A Lower Bound In Geometric Discrepancy Theory

TSS - 13/04/22

Deepanshu Kush

Dedicated to the memory of

Jiří Matoušek

What is Discrepancy Theory?

What is Discrepancy Theory?

> The theory of "irregularities of distribution"

What is Discrepancy Theory?

> The theory of "irregularities of distribution"

> What is the "most uniform" way of distributing n pts in a box?

What is Discrepancy Theory?

> The theory of "irregularities of distribution"

> What is the "most uniform" way of distributing n pts in a box?

> How much "irregularity" is necessary?

What is Discrepancy Theory?

> The theory of "irregularities of distribution"

> What is the "most uniform" way of distributing n pts in a box?

> How much "irregularity" is necessary?

> For precision, need to quantify this notion - discrepancy is this numerical parameter.

History

History.

> Such questions were first tackled in 30s

- motivation came from number theory.

History.

> Such questions were first tackled in 30s
- motivation came from number theory.

> A satisfactory solution to the basic disc. problem in the plane was completed in the 60s

History.

> Such questions were first tackled in 30s
- motivation came from number theory.
> A satisfactory solution to the basic disc. problem in the plane was completed in the 60s (we'll see this!)

History.

> Such questions were first tackled in 30s
- motivation came from number theory.
> A satisfactory solution to the basic disc. problem in the plane was completed in the 60s
> But the analogous high dim. problem is far from solved even today.

History (contd.)

> In the meantime, discr. theory has blossomed into a field of great depth & diversity

History (contd.)

- In the meantime, disc. theory has blossomed into a field of great depth & diversity
- original connections to number theory / Ramsey theory still going strong

History (contd.)

> In the meantime, disc. theory has blossomed into a field of great depth & diversity

> original connections to number theory / Ramsey theory still going strong

> given rise to practical methods & algos for numerical integration

History (contd.)

> In the meantime, disc. theory has blossomed into a field of great depth & diversity

> original connections to number theory / Ramsey theory still going strong

> given rise to practical methods & algos for numerical integration

> applications include financial math., computer graphics, computational physics etc.

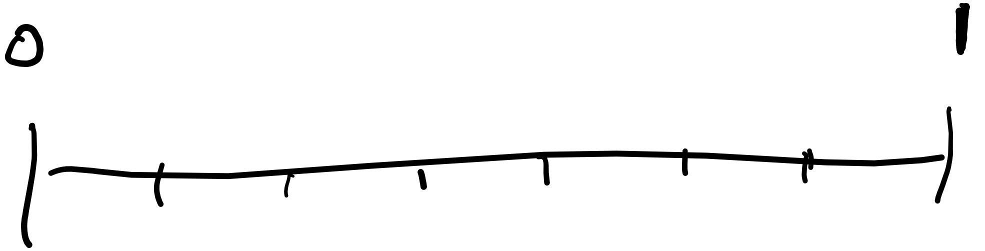
Definition of Uniformity (?).

Definition of Uniformity (?).

Main question: How "uniformly" can an n-pt. set be distributed in $[0,1]^d$?

Definition of Uniformity (?).

Main question: How "uniformly" can an n-pt. set be distributed in $[0,1]^d$?

$d=1$:  has a pretty dead easy 'answer' (contd.). Definition of Uniformity (?).

But already $d=2$ has several reasonable criteria of uniform distribution: sets that are good for some may be quite bad for others (contd.). Definition of Uniformity (?).

But already $d=2$ has several reasonable criteria of uniform distribution: sets that are good for some may be quite bad for others

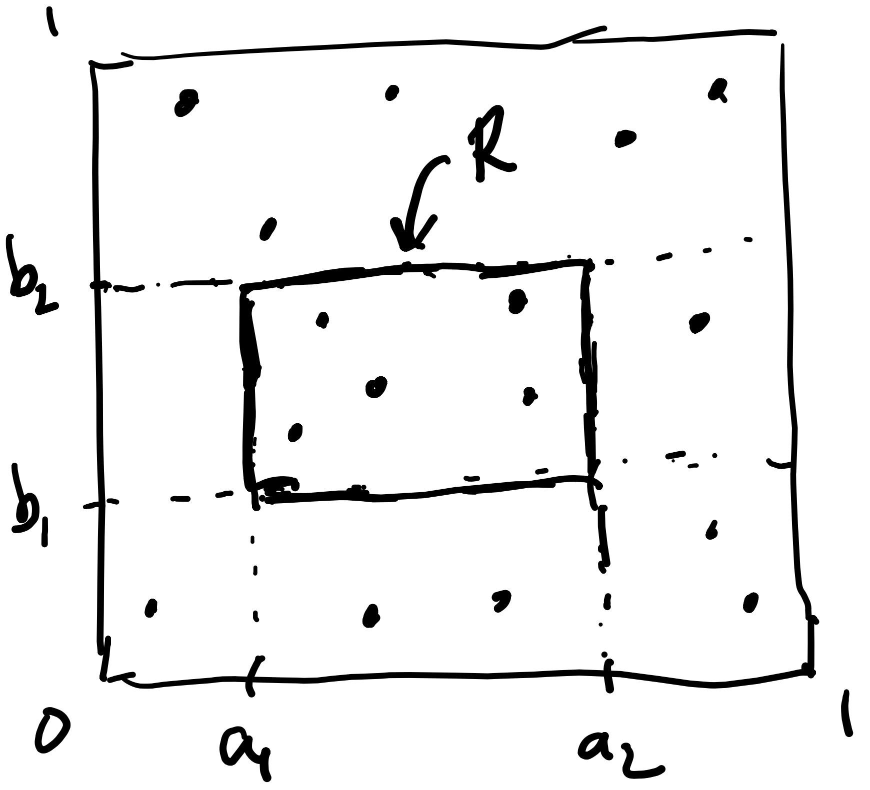
In this talk, we'll focus on one:
"uniformity" means "uniformly wrt axis-parallel rectangles"

(still contd.). <u>Definition</u> of <u>Uniformity</u> (?).

P: n pt set in the unit sq. $[0,1]^2$

(still contd.). <u>Definition of Uniformity (?)</u>.

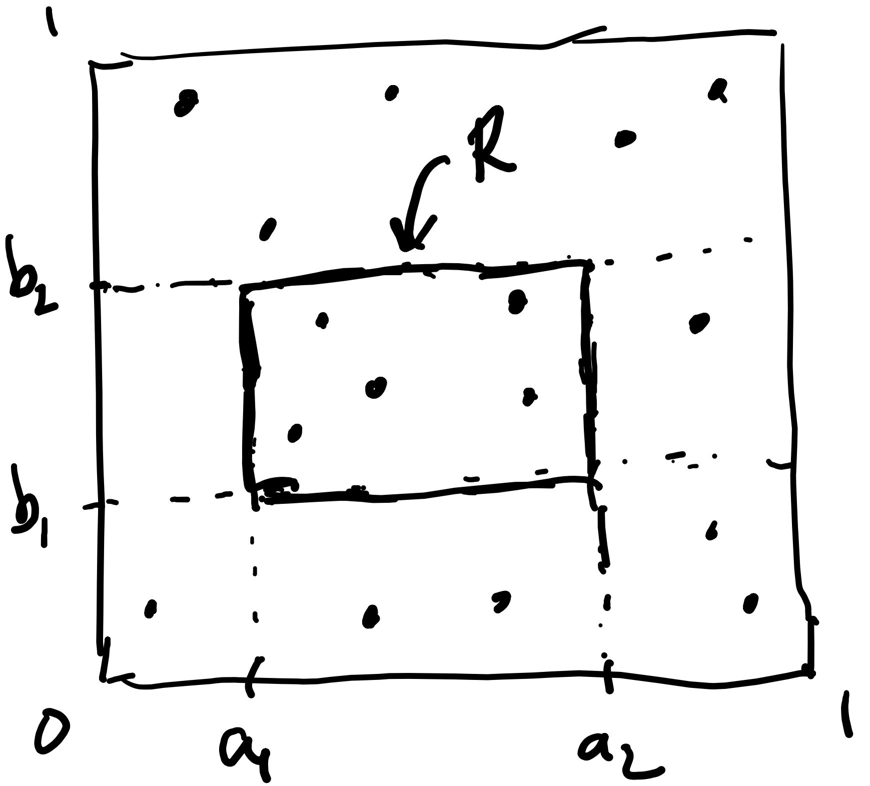
P: n pt set in the unit sq. $[0,1]^2$



(still contd.). Definition of Uniformity (?).

P: n pt set in the unit sq. $[0,1]^2$

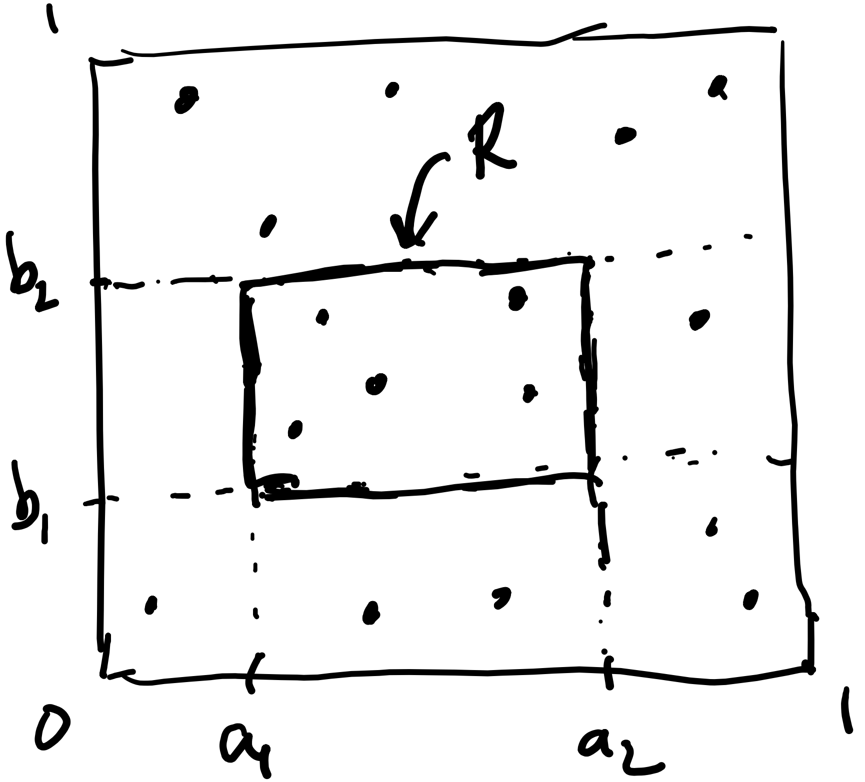
Fix a rectangle R.



