PRGs For Two-party Communication Protocols

TSS

19/04/23

Outline of The Talk.

- Majority of the time: motivating the theory of unconditional pseudo-randomness
- Something about communication protocols
- Something about expander graphs
- Main Result (sketch)

* All the credit to recent survey by Hatami-Hoza (ECCC '23)

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- What to do when limited supply of random bits?
- A PRG uses small amt. of 'true' randomness (aka 'seed') - generates long seq. appearing random.
- 'Observer' - to be modelled as a function.

Tooling.

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> PRG's job: generate such an X with few truly random bits.

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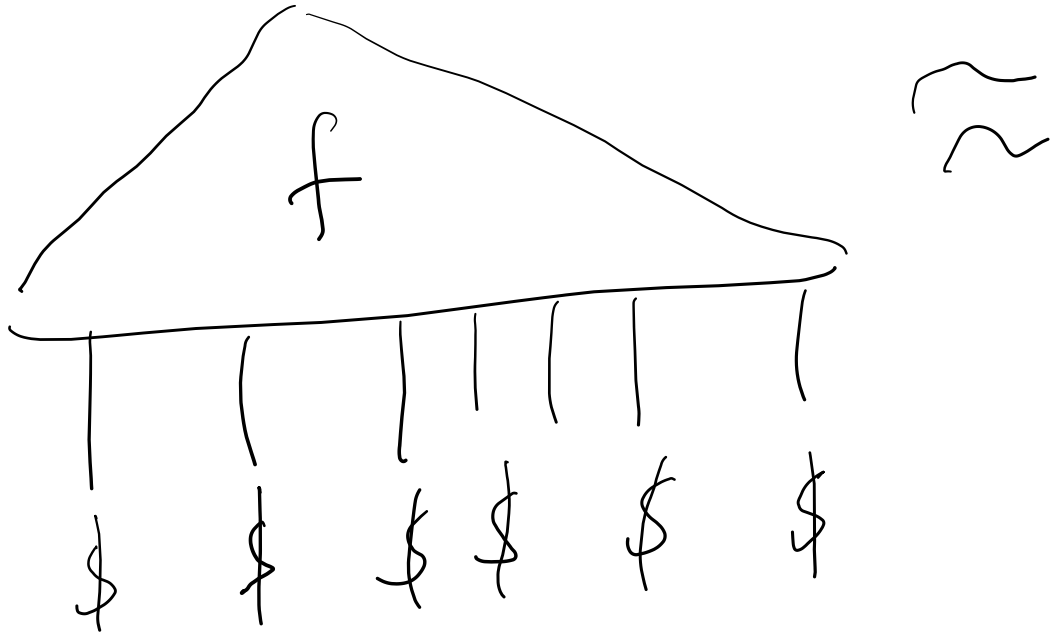
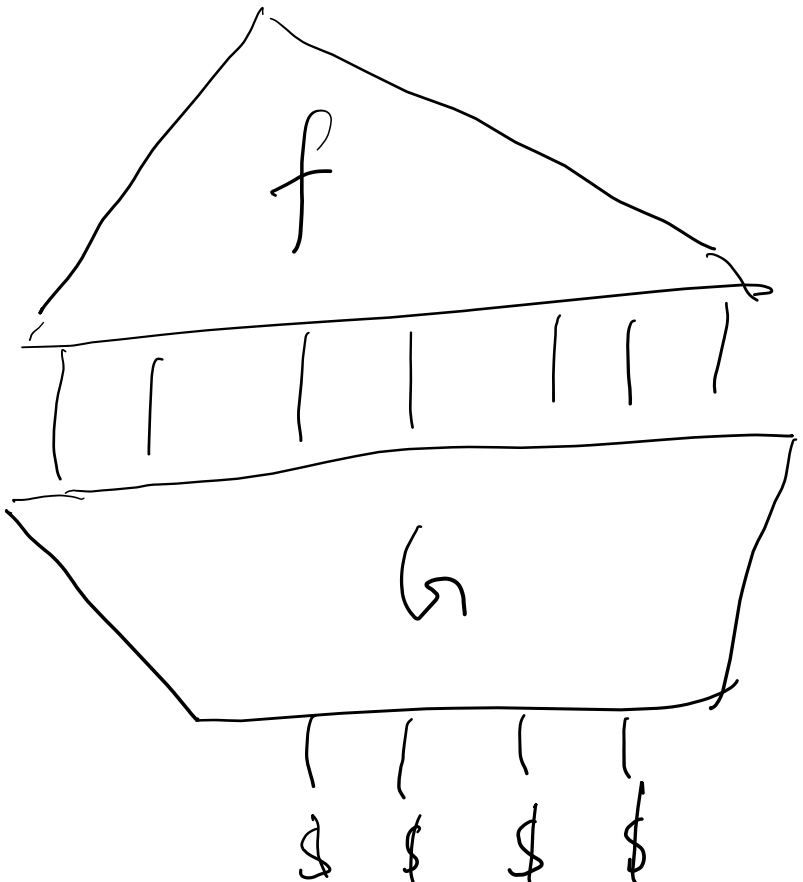


