Near-Neighbour Search & Locality-Sensitive Hashing.

TSS 20/09/23

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Outline

- Problem Definition & Background
- Approximate Version
- LSH Families (Simple)
- Data Structure Construction
- LSH Families (Complex)
- Recent Developments

Nearest Neighbour Search

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1. Database $X = \{x_1, \ldots, x_n\} \subset U \subseteq \mathbb{R}^d$ (eg. $U = \{0,1\}^d$)
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1. Construct a data structure for X (pre-processing, extra space allowed)
2. Given any query pt. $q \in U$, find a nearest pt. in X

ie. $\operatorname{argmin}_{x \in X} d(q,x)$

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⚠ minimize pre-processing time for practical apps

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> query-time: $O(\log n)$, space $O(n)$, pre-processing $O(n \log n)$

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> Moreover, in all known approaches, size of DS / q. time grows like $\exp(d)$

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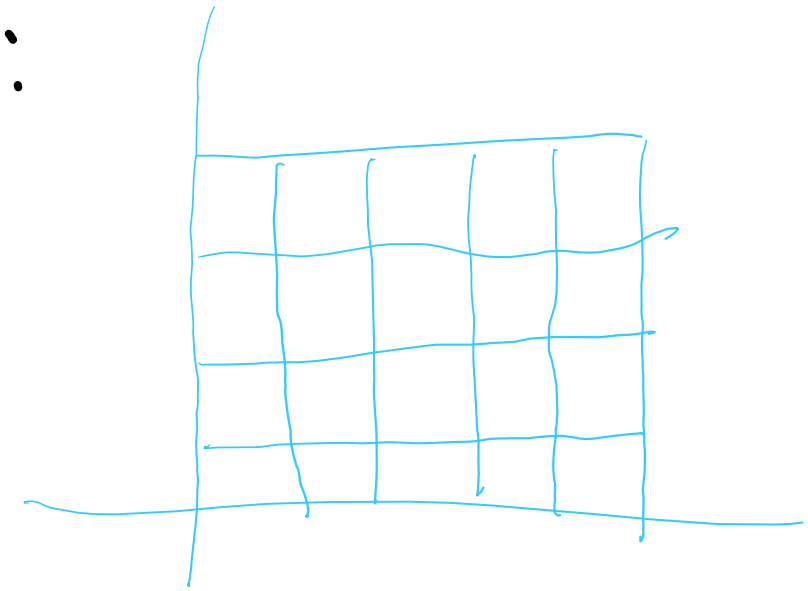

grid of side a.

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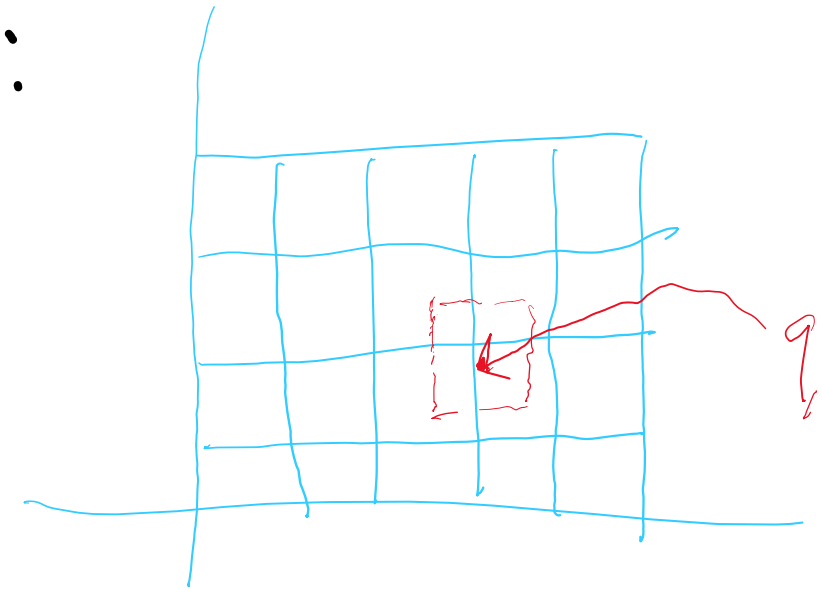