(still contd.) Definition of Uniformity (?).

P: n pt set in the unit sq. $[0,1]^2$

Fix a rectangle R.

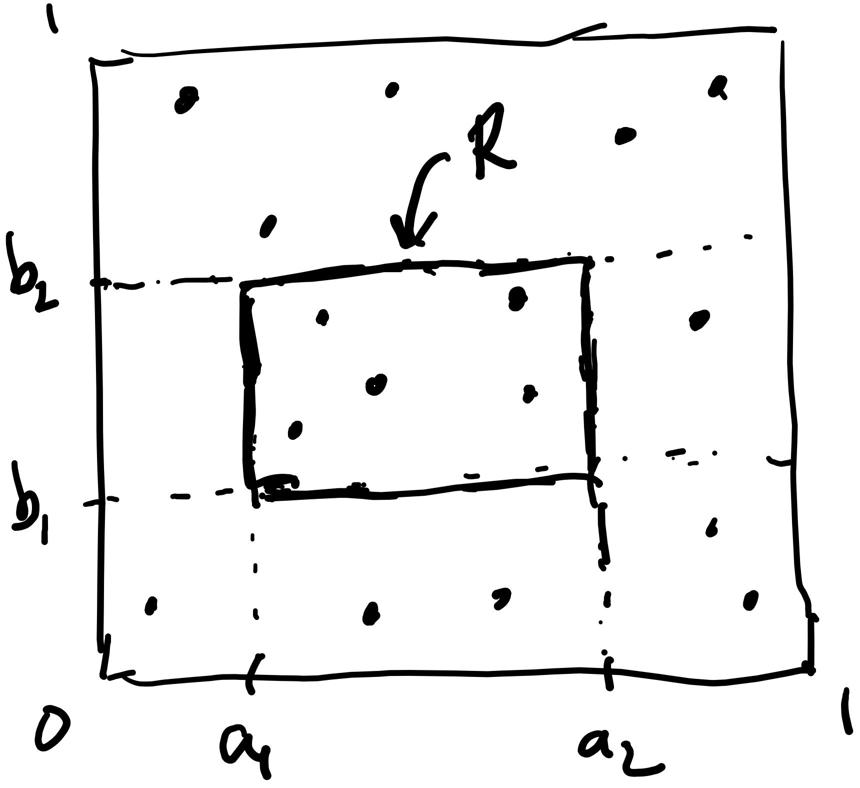


For a unif. distributed set P, the exp. # of pts of P that fall in R is $n \cdot vol(R)$.

(still contd.). <u>Definition of Uniformity (?)</u>.

P: n pt set in the unit sq. $[0,1]^2$

Fix a rectangle R.

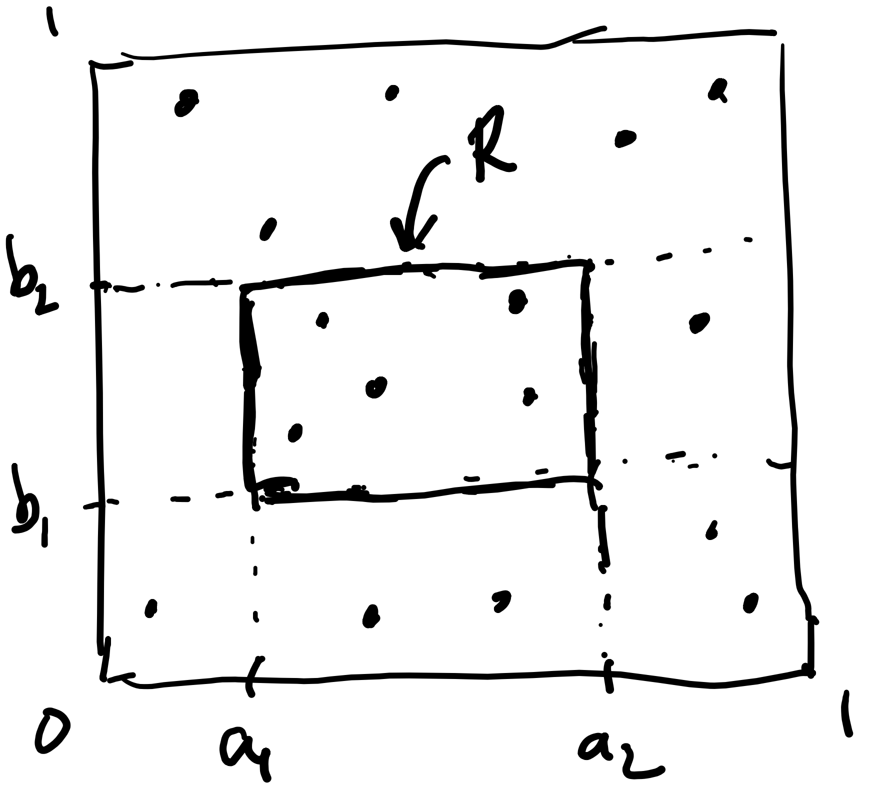


For a unif. distributed set P, the exp. # of pts of P that fall in R is $n \cdot vol(R)$.

So: $\left| n \cdot vol(R) - |P \cap R| \right|$ is the deviation.

(still contd.). <u>Definition of Uniformity (?)</u>.

P: n pt set in the unit sq. $[0,1]^2$

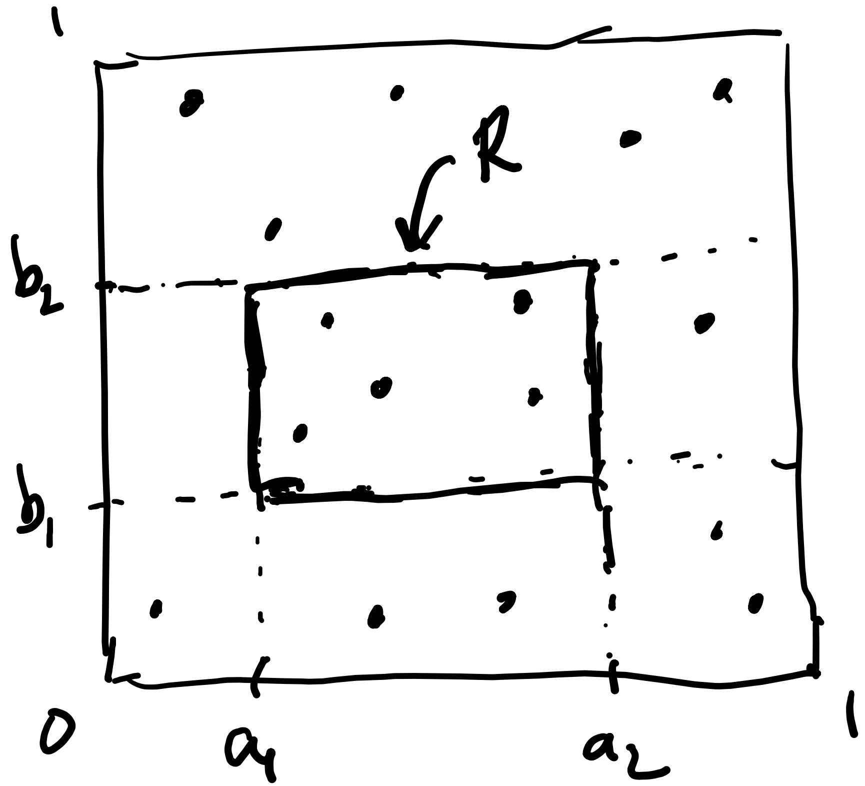


<u>Central ques</u>: Is there a P s.t. the deviation

$$\left| n.\mathrm{vol}(R) - |P \cap R| \right| \le 5 \ ?$$

(still contd.). Definition of Uniformity (?).

P: n pt set in the unit sq. $[0,1]^2$



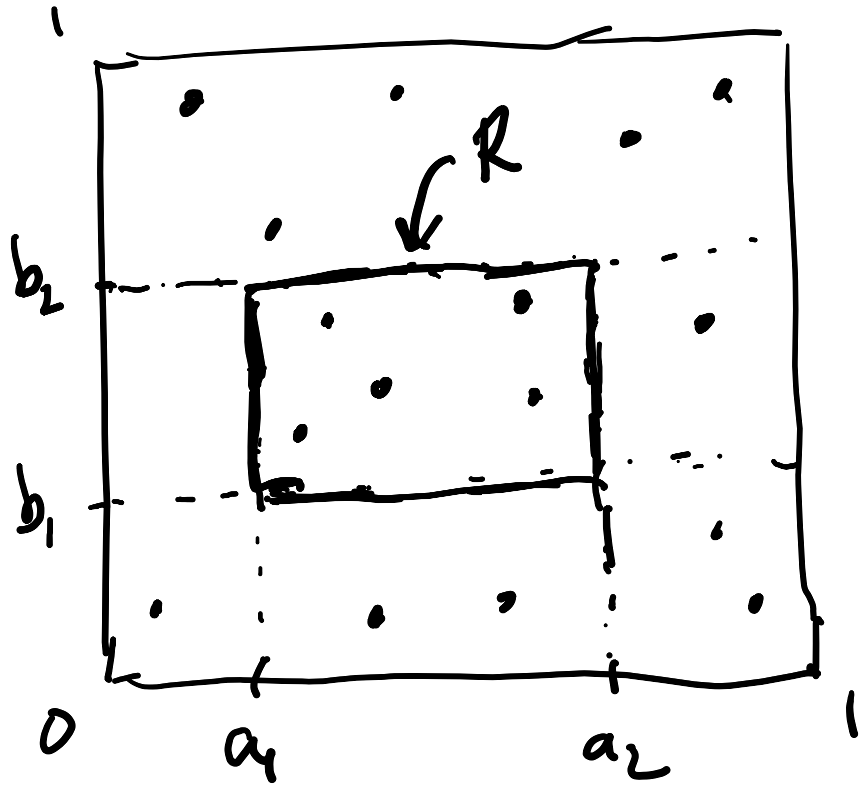
Central ques: Is there a P s.t. the deviation $\forall R$.

$$\big| n \cdot \mathrm{vol}(R) - |P \cap R| \big| \leq 5 \,?$$

(or any const.)