$\approx$



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Clm: Let $G: \{0,1\}^s \to \{0,1\}^n$, $s < n$. $\exists f: \{0,1\}^n \to \{0,1\}$ s.t. G does not 0.49-fool f.

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Pf: f: ind. fn. of img. of G. Then $\mathbb{E}[f(G(U_s))] = 1$, but

$\mathbb{E}[f] \leq 1/2$ as $s < n$.

$\left(|\{0,1\}^s| \text{ as a fraction of } |\{0,1\}^n| \right)$

(i.e. $f = \chi_{G(U_s)}$)

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So, back to: which observers shall we fool? 3 paradigms:

1. Everyday non-adversarial applications
2. All efficient observers
3. 'Restricted' models of computation

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↳ So in practice, this is only one of the multiple components of a practical random system.

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> But these are all heuristics with no firm mathematical guarantee. Unsatisfactory for us theoreticians!

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> Us in theory: success of practical PRGs largely a mystery.

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> Example (BBSB6). Pick $X_0 \in \{0, 1, \ldots, M-1\}$ rar.

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> Belief: Fools all poly-time algorithms!

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> Conditional results exist: for e.g., BBS has it assuming 'quadratic residuosity' is hard. Other examples - - -
> Genuine room for doubt whether known PRGs work, and even if they do, don't know why they work.
> Conditional proofs are partial explanations at best.

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- when u_1, u_2 are uar, the 3 o/p bits are correlated but any 2 of the bits are random, indep.

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> Unconditional PRGs are invaluable within theory of computing (in particular in attacking: L vs RL

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Pf: Pick $G: \{0,1\}^\Delta \to \{0,1\}^n$ uar (Δ: TBD). Notice
for any $f \in \mathcal{Y}$, for any seed y, $\mathop{\mathbb{E}}_{G}[f(G(y))] = \mathop{\mathbb{E}}_{U_n}[f(U_n)]$.

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$\Rightarrow f(G(y))$ are indep, so by Chernoff, $\Pr_G\left[\left|\mathbb{E}[f] - 2^{-\Delta}\sum_{y\in\{0,1\}^\Delta} f(G(y))\right| > \varepsilon\right]$
$\leq 2e^{-2\varepsilon^2 \cdot 2^\Delta}$

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Defⁿ (explicitness) A PRG $G: \{0,1\}^s \to \{0,1\}^n$ is explicit if it can be computed in poly(n) time.

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— assuming some complexity theoretic conjectures.

Practice: Triangle Inequality for PRGs.

