f_{uv} - \sum_{v: w \to v} f_{wv} = 1, -1 \quad \text{for } v = s, t$$

Solving the linear programming problem through subproblem





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ℓ_2 minimiser

$O(n)$ time *

Laplacian Solver

$O(\sqrt{n})$ times

MWU

Iteratively refine