Degree vs Approximate Degree

$$f : \{0,1\}^n \to \{0,1\}$$

$f = OR_2$: $\quad f(x) = x_1 + x_2 - x_1 x_2$

unique representation $\quad \deg(f) = 2$

$$g(x) := 0.3 + 0.4(x_1 + x_2)$$

$\forall x \quad g(x) \in f(x) \pm 0.3$ $\qquad \widetilde{\deg}_{0.3}(f) \leq 1$

$\deg(OR_n) = n \qquad \widetilde{\deg}_{0.3}(OR_n) \leq O(\sqrt{n})$

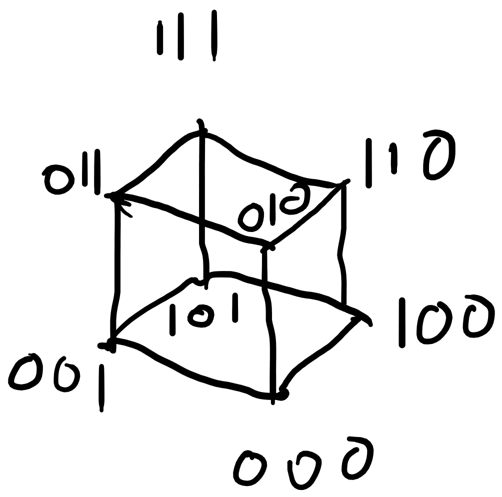
Using $\{1,-1\}^n$ instead of $\{0,1\}^n$

$$f(x) = \frac{1-x_1}{2} + \frac{1-x_2}{2} - \left(\frac{1-x_1}{2}\right)\left(\frac{1-x_2}{2}\right)$$

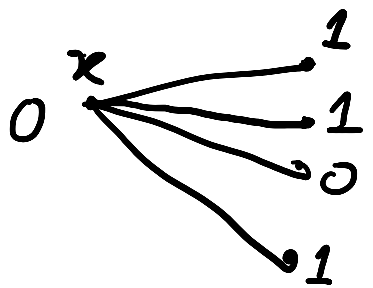
Same polynomial behaviour, degree does not change

Degree vs Approximate Degree

$f : \{0,1\}^n \to \{0,1\}$



Sensitivity of f:

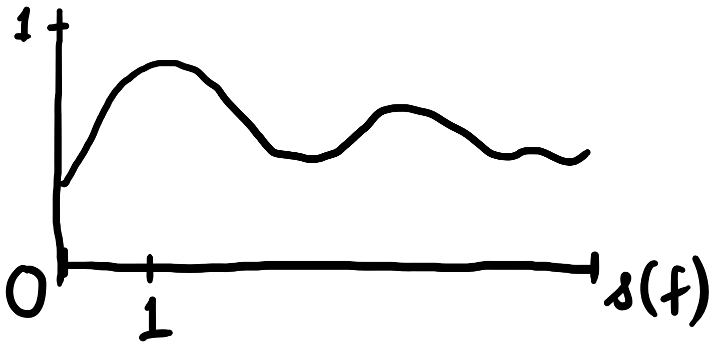


$s(f,x) = 3$

$s(f) = \max_x s(f,x)$

Brilliant topic for a TSS

Slope $\Omega(1) \Rightarrow$ degree $> \Omega(\sqrt{s})$

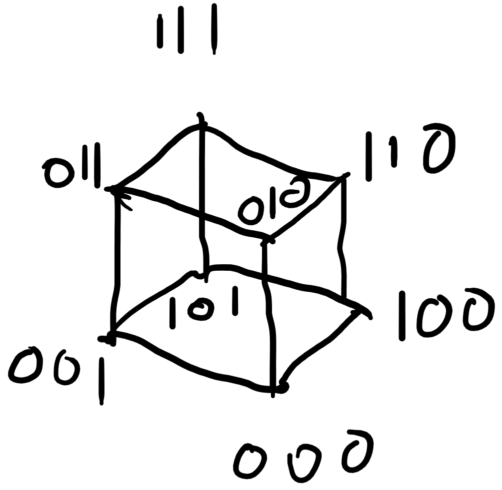


Polynomials computing f have to change value a lot

$\Downarrow$

$\exists$ univariate polynomial of the same degree that changes rapidly

Degree vs Approximate Degree



$f:\{0,1\}^n \to \{0,1\}$

[Nisan Szegedy 92]