(still contd.). <u>Definition</u> of <u>Uniformity</u> (?).

P: n pt set in the unit sq. $[0,1]^2$



<u>Central ques</u>: Is there a P s.t. the deviation $\forall R$.

$$\big| n.\mathrm{vol}(R) - |P \cap R| \big| \leq 5 ?$$

(or any const.).

Do arbitrarily large such sets exist?

(still contd.). <u>Definition</u> of <u>Uniformity</u> (?).

> this is one of the founding questions giving birth to the field.

(still contd.). <u>Definition</u> of <u>Uniformity</u> (?).

> this is one of the founding questions giving birth to the field.

> Answer: NO!

(still contd.). Definition of Uniformity (?).

> this is one of the founding questions giving birth to the field.

> Answer: NO! Any distribution of n pts in $[0,1]^2$ has to display a significant irregularity for some rectangle R

(still contd.). Definition of Uniformity (?).

> this is one of the founding questions giving birth to the field.

> Answer: NO! Any distribution of n pts in $[0,1]^2$ has to display a significant irregularity for *some* rectangle R – and the magnitude of this irreg- grows to ∞ as $n \to \infty$.

(still contd.). Definition of Uniformity (?).

> this is one of the founding questions giving birth to the field.

> Answer: NO! Any distribution of n pts in $[0,1]^2$ has to display a significant irregularity for *some* rectangle R — and the magnitude of this irreg- grows to ∞ as $n \to \infty$.

> For $d=2$: we know exactly how.

Precise Definitions & Notation

Precise Definitions & Notation

> Given an n pt set P & rect. R,

$$D(P,R) = n \cdot vol(R) - |P \cap R|$$

Precise Definitions & Notation

> Given an n pt set P & rect. R,
$$D(P, R) = n \cdot vol(R) - |P \cap R|$$

> $\mathcal{R}_2$: set of all axis-parallel rect.s in $[0,1]^2$

Precise Definitions & Notation

> Given an n pt set P & rect. R,

$$D(P, R) = n \cdot vol(R) - |P \cap R|$$

> $\mathcal{R}_2$: set of all axis-parallel rect.s in $[0,1]^2$

$$D(P, \mathcal{R}_2) = \sup_{R \in \mathcal{R}_2} |D(P, R)|$$

is the discrepancy of P for axis-parallel rectangles (contd). Precise Definitions & Notation

> The function

$$D(n, \mathcal{R}_2) = \inf_{\substack{P \subset [0,1]^2 \\ |P| = n}} D(P, \mathcal{R}_2)$$

quantifies the smallest possible discrepancy of an n-pt set.

Central Question Restated:

Is $D(n, R_2)$ bdd above by a constant for all n, or is $\limsup_{n \to \infty} D(n, R_1) = \infty$?

Small Extention to the Definition

Small Extention to the Definition

> Discrepancy is also studied for classes of geometric figures other than axis-|| rects, such as set of balls, set of boxes etc.

Small Extention to the Definition

- Discrepancy is also studied for classes of geometric figures other than axis-|| rects, such as set of balls, set of boxes etc.
- Let $\mathcal{A}$ be a class of sets in $[0,1]^d$.

Small Extention to the Definition

> Discrepancy is also studied for classes of geometric figures other than axis-|| rects, such as set of balls, set of boxes etc.

> Let $\mathcal{A}$ be a class of sets in $[0,1]^d$. For P & $A \in \mathcal{A}$,

$$D(P, A) = n \cdot vol(A) - |P \cap A|.$$

$$D(P, \mathcal{A}) = \sup_{A \in \mathcal{A}} |D(P, A)|$$

Small Extension to the Definition

> Discrepancy is also studied for classes of geometric figures other than axis-$\parallel$ rects, such as set of balls, set of boxes etc.

> Let $\mathcal{A}$ be a class of sets in $[0,1]^d$. For P & $A \in \mathcal{A}$,

$$D(P, A) = n \cdot vol(A) - |P \cap A|.$$

$$D(P, \mathcal{A}) = \sup_{A \in \mathcal{A}} |D(P, A)|$$

> Similarly, the disc. for $D(n, \mathcal{A}) = \inf_{\substack{P \subset [0,1]^d \\ |P| = n}} D(P, \mathcal{A}).$

Hence, to show that $D(n, \mathcal{A})$ is small, we need to exhibit an n-pt set and verify that it is good $\forall A \in \mathcal{A}$.

Hence, to show that $D(n, \mathcal{A})$ is small, we need to exhibit an n-pt set and verify that it is good $\forall A \in \mathcal{A}$.

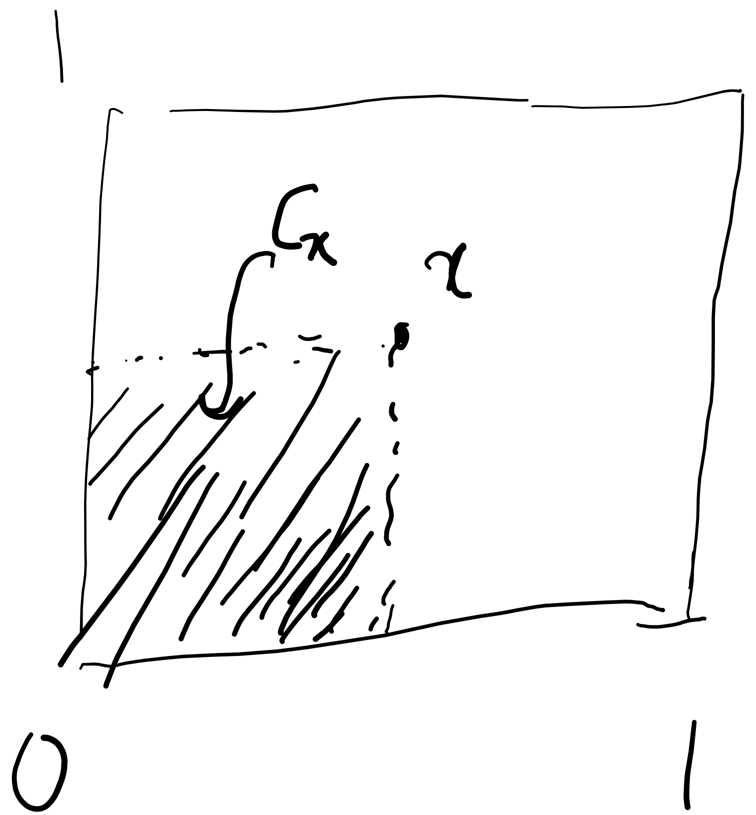
To prove that $D(n, \mathcal{A})$ is large (ie. show a lower bd), need to show that for <u>any</u> pt set P given by an adversary, $\exists$ a bad $A \in \mathcal{A}$.

A Proxy for R_2

A Proxy for $\mathcal{R}_2$

For $x \in [0,1]^d$, define $\mathcal{C}_2$ (or $\mathcal{C}_d$ in gen.) as the union of all corners C_x

A Proxy for $\mathcal{R}_2$



For $x \in [0,1]^d$, define $\mathcal{C}_2$ (or $\mathcal{C}_d$ in gen.) as the union of all corners C_x

Obs: $D(P, \mathcal{C}_d) \leq D(P, \mathcal{R}_d)$

$\leq 2^d D(P, \mathcal{C}_d)$

Average Discrepancy.

Average Discrepancy

> In def. of $D(P, \mathcal{A})$, we take a sup over all $A \in \mathcal{A}$ — worst case

Average Discrepancy.

> In def. of $D(P, \mathcal{A})$, we take a sup over all $A \in \mathcal{A}$ – worst case
> Makes sense to define an avg over $\mathcal{A}$ instead.

Average Discrepancy

> In def. of $D(P, \mathcal{A})$, we take a sup over all $A \in \mathcal{A}$ – worst case

> Makes sense to define an avg over $\mathcal{A}$ instead

> This can help in lower bds – we only need to exhibit a single 'bad' $A \in \mathcal{A}$

Average Discrepancy.

> In def. of $D(\mathcal{P}, \mathcal{A})$, we take a sup over all $A \in \mathcal{A}$ – worst case

> Makes sense to define an avg over $\mathcal{A}$ instead

> This can help in lower bds – we only need to exhibit a single 'bad' $A \in \mathcal{A}$

> In most lb proofs, one actually shows that a 'random' or 'avg' set from $\mathcal{A}$ is bad.

(contd.)

Average Discrepancy.

(contd.)

Average Discrepancy.

> One std. kind of avg is L_p-discrepancy:

(for corners):

$$D_p(P, \mathcal{C}_d) = \left(\int_{[0,1]^d} |D(P, C_x)|^p \, dx \right)^{1/p}$$

(contd.)

Average Discrepancy

> One std. kind of avg is L_p-discrepancy:

(for corners):

$$D_p(P, \mathcal{C}_d) = \left(\int_{[0,1]^d} |D(P, C_x)|^p \, dx \right)^{1/p}$$

Note: $D_p(P, \mathcal{C}_d) \leq D(P, \mathcal{C}_d)$

& $D_p(P, \mathcal{C}_d) \leq D_{p'}(P, \mathcal{C}_d)$ if $p \leq p'$.

Central Question: What is $D(n, \ell_2)$?

Central Question: What is $D(n, \ell_2)$?

Answer: It's $\Theta(\log n)$.

Central Question: What is $D(n, \ell_2)$?

Answer: It's $\Theta(\log n)$.