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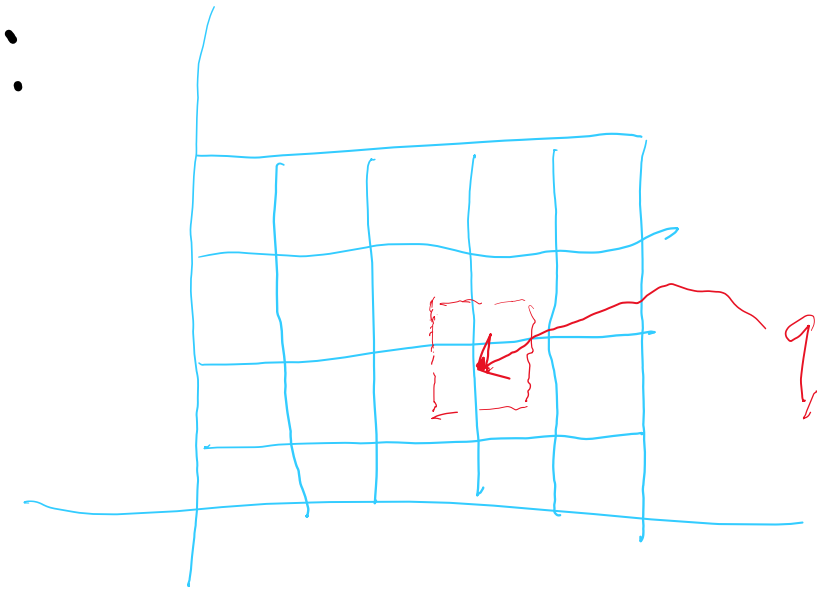

- NNS based on k-d trees will need $\Omega(2^d)$ lookup time

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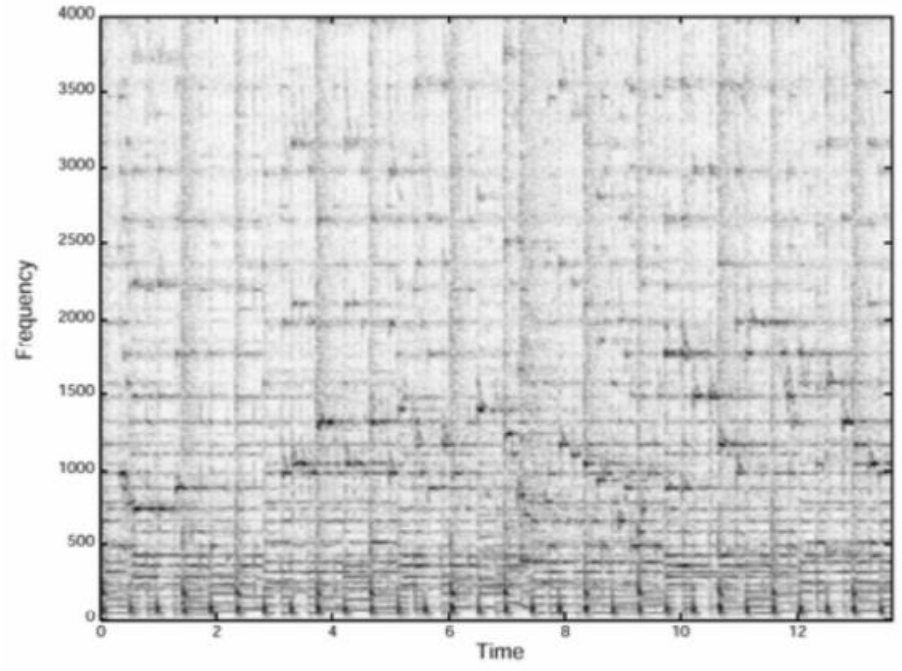
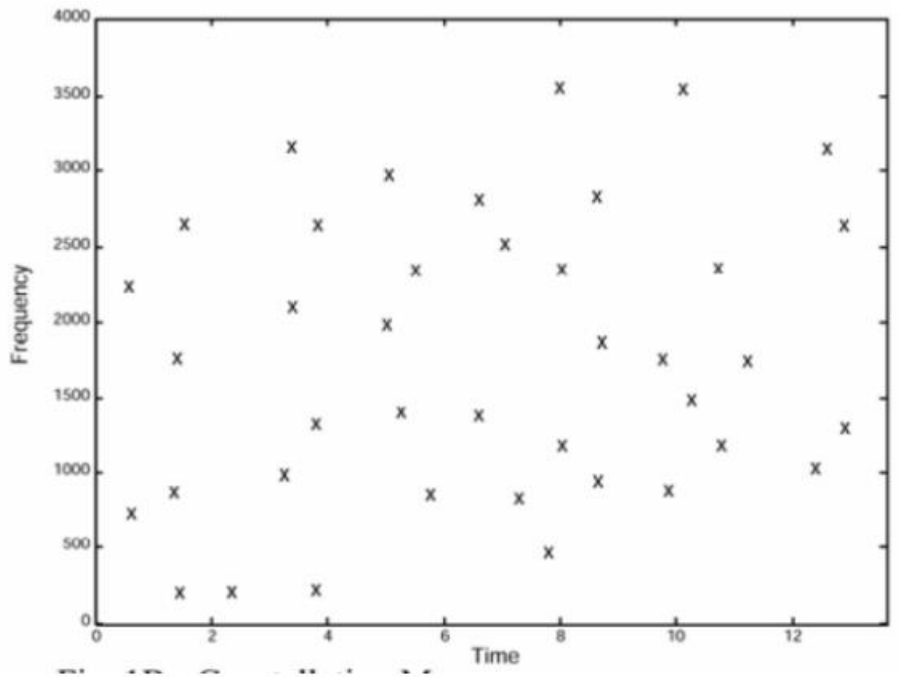
> Idea (from Avery Li-Chun Wang '03, ICMIR): First construct a "spectrogram" (some windowed FFT). Then identify the clip. From "peaks" in the spectrogram to construct $q \in \mathbb{R}^d$. Then apply NNS for a huge database of already processed clips.