$$\widetilde{\deg}_{1/3}(f) \geq \Omega\left(\sqrt{s(f)}\right)$$

$$\cdots \quad \widetilde{\deg}_{1/3}(f) \geq \sqrt[6]{\deg(f)}$$

Sensitivity Conjecture: Is $s(f) \geq \sqrt[c]{\deg(f)}$?

[Gotsman Linial] In any unbalanced R/B col. of $\{0,1\}^n$, max monochromatic degree $\geq \sqrt[c]{n}$?

[Huang 19]

$$s(f) \geq \sqrt{\deg(f)} \qquad \left[\Rightarrow \widetilde{\deg}(f) \geq \sqrt[4]{\deg(f)}\right]$$

[Aaronson Ben-David Kothari Rao Tal 20]

[Huang] $\|S_f\| \geq \sqrt{\deg(f)}$

$$\|S_f\| \leq \frac{1}{1-2\epsilon}\, \widetilde{\deg}_{\epsilon}(f)$$

$$\begin{array}{ccc} & \overbrace{\hphantom{\quad\quad\quad\quad}}^{2^n} & \\ 2^n \left\{ \vphantom{\begin{bmatrix} \\ \\ \\ \end{bmatrix}} \right. & \begin{bmatrix} & & \\ & \square & \\ & & \end{bmatrix} & - x \\ & y & \end{array}$$

$S_f[x,y] :=$
1 if $x — y$
and $f(x) \neq f(y)$

A_f

$$\begin{array}{cc} x & \begin{bmatrix} & & \\ & \square & \\ & & \end{bmatrix} \\ & \quad y \end{array}$$

$$A_f[x,y] := \begin{cases} f(x)-f(y) & \text{if } x - y \\ 0 & \text{o/w} \end{cases}$$

$$\begin{array}{cc} f^{-1}(0) & \multirow{2}{*}{$\left[\begin{array}{c|c} 0 & -A_f'^{T} \\ \hline A_f' & 0 \end{array}\right]$} \\ f^{-1}(1) & \end{array}$$

$$\|A_f\| = \|A_f'\|$$

Proof layout for $\|A_f\| \leq \frac{1}{1-2\epsilon} \widetilde{\deg}_\epsilon(f)$

- Move from values of f to values of g

- Move from values of g to coefficients of g.

- Play around with operator norm.

A_g

$$\begin{array}{cc} x & \begin{bmatrix} & \square & \\ & & \end{bmatrix} \\ & y \end{array}$$

$$A_g[x,y] := g(x) - g(y) \quad \text{if } x - y$$

$$\begin{array}{cc} f^{-1}(0) & \left[\begin{array}{c|c} [-2\epsilon, 2\epsilon] & [1-2\epsilon, 1+2\epsilon] \\ \hline [1-2\epsilon, 1+2\epsilon] \;\; A_g' & [2\epsilon, 2\epsilon] \end{array}\right] \\ f^{-1}(1) & \end{array}$$

$$\|A_g\| \geqslant \|A_g'\|$$

$$\|A_g'\| \geqslant (1-2\epsilon)\, \|A_f'\|$$

Will show $\|A_g\| \leq (1+\epsilon)\deg(g)$

A_g

$$\begin{array}{cc} x & \begin{bmatrix} & & \\ & \square & \\ & & \end{bmatrix} \\ & y \end{array} \qquad A_g[x,y] := g(x) - g(y) \quad \text{if } x \text{---} y$$