It's possible to construct a set $P \subseteq [0,1]^2$ whose discrepancy w.r.t. any corner C_x is $\leq \log n$.

Central Question: what is $D(n, \ell_2)$?

Answer: It's $\Theta(\log n)$.

It's possible to construct a set $P \subseteq [0,1]^2$ whose discrepancy w.r.t. any corner C_x is $\leq \log n$.

Van Der Corput Set

Central Question: What is $D(n, \mathcal{C}_2)$?

Answer: It's $\Theta(\log n)$.

It's possible to construct a set $P \subseteq [0,1]^2$ whose discrepancy w.r.t. any corner C_x is $\leq \log n$.

Van Der Corput Set

What is $D_2(n, \mathcal{C}_2)$?

<u>Central Question</u>: What is $D(n, \mathcal{C}_2)$?

<u>Answer</u>: It's $\Theta(\log n)$.

It's possible to construct a set $P \subseteq [0,1]^2$ whose discrepancy w.r.t. any corner C_x is $\leq \log n$.

Van Der Corput Set

What is $D_2(n, \mathcal{C}_2)$? It's $\Theta(\sqrt{\log n})$.

Thm (Roth) $D_2(n, \ell_2) = \Omega(\sqrt{\log n})$.

Thm (Roth) $D_2(n, \mathcal{C}_2) = \Omega(\sqrt{\log n})$.

(Note this implies $D(n, \mathcal{C}_2), D(n, \mathcal{R}_2) = \Omega(\sqrt{\log n})$,

thereby resolving the basic disc. problem in the plane).

Thm (Roth) $D_2(n, \mathcal{C}_2) = \Omega(\sqrt{\log n})$. (1954)

Pf. Recall $D_2(P, \mathcal{C}_2) = \left(\int_{[0,1]^2} D(P, C_x)^2 \, dx \right)^{1/2}$

where $D(P, C_x) = n \cdot \mathrm{vol}(C_x) - |P \cap C_x|$.

Thm (Roth) $D_2(n, \mathcal{C}_2) = \Omega(\sqrt{\log n})$. (1954)

Pf. Recall $D_2(P, \mathcal{C}_2) = \left(\int_{[0,1]^2} D(P, C_x)^2 \, dx \right)^{1/2}$

where $D(P, C_x) = n \cdot \mathrm{vol}(C_x) - |P \cap C_x|$. Call this $D(x)$ (P is fixed now).

Thm (Roth) $D_2(n, \mathcal{C}_2) = \Omega(\sqrt{\log n})$. (1954)

Pf: Recall $D_2(P, \mathcal{C}_2) = \left(\int_{[0,1]^2} D(P, C_x)^2 \, dx \right)^{1/2}$

where $D(P, C_x) = n \cdot \mathrm{vol}(C_x) - |P \cap C_x|$. ← Call this $D(x)$

(P is fixed now).

Idea: Use an auxiliary 'averaging' fn $F : [0,1]^2 \to \mathbb{R}$

& Cauchy-Schwartz: $\int FD \leq \sqrt{\int F^2} \cdot \sqrt{\int D^2}$

Thm (Roth) $D_2(n, \mathcal{C}_2) = \Omega(\sqrt{\log n})$. (1954)

Pf: Recall $D_2(P, \mathcal{C}_2) = \left(\int_{[0,1]^2} D(P, C_x)^2 \, dx \right)^{1/2}$

where $D(P, C_x) = n \cdot \mathrm{vol}(C_x) - |P \cap C_x|$. ← Call this $D(x)$

(P is fixed now).

Idea: Use an auxiliary 'averaging' fn $F: [0,1]^2 \to \mathbb{R}$

& Cauchy-Schwartz: $\int FD \leq \sqrt{\int F^2} \cdot \sqrt{\int D^2}$

So: $D(P, \mathcal{C}_2) = \sqrt{\int D^2} \geq \dfrac{\int FD}{\sqrt{\int F^2}}$

Thm (Roth) $D_2(n, \mathcal{C}_2) = \Omega(\sqrt{\log n})$. (1954)

Pf. Recall $D_2(P, \mathcal{C}_2) = \left(\int_{[0,1]^2} D(P, C_x)^2 \, dx \right)^{1/2}$

where $D(P, C_x) = n \cdot \mathrm{vol}(C_x) - |P \cap C_x|$. ← Call this $D(x)$ (P is fixed now).

Idea: Use an auxiliary 'averaging' fn $F : [0,1]^2 \to \mathbb{R}$

& Cauchy-Schwartz: $\int FD \le \sqrt{\int F^2} \cdot \sqrt{\int D^2}$

So: $D(P, \mathcal{C}_2) = \sqrt{\int D^2} \ge \dfrac{\int FD}{\sqrt{\int F^2}}$ → want $\Omega(\log n)$

Thm (Roth) $D_2(n, \mathcal{C}_2) = \Omega(\sqrt{\log n})$. (1954)

Pf. Recall $D_2(P, \mathcal{C}_2) = \left(\int_{[0,1]^2} D(P, C_x)^2 \, dx \right)^{1/2}$

where $D(P, C_x) = n \cdot \mathrm{vol}(C_x) - |P \cap C_x|$. ← Call this $D(x)$ (P is fixed now).

Idea: Use an auxiliary 'averaging' fn $F: [0,1]^2 \to \mathbb{R}$ & Cauchy-Schwartz: $\int FD \leq \sqrt{\int F^2} \cdot \sqrt{\int D^2}$

So: $D(P, \mathcal{C}_2) = \sqrt{\int D^2} \geq \dfrac{\int FD}{\sqrt{\int F^2}}$

$\int FD \rightsquigarrow$ want $\Omega(\log n)$

$\sqrt{\int F^2} \rightsquigarrow$ want $O(\sqrt{\log n})$

Thm (Roth) $D_2(n, \mathcal{C}_2) = \Omega(\sqrt{\log n})$. (1954)

We'll have F depend on P, so that it 'collects' its discrepancy in a suitable sense

Thm (Roth) $D_2(n, \mathcal{C}_2) = \Omega(\sqrt{\log n})$. (1954)

We'll have F depend on P, so that it 'collects' its discrepancy in a suitable sense

Main Idea: Define F as a sum of m $(= \Theta(\log n))$ orthogonal functions

Thm (Roth) $D_2(n, \mathcal{C}_2) = \Omega(\sqrt{\log n})$. (1954)

We'll have F depend on P, so that it 'collects' its discrepancy in a suitable sense

Main Idea: Define F as a sum of m $(= \Theta(\log n))$. orthogonal functions

Let $2n \leq 2^m \leq 4n$. For each $i = 0, 1, \dots, m$, define

$$f_i : [0,1]^2 \longrightarrow \{-1, 0, 1\}.$$

Thm (Roth) $D_2(n, \mathcal{C}_2) = \Omega(\sqrt{\log n})$. (1954)

We'll have F depend on P, so that it 'collects' its discrepancy in a suitable sense

Main Idea: Define F as a sum of m $(= \Theta(\log n))$. orthogonal functions

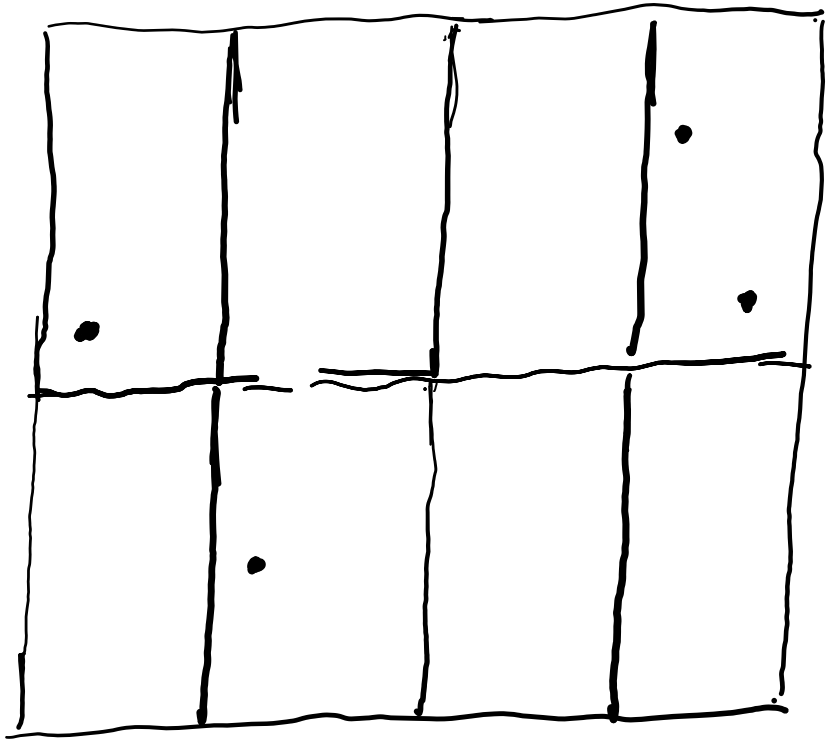
Let $2n \leq 2^m \leq 4n$. For each $i = 0, 1, \ldots, m$, define

$$f_i : [0,1]^2 \longrightarrow \{-1, 0, 1\}.$$

by breaking up $[0,1]^2$ into sub-rectangles.