Picture:



(a) Spectrogram extracted from clip of an audio file.



(b) Post-processed spectrogram which can be used to construct an audio "fingerprint" $q \in \mathbb{N}^d$ for identifying a song clip.

Figure 1: Images from the paper outlining Shazam's approach to audio retrieval [2]. Once features are extracted from an audio clip, a hashing based algorithm is used to perform approximate nearest neighbor search in a database of known songs.

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> Again, spectrogram of a seismogram is computed in real-time & compared (quickly) against a database of surrounding major seismic events
> Also a very high-dim problem

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> (c, r)-NNS: Given $X \subset U \subseteq \mathbb{R}^d$ of n pts, construct a DS s.t. for any query pt q, if $\exists p \in X$ s.t. $d(p, q) \leq r$, output a pt p' s.t. $d(p', q) \leq cr$.

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> if there is no such p, no requirements.

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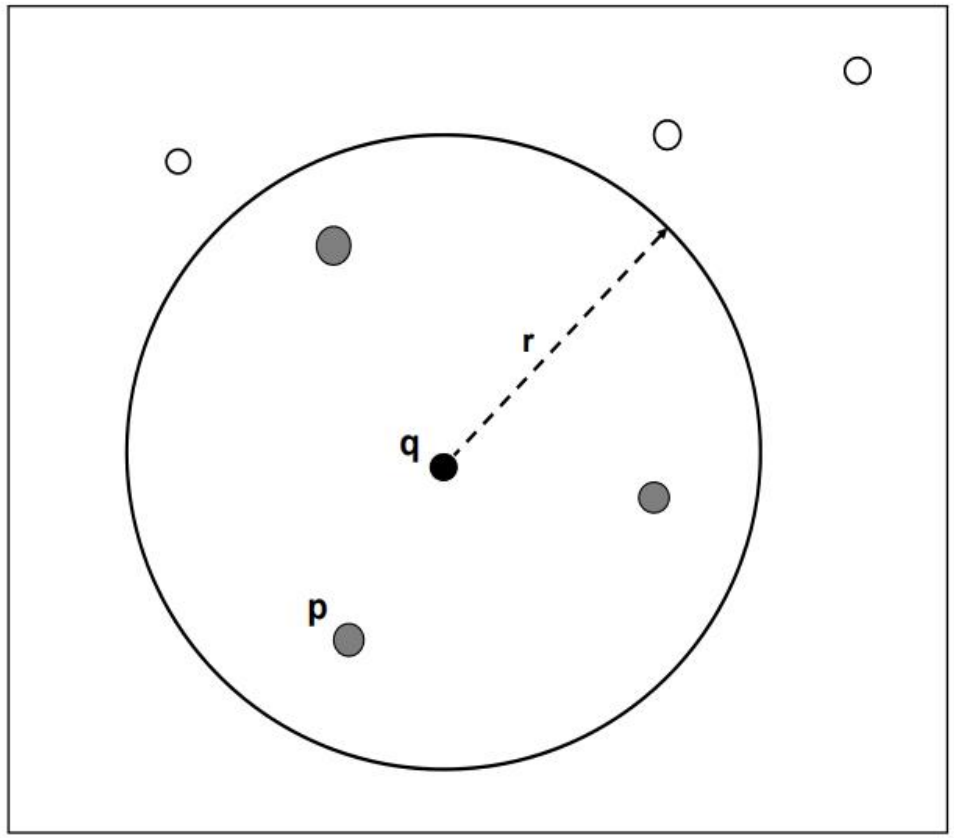