Let $X :=$ Adj matrix of hypercube, $V_g = \begin{bmatrix} g(000) & & & \\ & g(001) & & \\ & & \ddots & \\ & & & g(111) \end{bmatrix}$

$$A_g = V_g X - X V_g$$

Fourier basis

$$\chi_S(x) := \begin{cases} 1 & \text{if } x|_S \text{ has even parity} \\ -1 & \text{if } x|_S \text{ has odd parity} \end{cases}$$

Same as $\prod_{i \in S} x_i$ if $x \in \{1,-1\}^n$

Each $\chi_S \in \{1,-1\}^{2^n} \subseteq \mathbb{R}^{2^n}$

- $\langle \chi_S, \chi_S \rangle = 2^n$
- $\langle \chi_S, \chi_T \rangle = 0$
- Every $g(x) = \sum \hat{g}(S)\, \chi_S(x)$

$$\deg(g) = \max_{S:\ \hat{g}(S)\neq 0} |S|$$

$$H = \frac{1}{2^{n/2}} \begin{bmatrix} | & | & & | \\ \chi_{\emptyset} & \chi_{\{1\}} & \cdots & \chi_{[n]} \\ | & | & & | \end{bmatrix}$$

$$\begin{array}{cc} & \\ S & \begin{bmatrix} & & \\ & \boxed{\;} \to \hat{g}(S\Delta T) & \\ & & \end{bmatrix} \\ & T \end{array}$$

- $H = H^T = H^{-1}$
- $H \begin{bmatrix} v \\ a \\ l \\ u \\ e \\ s \end{bmatrix} = \begin{bmatrix} c \\ o \\ e \\ f \\ f \\ s \end{bmatrix}$
- $H \begin{bmatrix} v & & & & & \\ & a & & & & \\ & & l & & & \\ & & & u & & \\ & & & & e & \\ & & & & & s \end{bmatrix} H = \begin{bmatrix} c & o & e & f & f & s \\ o & e & c & s & f & f \\ e & f & o & f & c & s \\ c & e & f & o & s & f \end{bmatrix}$

$$A_g = V_g X - X V_g$$

$$H A_g H = \underbrace{H V_g X H} - H X V_g H$$

$$= H V_g H H X H$$

$$= C_g \; H X H$$

$$= C_g \; V_h$$

$$X = \begin{array}{cc} \begin{bmatrix} & & \\ & & \\ & \boxed{} & \end{bmatrix} & T \\ S & \end{array} \qquad \boxed{} \to 1 \text{ if } |S \Delta T| = 1$$

$$X = C_h \quad \text{where} \quad h = \sum_{i \in [n]} \chi_{\{i\}}$$

$$= n - 2 \sum x_i$$

$$= n - 2|x|$$

$$H A_g H = C_g V_h - V_h C_g$$

$$HA_gH = C_gV_h - V_hC_g$$

$$HA_gH[x,y] = C_g[x,y]\left(n-2|y| - (n-2|x|)\right)$$
$$= 2C_g[x,y]\left(|x|-|y|\right)$$

$$HA_gH = 2C_g \circ D \quad \text{where} \quad D = \begin{array}{cc} & y \\ x & \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \end{array} \longrightarrow |x|-|y|$$

Note that $\|C_g\| = \|V_g\| \leq 1+\epsilon$

How much can $\circ D$ change things?

How much can $\circ D$ change things?

a rectangle, like $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

If $D = \sum \alpha_i R_i$,

then $\|M \circ D\| \leq \sum |\alpha_i| \, \|M \circ R_i\| \leq \sum |\alpha_i| \, \|M\|$

The μ-norm:

$$\mu(D) := \min \sum |\alpha_i| \quad \text{s.t.} \quad D = \sum \alpha_i R_i$$

$$\|M \circ D\| \leq \mu(D) \, \|M\|$$

D is essentially

$$\begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 & \cdots & n \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ n \end{matrix} & \begin{bmatrix} 0 & -1 & -2 & -3 & \cdots & -n \\ 1 & 0 & -1 & -2 & \cdots & -n+1 \\ 2 & 1 & 0 & -1 & & \\ 3 & 2 & 1 & 0 & & \\ \vdots & \vdots & & & \ddots & \\ n & n-1 & & & & \end{bmatrix} \end{array}$$