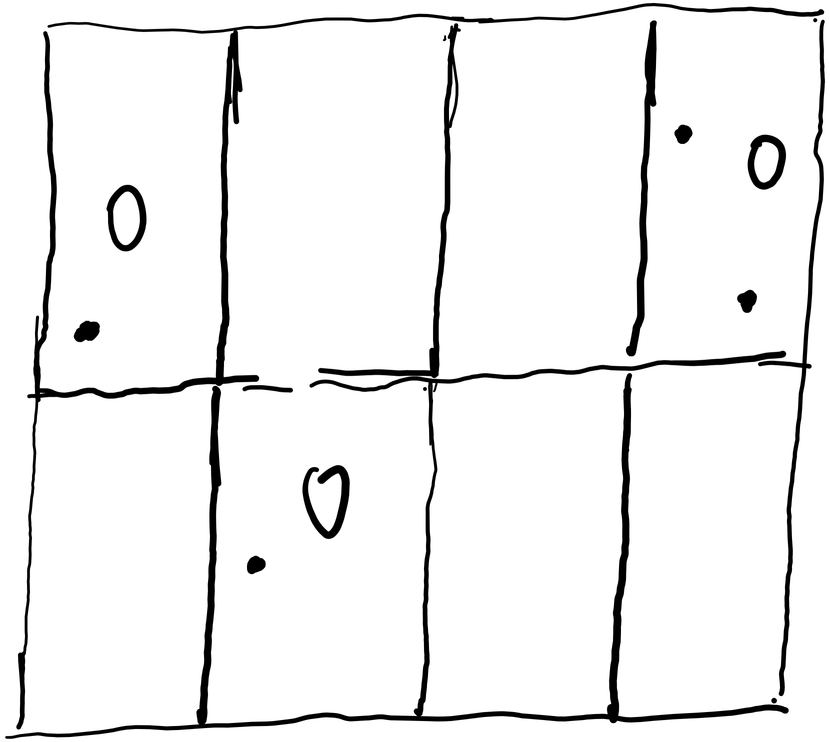
To define f_j, divide $\square$ into 2^j lines horizontally and 2^{m-j} lines vertically to form a rectangular grid:

To define f_j, divide $\square$ into 2^j lines horizontally and 2^{m-j} lines vertically to form a rectangular grid:



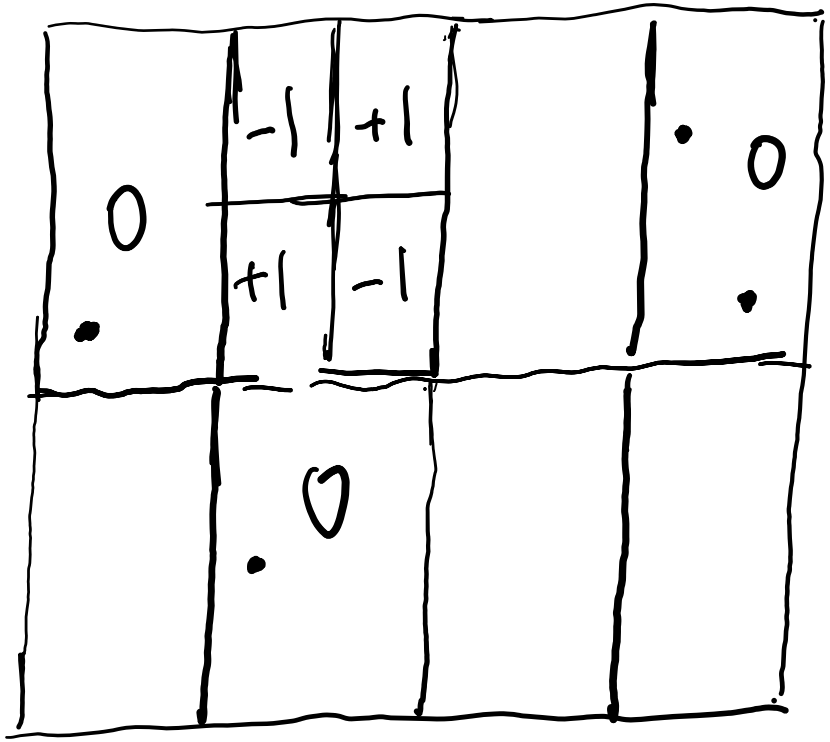
$n=4$, $m=3$, $j=2$

To define f_j, divide $\square$ into 2^j lines horizontally and 2^{m-j} lines vertically to form a rectangular grid:



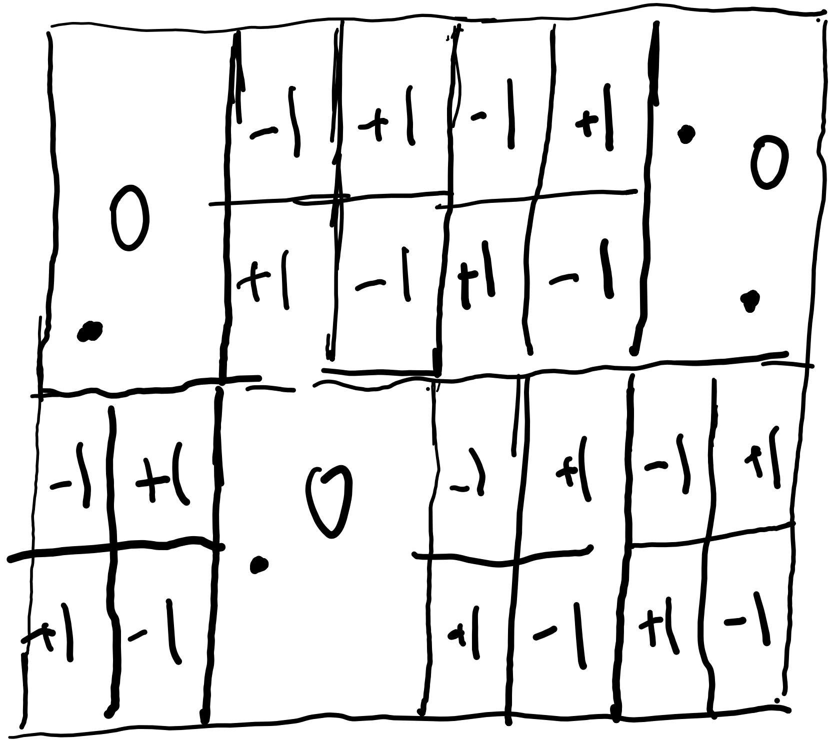
$n=4$, $m=3$, $j=2$

To define f_j, divide $\square$ into 2^j lines horizontally and 2^{m-j} lines vertically to form a rectangular grid:



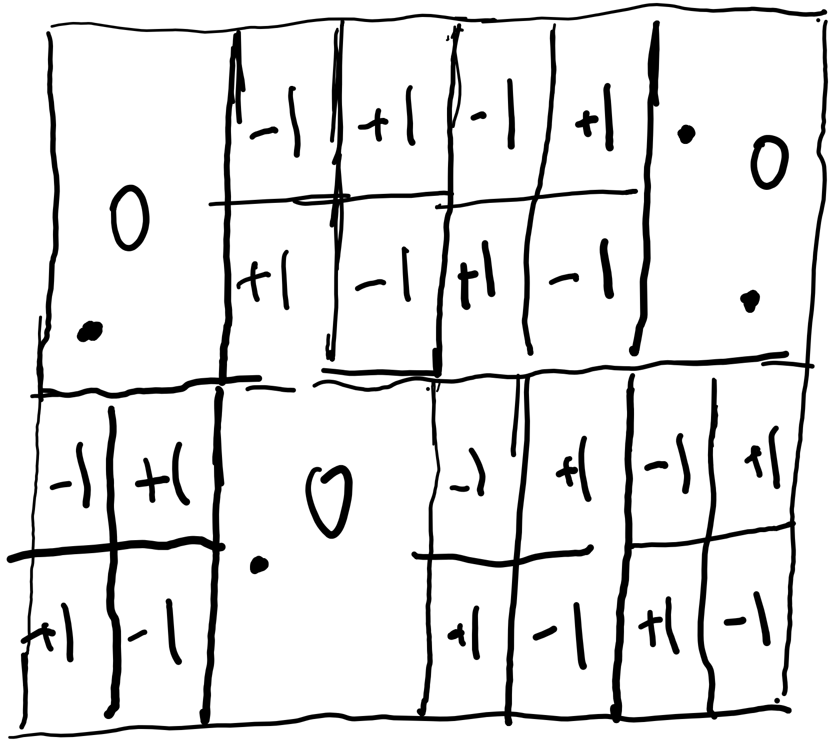
$n=4$, $m=3$, $j=2$

To define f_j, divide $\square$ into 2^j lines horizontally and 2^{m-j} lines vertically to form a rectangular grid:



$n=4,\ m=3,\ j=2$

To define f_j, divide $\square$ into 2^j lines horizontally and 2^{m-j} lines vertically to form a rectangular grid:

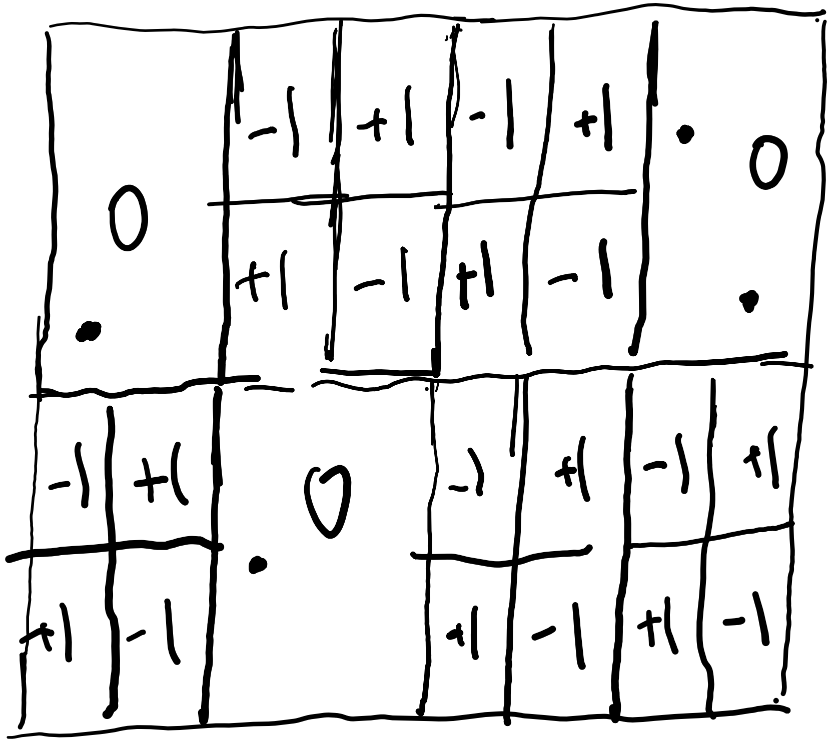


$n=4$, $m=3$, $j=2$

Main property: for any $i<j$,

$$\int f_i f_j = 0$$

To define f_j, divide $\square$ into 2^j lines horizontally and 2^{m-j} lines vertically to form a rectangular grid:



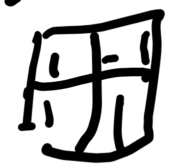
$n=4,\ m=3,\ j=2$

Main property: for any $i<j$,

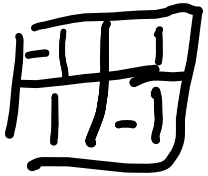
$\int f_i f_j = 0$

<u>Sketch</u>: i & j gives rise to diff. tilings of $[0,1]^2$. Ignore cells that have pts of P. Otherwise, we obtain a $2^j \times 2^{m-i}$ grid whose cells look like

mesh.



or



Define $F = f_0 + \dots + f_m$. So,

$$\int F^2 = \sum_{i,j=0}^{m} \int f_i f_j = \sum_{i=0}^{m} \int f_i^2 \leq \sum_{i=0}^{m} 1 \leq \log n + 2$$

Define $F = f_0 + \dots + f_m$. So,

$$\int F^2 = \sum_{i,j=0}^{m} \int f_i f_j = \sum_{i=0}^{m} \int f_i^2 \leq \sum_{i=0}^{m} 1 \leq \log n + 2$$

$\Rightarrow$ the 'denominator' $\sqrt{\int F^2} \leq O(\sqrt{\log n})$.

Define $F = f_0 + \dots + f_m$. So,

$$\int F^2 = \sum_{i,j=0}^{m} \int f_i f_j = \sum_{i=0}^{m} \int f_i^2 \leq \sum_{i=0}^{m} 1 \leq \log n + 2$$

$\Rightarrow$ the 'denominator' $\sqrt{\int f^2} \leq O(\sqrt{\log n})$.

Next, to show: $\int F D = \Omega(\log n)$.

Define $F = f_0 + \dots + f_m$. So,

$$\int F^2 = \sum_{i,j=0}^{m} \int f_i f_j = \sum_{i=0}^{m} \int f_i^2 \leq \sum_{i=0}^{m} 1 \leq \log n + 2$$

$\Rightarrow$ the 'denominator' $\sqrt{\int F^2} \leq O(\sqrt{\log n})$.

Next, to show: $\int F D = \Omega(\log n)$.

ie. want to bound $\int f_j D >$ some const > 0.

Define $F = f_0 + \dots + f_m$. So,

$$\int F^2 = \sum_{i,j=0}^{m} \int f_i f_j = \sum_{i=0}^{m} \int f_i^2 \leq \sum_{i=0}^{m} 1 \leq \log n + 2$$

$\Rightarrow$ the 'denominator' $\sqrt{\int F^2} \leq O(\sqrt{\log n})$.

Next, to show: $\int F D = \Omega(\log n)$.

ie. want to bound $\int f_j D >$ some const > 0.

Consider a rectangle R in the tiling of f_j

Define $F = f_0 + \dots + f_m$. So,

$$\int F^2 = \sum_{i,j=0}^{m} \int f_i f_j = \sum_{i=0}^{m} \int f_i^2 \leq \sum_{i=0}^{m} 1 \leq \log n + 2$$

$\Rightarrow$ the 'denominator' $\sqrt{\int F^2} \leq O(\sqrt{\log n})$.

Next, to show: $\int F D = \Omega(\log n)$.

i.e. want to bound $\int f_j D >$ some const > 0.

Consider a rectangle R in the tiling of f_j.

Only care about empty rectangles - and there are at least n of them (as m was chosen s.t. $2^m \geq 2n$).

Define $F = f_0 + \dots + f_m$. So,

$$\int F^2 = \sum_{i,j=0}^{m} \int f_i f_j = \sum_{i=0}^{m} \int f_i^2 \leq \sum_{i=0}^{m} 1 \leq \log n + 2$$

$\Rightarrow$ the 'denominator' $\sqrt{\int F^2} \leq O(\sqrt{\log n})$.

Next, to show: $\int F D = \Omega(\log n)$.

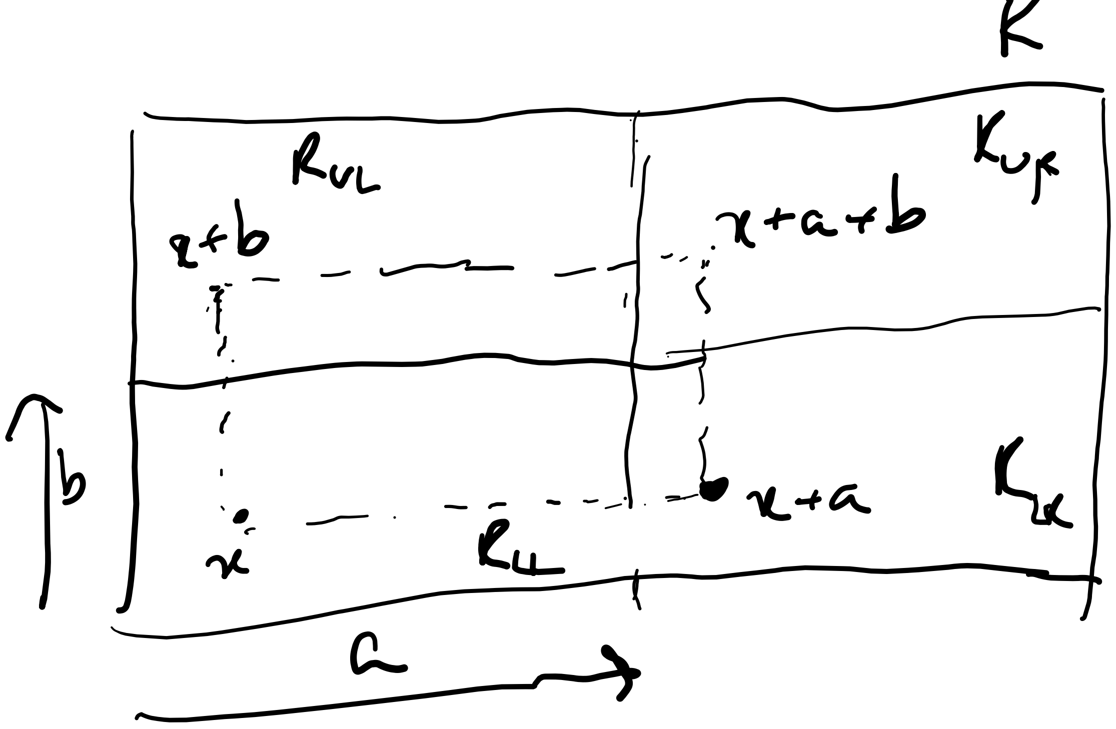
i.e. want to bound $\int f_j D >$ some const > 0.

Consider a rectangle R in the tiling of f_j.

Only care about empty rectangles - and there are at least n of them (as m was chosen s.t. $2^m \geq 2n$).

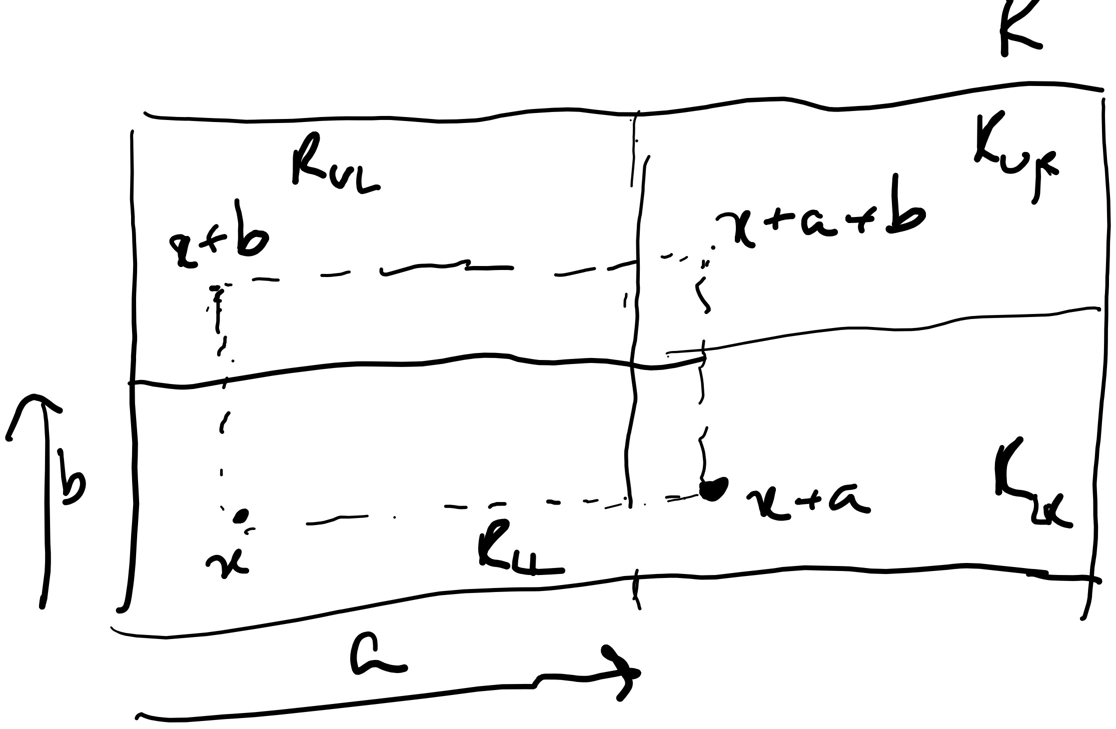
Suffices to show $\int_R f_j D = \Omega(1/n)$