Figure 1: Near Neighbor problem, given q find p s.t. $D(q, p) \leq r$

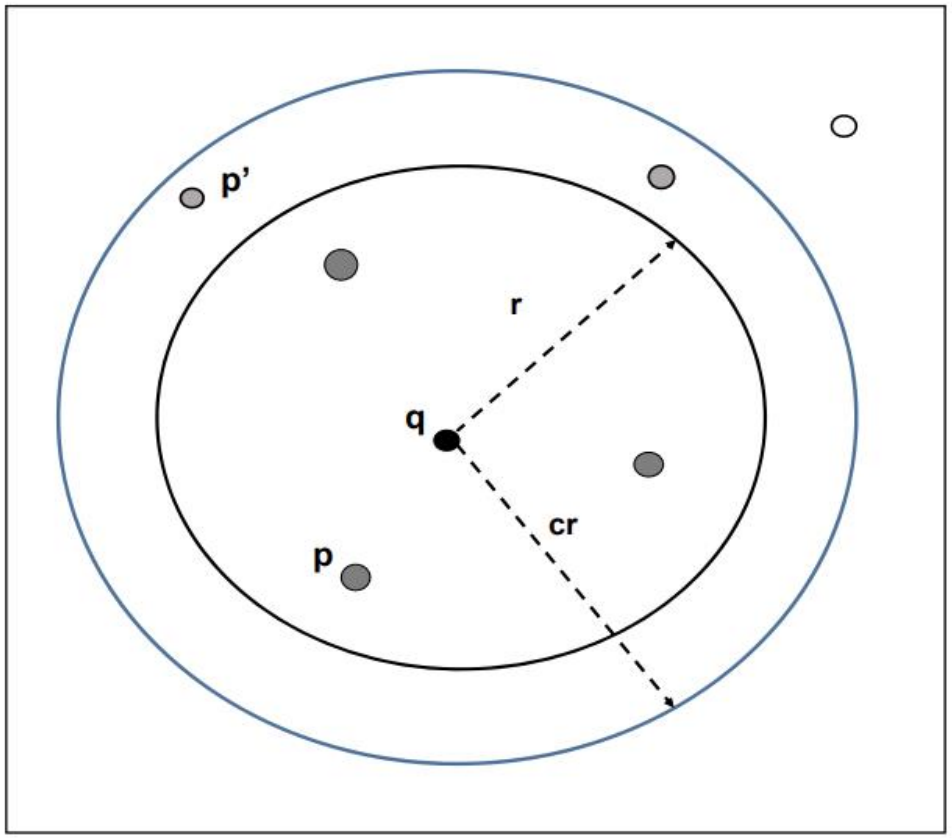

Figure 2: Approx Near Neighbor problem, given q find a p' s.t. $D(q, p') \leq cr$

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- Report the minimal value of r for which we find a pt at dist $c(1-\varepsilon)$ of q.

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  (As we maintain a separate DS for each value of r above)
- So, for the rest of the talk, we focus on (c, r)-NNS !

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> Main idea of IM98; even if $p_1 > p_2$ slightly, can boost $p_1 \approx 1$ & $p_2 \approx 1/n$.