$\mu(D) \geqslant n$

But $C_g[S,T] = 0$ when $|S \Delta T| > \deg(g)$ $(=d)$

Hence $= 0$ when $\big||S| - |T|\big| > d$

$$D' = \begin{bmatrix}
0 & -1 & -2 & -3 & \cdots & -(d-1) & -d & -(d-1) & \cdots & -1 & 0 & 0 & 0 & \cdots \\
1 & 0 & -1 & -2 & -3 & \cdots & -(d-1) & -d & -(d-1) & \cdots & -1 & \ddots & & \\
2 & 1 & 0 & -1 & -2 & 3 & & & & \ddots & & & \ddots & \\
3 & 2 & 1 & 0 & -1 & & & & & & \ddots & & & \\
\vdots & 3 & 2 & 1 & 0 & \ddots & & & & & & & & \\
\vdots & \vdots & 3 & & & & \ddots & & & & & & & \\
d-1 & \vdots & \vdots & & & & & \ddots & & & & & & \\
d & d-1 & \vdots & & & & & & & & & & & \\
d-1 & d & d-1 & & & & & & & & & & & \\
\vdots & d-1 & d & \ddots & & & & & & & & & & \\
1 & & & & \ddots & & & & & & & & & \\
0 & 1 & & & & \ddots & & & & & & & & \\
0 & 0 & 1 & & & & & & & & & & & \\
\vdots & & & \ddots & & & & & & & & & &
\end{bmatrix}$$

$$C_g \circ D = C_g \circ D'$$

$$\mu\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2 \qquad \mu(Id_{n\times n}) \leq 5$$

D' is the sum of $2d$ Id matrices.

$$\mu(D') \leq 10d$$

$$\|A_f\| \leq \frac{1}{1-2\epsilon}\|A_g\|$$

$$\leq \frac{1}{1-2\epsilon} 2\|C_g\| \mu(D') \leq \frac{20(1+\epsilon)}{1-2\epsilon}\widetilde{\deg}_\epsilon(f)$$

$$\mu\left(Id_{n\times n}\right) \le 5$$

$$M_S = \begin{array}{c|c} & \begin{array}{cc} |y|_S| \text{ even} & |y|_S| \text{ odd} \end{array} \\ \begin{array}{c} |x|_S| \text{ even} \\ \\ |x|_S| \text{ odd} \end{array} & \left[\begin{array}{c:c} 1 \; 1 \; \cdots \; 1 & 0 \\ \hdashline 0 & 1 \; 1 \; \cdots \; 1 \end{array}\right] \end{array} \qquad \mu(M_s) = 2$$

$$M := \frac{1}{2^n} \sum_S M_S = \begin{bmatrix} 1 & & & .5 \\ & 1 & & \\ & & \ddots & \\ .5 & & & 1 \end{bmatrix} \qquad \mu(M) \le 2$$

$$Id = 2M - J \qquad \mu(Id) \le 5$$

D' is the sum of $2d$ Id matrices.

$$\begin{bmatrix} 1 & 1 & 1 & & & \\ 1 & 1 & 1 & & & \\ 1 & 1 & 1 & & & \\ & & & 1 & 1 & 1 \\ & & & 1 & 1 & 1 \\ & & & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & & & & & \\ & 1 & 1 & 1 & & \\ & 1 & 1 & 1 & & \\ & 1 & 1 & 1 & & \\ & & & & 1 & 1 \\ & & & & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & & & & \\ 1 & 1 & & & & \\ & & 1 & 1 & 1 & \\ & & 1 & 1 & 1 & \\ & & 1 & 1 & 1 & \\ & & & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 1 & & & \\ 2 & 3 & 2 & 1 & & \\ 1 & 2 & 3 & 2 & 1 & \\ & 1 & 2 & 3 & 2 & 1 \\ & & 1 & 2 & 3 & 2 \\ & & & 1 & 2 & 3 \end{bmatrix}$$