So, consider a fixed, empty rectangle



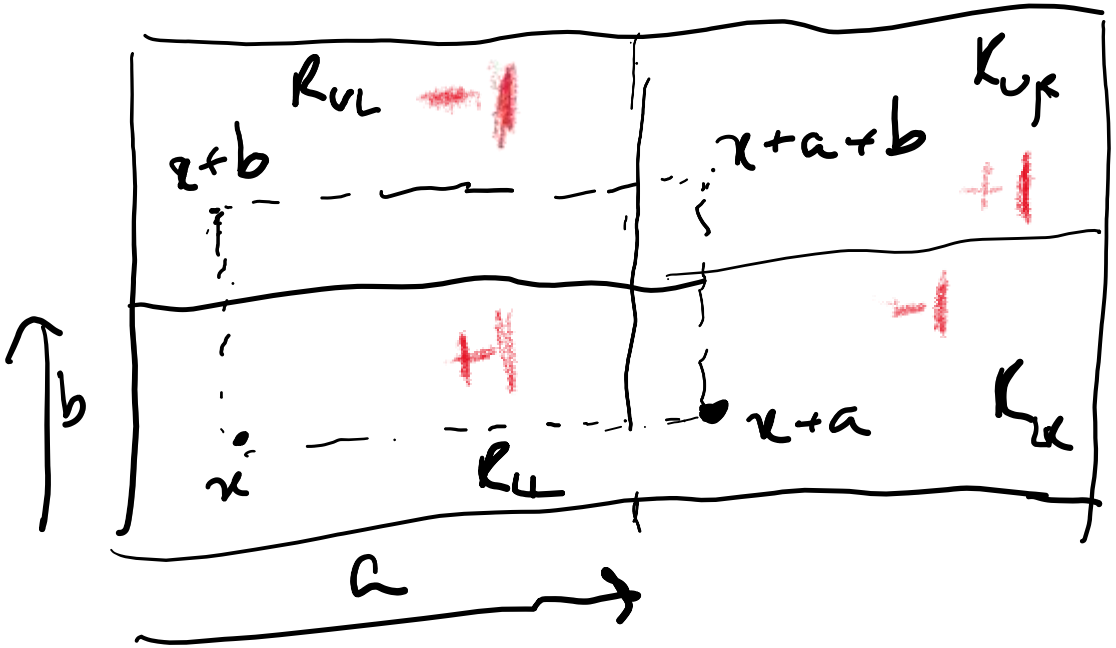
So, consider a fixed, empty rectangle

Note $a = \frac{1}{2^{j+1}}$, $b = \frac{1}{2^{m-j+1}}$

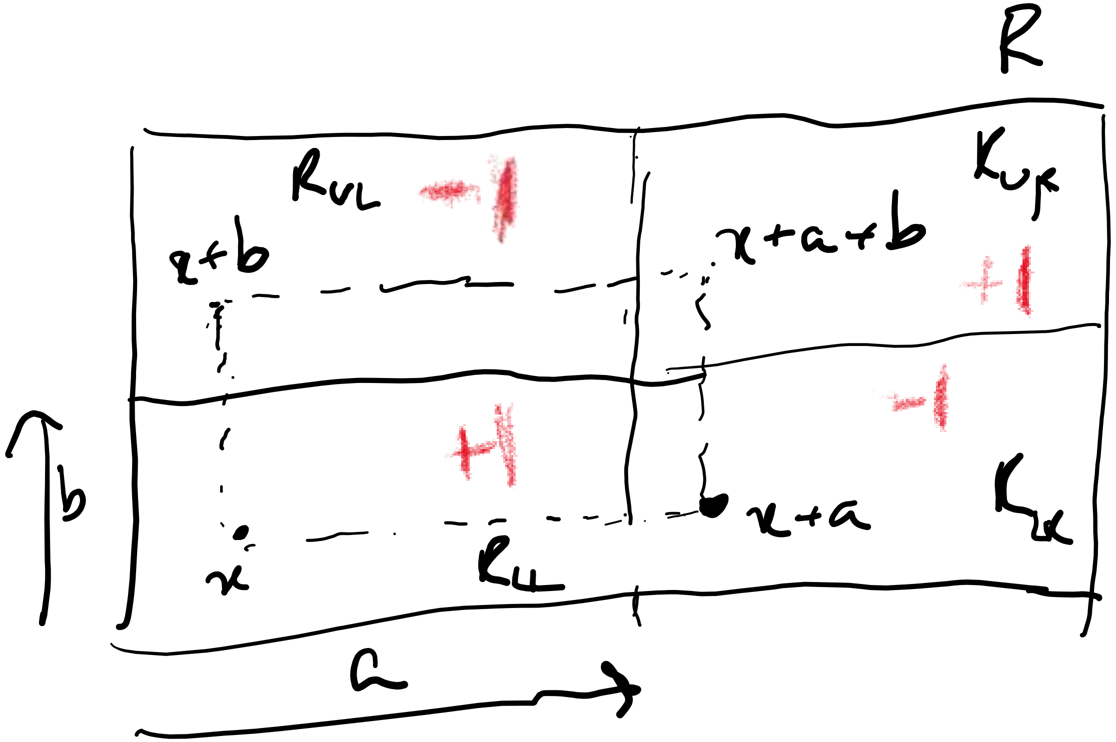


So, consider a fixed, empty rectangle R

Note $a = \frac{1}{2^{j+1}}$, $b = \frac{1}{2^{m-j+1}}$



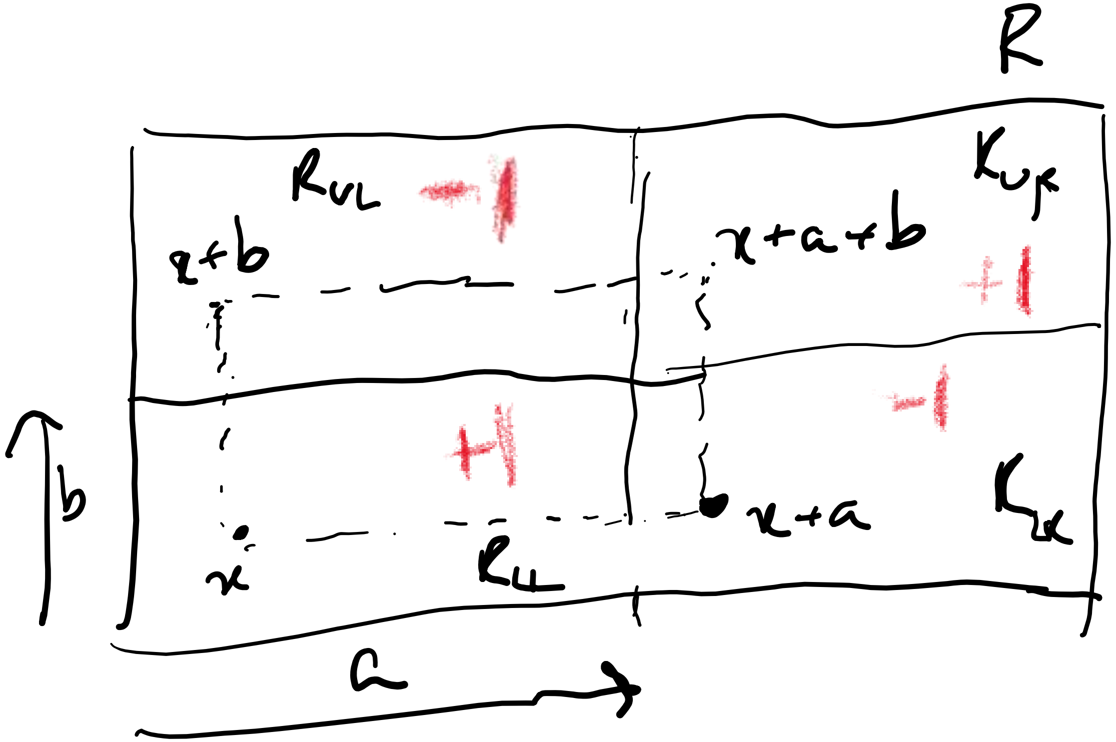
So, consider a fixed, empty rectangle



Note $a = \frac{1}{2^{j+1}}$, $b = \frac{1}{2^{m-j+1}}$

$$\int_R f_j D = \int_{R_{LL}} D - \int_{R_{LR}} D - \int_{R_{UL}} D + \int_{R_{UR}} D$$

So, consider a fixed, empty rectangle

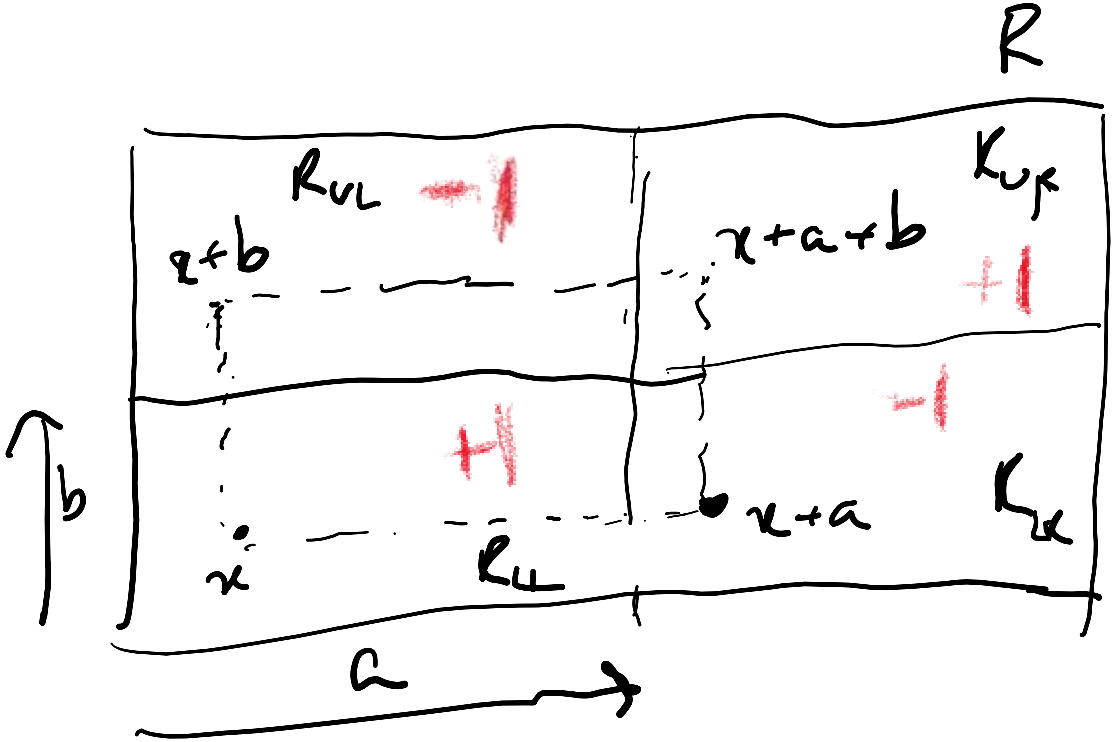


Note $a = \frac{1}{2^{j+1}}$, $b = \frac{1}{2^{m-j+1}}$

$$\int_R f_j D = \int_{R_{LL}} D - \int_{R_{LR}} D - \int_{R_{UL}} D + \int_{R_{UR}} D$$