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> So, $Pr[h(p)=h(p')] = \begin{cases} \geq 1 - \frac{r}{d} \approx e^{-r/d} & \text{if } d(p,p') \leq r \\ \leq 1 - \frac{cr}{d} \approx e^{-cr/d} & \text{if } d(p,p') \geq cr \end{cases}$

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Conclusion: $\mathcal{H}$ is $(c, cr, e^{-r/d}, e^{-cr/d})$-LSH

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> So, only need to show: given (r, cr, p_1, p_2)-LSH fam with $p_1 > p_2$, we can "boost" it to get $p_1 \approx 1$, $p_2 \approx 0$.

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> Then $\forall p, p'$, $d(p,p') \geq cr \Rightarrow \Pr[h(p) = h(p')] \leq p_2^k$.

> Great, but this doesn't $\uparrow p_1$! In fact, 2 "close" pts are mapped to same vector only with prob. p_1^k.

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$$\Pr[\exists i \mid f_i(p) = f_i(q)] = 1 - \Pr[\forall i \mid f_i(p) \neq f_i(q)]$$
$$= 1 - \Pr[f_i(p) \neq f_i(q)]^l$$
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gotta make sure this is small!

The Algorithm

Algorithm 1 LSH Algorithm

Preprocessing:

Choose $k \cdot \ell$, $h_{1,1}, \ldots, h_{\ell,k}$ functions uniformly at random from $\mathcal{H}$.

Construct ℓ hash tables; for all $1 \leq i \leq \ell$ store $f_i(p) = (h_{i,1}(p), \ldots, h_{i,k}(p))$ for all $p \in P$ in the i-th has table.

Query(q)**:**

for $i = 1 \to \ell$ **do**

Compute $f_i(q)$.

Go over all points p where $f_i(p) = f_i(q)$. For all such points if $\text{dist}(p, q) \leq c \cdot r$, output p.

end for

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- Fix a query $q \in U$. By lin. of exp., $\forall i$,

$$\Pr[\exists p : d(p,q) > cr, f_i(p) = f_i(q)] = n \cdot p_2^k \leq 1$$

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$$\Pr[\exists p : d(p,q) > cr, f_i(p) = f_i(q)] = n \cdot p_2^k \leq 1$$

- Summing over i, in exp, $O(\ell)$ far pts hash to same value as q for some i. $\Rightarrow O(\ell)$ overhead in q time.

Parameter Tuning (cont'd)

- If $d(p,q) \leq r$ for some $p \in X$, then,

$$\Pr[\exists i : f_i(p) = f_i(q)] \geq 1 - (1 - p_1^k)^l$$

$$= 1 - (1 - p_2^{ck})^l$$

$$= 1 - (1 - n^{-\rho})^l$$

$$\approx 1 - e^{ln^{-\rho}} = 1 - 1/n.$$

Parameter Tuning (cont'd)

> If $d(p,q) \le r$ for some $p \in X$, then,

$$\Pr[\exists i : f_i(p) = f_i(q)] \ge 1 - (1 - p_1^k)^\ell$$
$$= 1 - (1 - p_2^{\delta k})^\ell$$
$$= 1 - (1 - n^{-\delta})^\ell$$
$$\approx 1 - e^{\ell n^{-\delta}} = 1 - 1/n.$$

> Summary: for a pt p with $d(p,q) \le r$, our algo outputs p w.p. $\ge 1 - \frac{1}{n}$. In exp., $O(\ell \cdot d)$ overhead to examine $O(\ell)$ far pts with some hash value.

Space Analysis

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> Maintain $O(1)$ hash tables.

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- Maintain $O(l)$ hash tables.
- In each table, store $n = |X|$ hash values, each a k-dim vector.

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- In each table, store $n = |X|$ hash values, each a k-dim vector.
- Overall, $O(l \cdot n \cdot k) = O(n^{1+\rho} \log \frac{n}{p_2})$

Query Time Analysis

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- $O(lq)$ time to store $f_i(q)$ $\forall$ $i \in [l]$

Query Time Analysis

- $O(l_q)$ time to some $f_i(q)$ $\forall$ $i \in [l]$
- any candidate close pt p, spend $O(d)$ time to calculate $d(p, q)$.

