Theory Student Seminar Trivia 2.0

November 30 2022

1 True/False/Open

1. Consider the debt-repayment problem. There are n individuals, who owe each other some W_{ij} money, depicted by a weighted directed edge. There is a polynomial time algorithm (in total input size) to determine the minimal number of transactions required to settle all debts.

2. Let X_1, X_2, X_3 be three independent uniformly random variables over [0,1]. The mean of the three has lower variance than the median of these three variables.

3. There exists some k such that the following is true for any n points: Suppose there are n points on the plane, some of which are blue, and some of which are red. Suppose that there exists a straight line that partitions these points into two distinct groups. Then there exists a subset of just k points, so that any line partitioning these k points also partitions the original set.

4. Let L be a regular language. There exists a k such that for any string string $s = w_1 w_2 w_3 ... w_k, s \in L, w_i \in \Sigma^*$, there exists some $i < j$ such that:

$$w_1 w_2 ... (w_i ... w_{j-1})^r w_j ... w_k \in L$$

   for all $r >= 0$.

5. The entrywise product of two convex functions from $\mathbb{R} \to \mathbb{R}$ is convex.

6. Consider the balls and bins problem, with n balls and n bins. The expected size of the largest bin is $\Theta(\log n)$.

7. Let $\boldsymbol{D}$ be a diagonal matrix with entries in $\{0, 1\}$. Consider $\boldsymbol{M}$ to be any matrix with real eigenvalues. Let $\lambda_i(\boldsymbol{A})$ be the i-th smallest eigenvalue of matrix $\boldsymbol{A}$. We have:

$$\lambda_i(\boldsymbol{DM}) \leq \lambda_i(\boldsymbol{M})$$

   for all i.

8. There exists intervals $I_1, I_2, I_3, ...,$ such that the total sum of the lengths of the intervals is less than 1, but the union of the intervals covers every single rational number in $[0, 1]$.

9. Consider the Rubiks cube. The total number of possible positions(obtained from taking the cube apart) divided by the total number of positions reachable from a solved state is 6.

10. Call a matching stable if there are no two members that prefer each other over their current pairing. Recall that a stable perfect matching always exists between two equally sized sets. True/False: A stable perfect matching always exists in an even sized set, even when all members can be matched with everyone else. (ie, if there are n members, each member constructs a preference list for the other $n-1$ members, not just the $n/2$ members in the other group.)

11. If a Markov chain has a stationary distribution, then the random walk on its graph converges to the stationary distribution.

12. The following is a permutation of authors and results. 3 of them are right.

Authors	Result
Misra-Gries	Stream frequent element counting
Valiant–Vazirani	$A + A - A - A$ has a large subspace
Bogolyubov-Ruzsa	e is transcendental
Lindemann-Weierstrass	$p_c(\mathcal{F}) \leq Kq(\mathcal{F}) \log \ell(\mathcal{F})$
Kahn-Kalai	$\text{NP} \subseteq \text{RP}^{\text{Promise-UP}}$
Sylvester–Gallai	$\exists$ line passes through 2 or all
Schwartz–Zippel	$Pr[P(r_1, r_2, ...r_n) = 0] \leq \frac{d}{\vert S\vert}$

13. Call a word square free if it does not contain a substring of the form XX. There are arbitrarily long square free words that can be constructed from a character set of size 3.

14. Alice and Bob play the following game: They begin with a positive integer n, and take turns subtracting a divisor of n that isn't 1 or n from n. The player who cannot make a move on his/her turn loses, and Alice always goes first. True/False: Of the integers from 1 to 100, Alice wins on 50 of them if they are the starting number.

15. There exists some $n > 4$ such that there exists n points on the 3 dimensional sphere that are all equidistant from one another.

16. There exists 6 points on the plane, so that the distance between any two is an integer, and no three are collinear.

17. There exists 3 non-zero integers a, b, c such that if placed as the coefficients of a quadratic in any order, it will have at least one integer root.

18. Let G be a graph such that each vertex has degree at least 3. Then it contains a simple cycle whose length is a power of 2.