$$= \int_{R_{LL}} \Big[D(x) - D(x+a) - D(x+b) + D(x+a+b) \Big] dx$$

So, consider a fixed, empty rectangle



Note $a = \frac{1}{2^{j+1}}$, $b = \frac{1}{2^{m-j+1}}$

$$\int_R f_j D = \int_{R_{LL}} D - \int_{R_{LR}} D - \int_{R_{UL}} D + \int_{R_{UR}} D$$

$$= \int_{R_{LL}} \Big[D(x) - D(x+a) - D(x+b) + D(x+a+b) \Big] dx$$

$$n\big(\mathrm{vol}(C_x) - \mathrm{vol}(C_{x+a}) - \mathrm{vol}(C_{x+b}) + \mathrm{vol}(C_{x+a+b})\big) -$$

$$\big(|P \cap C_x| - |P \cap C_{x+a}| - |P \cap C_{x+a}| + |P \cap C_{x+a+b}|\big)$$

$$n\left(\mathrm{vol}(C_x) - \mathrm{vol}(C_{x+a}) - \mathrm{vol}(C_{x+b}) + \mathrm{vol}(C_{x+a+b})\right) -$$

$$\left(|P \cap C_x| - |P \cap C_{x+a}| - |P \cap C_{x+a}| + |P \cap C_{x+a+b}|\right)$$

$$n\left(\mathrm{vol}(C_x) - \mathrm{vol}(C_{x+a}) - \mathrm{vol}(C_{x+b}) + \mathrm{vol}(C_{x+a+b})\right) -$$

$$\left(|P \cap C_x| - |P \cap C_{x+a}| - |P \cap C_{x+a}| + |P \cap C_{x+a+b}|\right)$$

First term is just the area of dotted rect.

$$= \mathrm{area}(R_{22}) = ab = \frac{1}{2^{m+2}}$$

$$n\left(\mathrm{vol}(C_x) - \mathrm{vol}(C_{x+a}) - \mathrm{vol}(C_{x+b}) + \mathrm{vol}(C_{x+a+b})\right) -$$

$$\left(|P \cap C_x| - |P \cap C_{x+a}| - |P \cap C_{x+a}| + |P \cap C_{x+a+b}|\right)$$

First term is just the area of dotted rect.

$= \mathrm{area}(R_{22}) = ab = \frac{1}{2^{m+2}}$

a pt in P contributes iff it lies within dotted $\square$

$$n\left(\mathrm{vol}(C_x) - \mathrm{vol}(C_{x+a}) - \mathrm{vol}(C_{x+b}) + \mathrm{vol}(C_{x+a+b})\right) -$$

$$\left(|P \cap C_x| - |P \cap C_{x+a}| - |P \cap C_{x+a}| + |P \cap C_{x+a+b}|\right)$$

First term is just the area of dotted rect.

$= \mathrm{area}(R_{LL}) = ab = \frac{1}{2^{m+2}}$

a pt in P contributes iff it lies within dotted $\square$

But it is empty!

<u>Conclusion:</u>

$$\int_R f_j D = \int_{R_{22}} n \cdot \mathrm{vol}(R_{22}) = n \cdot \mathrm{vol}(R_{22})^2$$

$$= n\,(ab)^2 = \frac{n}{2^{2m+2}} =$$

$$\Omega\left(\frac{1}{n}\right).$$

Done!

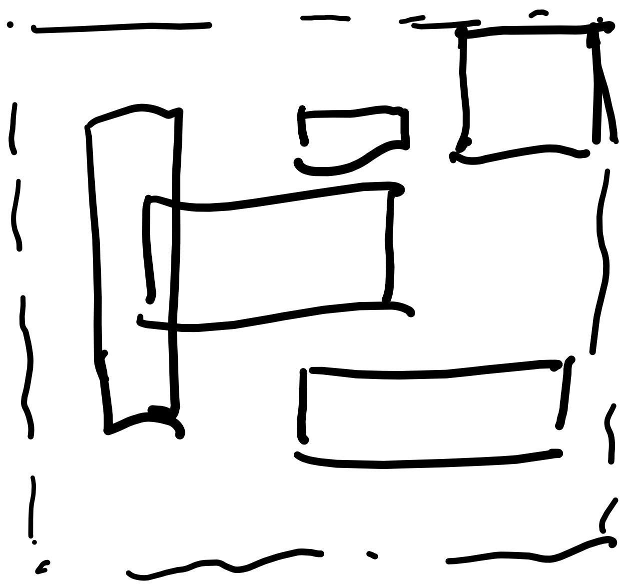
Discrepancy Functions of Various Geometric Shapes

Discrepancy Functions of Various Geometric Shapes

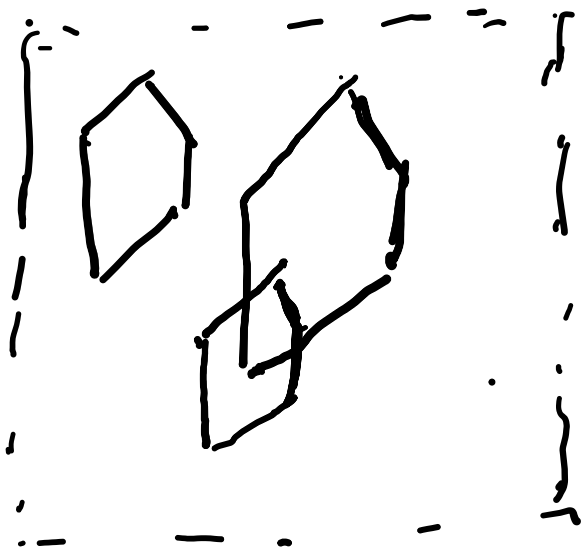
If the class consists of scaled & translated copies of a fixed polytope (eg. $\mathcal{R}_d$):

Discrepancy Functions of Various Geometric Shapes

If the class consists of scaled & translated copies of a fixed polytope (eg. $\mathcal{R}_d$):

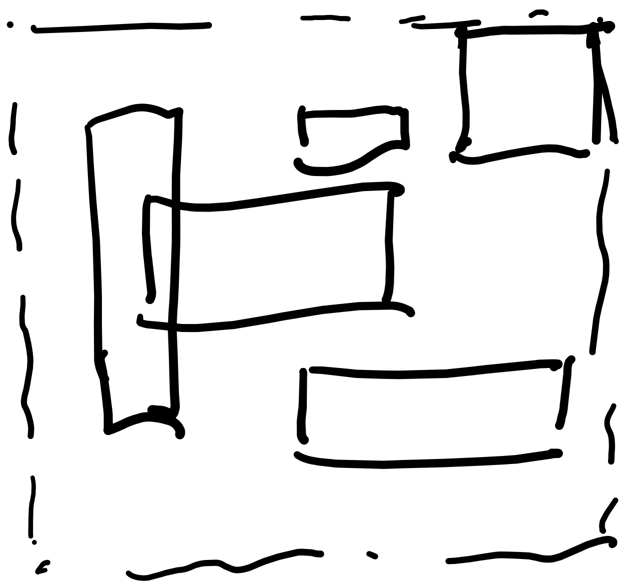


or

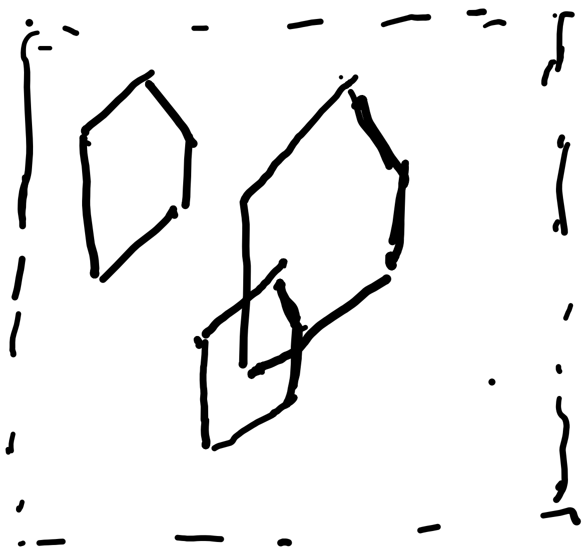


Discrepancy Functions of Various Geometric Shapes

If the class consists of scaled & translated copies of a fixed polytope (eg. $\mathcal{R}_d$):



or



discrepancy about polylog n)

Discrepancy Functions of Various Geometric Shapes

But for rotationally invariant classes (eg. halfspaces or boxes but in arb. rotated positions), or for objects with a smooth boundary. (discs/ellipsoids):

Discrepancy Functions of Various Geometric Shapes

But for rotationally invariant classes (eg. halfspaces or boxes but in arb. rotated positions), or for classes with a smooth boundary. (discs/ellipsoids) (only translated though):

Discrepancy Functions of Various Geometric Shapes

But for rotationally invariant classes (eg. halfspaces or boxes but in arb. rotated positions), or for classes with a smooth boundary. (discs/ellipsoids) (only translated though):

disc. is $n^{1/4}$. or n^{ε}.

Rule of Thumb: look at the bdry!

Rule of Thumb: Look at the bdry!

Thank You!