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- $O(lq)$ time to compute $f_i(q)$ $\forall$ $i \in [l]$
- Any candidate close pt p, spend $O(d)$ time to calculate $d(p, q)$.
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Query Time Analysis

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- Any candidate close pt p, spend $O(d)$ time to calculate $d(p, q)$.
- Let $|O|$ = size of output = # of pts at dist $\leq cr$ from q
- In exp., we examine $O(l)$ far pts that we don't output. So, overall, time is

$$O(d(l + |O|) + l \cdot k) = O(n^{\rho}(d + \log \tfrac{n}{p_2}) + |O|d)$$

Conclusion

> Memory $O(n^{1+\varepsilon})$, q-time $O(n^{\varepsilon})$

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> For the binary vector example, recall $p_1^{\rho} = p_2$ ⇒

$$\rho = \frac{\ln 1/p_1}{\ln 1/p_2} = \frac{r/d}{cr/d} = \frac{1}{c}$$

> For eg if $c=2$, need $O(n^{1.5})$ space for hash tables & $O(\sqrt{n})$ q-time.

> q-time & memory get significantly better as we relax c.

For other metrics

For other metrics

- For l', can 'reduce' to the binary vector case using a 'coordinate red^n' idea to get $\beta = 1/c$ as well.

For l_2:

- Use Johnson-Lindenstrauss to reduce dimension d to $\log n$.
- Assume all pts in X & query pts lie in S^d. Then

$h(x) = \text{sgn}(\langle g, x \rangle)$ where $g = (g_i)_{i=1}^{d}$ is sf.

$g_i \sim N(0,1)$

Picture

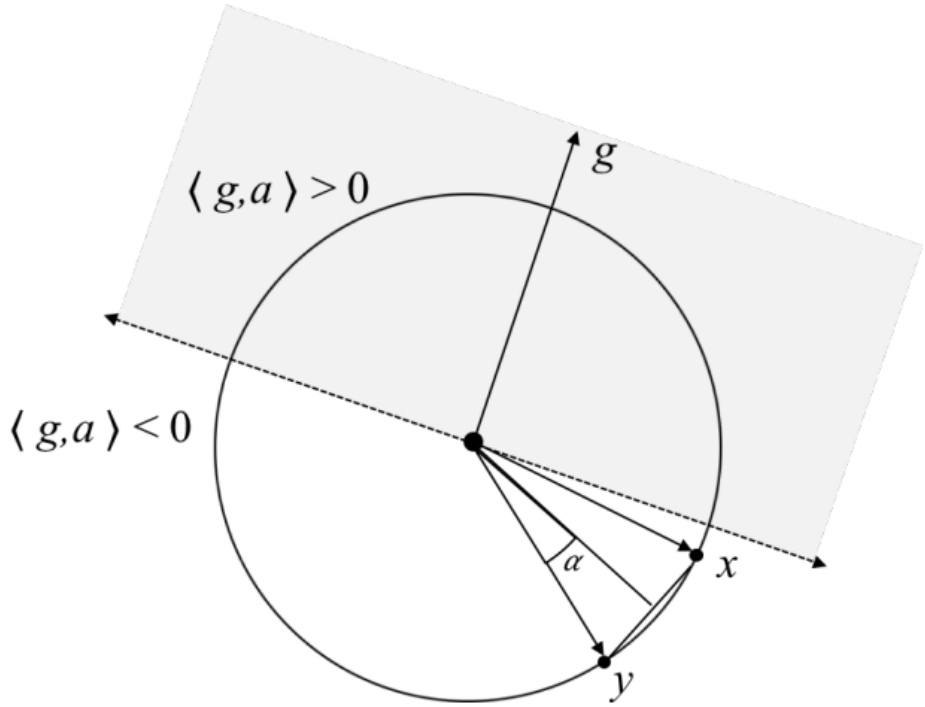

Figure 4: Our Euclidean locality sensitive hash function draws a random Gaussian vector g. Consider the hyperplane orthogonal to g and passing through $\{0\}^d$. All points above the hyperplane are hashed to 1 and all points below are hashed to -1.

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> this also gives a $\rho = 1/c$ construction.

> Possible to improve to (the optimal) $1/c^2$ using a more complex LSH scheme.

Recent Developments

(On board)

Thank You!