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Lemma: Let $f_1, \dots, f_k : \{0,1\}^n \to \mathbb{R}$ be fns, let $\lambda_0, \dots, \lambda_k \in \mathbb{R}$ and let $f(x) = \lambda_0 + \sum_{i=1}^{k} \lambda_i f_i(x)$. Let X be a dist. over $\{0,1\}^n$ and assume X fools f_i with error ε_i. Then X fools f with error $\varepsilon = \sum_{1}^{k} |\lambda_i| \cdot \varepsilon_i$.

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Pf:

$$\left| \mathbb{E}[f(x)] - \mathbb{E}[f] \right| = \left| \sum_{1}^{k} \lambda_i \mathbb{E}[f_i(x)] - \sum_{1}^{k} \lambda_i \mathbb{E}[f_i] \right|$$

$$\leq \sum_{1}^{k} |\lambda_i| \left| \mathbb{E}[f_i(x)] - \mathbb{E}[f_i] \right|$$

$$\leq \sum |\lambda_i| \cdot \varepsilon_i$$

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> In a protocol of cost m, after m rounds of communication, the protocol terminates and both parties output the same bit $F(x, y)$.

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Thm (INW94) For every $n, m \in \mathbb{N}$, $\forall \varepsilon > 0$, $\exists$ ε-PRG for 2-party protocols on n bits with communication cost m with seed length $\frac{n}{2} + O(m + \log(1/\varepsilon))$.

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→ optimal!

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Def^n G: regular undirected graph with transitional probability matrix $M \in [0,1]^{N \times N}$. Let the e.v.s of M be $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. Say G is an ε-spectral expander if $|\lambda_i| \leq \varepsilon \ \forall i = 2, \cdots, N$.

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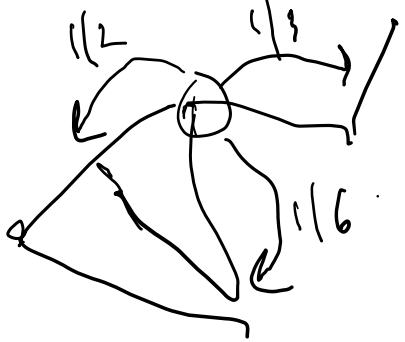
G: 

+ so on.

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Lemma n : even. $f : \{0,1\}^n \to \mathbb{R}$ of the form $f(x,y) = g(x) \cdot h(y)$ where $g, h : \{0,1\}^{n/2} \to \mathbb{R}$ satisfy.

$\mathrm{Var}(g) \leq 1$, $\mathrm{Var}(h) \leq 1$.

PRG Characterization of expander graphs.

Lemma $n \in \mathbb{N}$. $\forall f: \{0,1\}^n \to \mathbb{R}$ of the form $f(x,y) = g(x) \cdot h(y)$ where $g, h: \{0,1\}^{n/2} \to \mathbb{R}$ satisfy. $\mathrm{Var}(g) \leq 1$, $\mathrm{Var}(h) \leq 1$. G: regular undirected graph on $\{0,1\}^{n/2}$, sample a uar vertex X in G, sample a uar neighbour Y of X.

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Lemma $n \in \mathbb{N}$. $\mathcal{F}: f: \{0,1\}^n \to \mathbb{R}$ of the form $f(x,y) = g(x) \cdot h(y)$ where $g, h: \{0,1\}^{n/2} \to \mathbb{R}$ satisfy. $\mathrm{Var}(g) \le 1$, $\mathrm{Var}(h) \le 1$. G: regular undirected graph on $\{0,1\}^n$, sample a uar vertex X in G, sample a uar neighbour Y of X. Then $\forall \varepsilon > 0$,

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distribution (X, Y) fools $\mathcal{F}$ with error ε.

PRG Characterization of expander graphs.

Lemma ("combinatorial rectangles") $n \in \mathbb{N}$. Any $f: \{0,1\}^n \to \mathbb{R}$ of the form $f(x,y) = g(x) \cdot h(y)$ where $g, h: \{0,1\}^{n/2} \to \mathbb{R}$ satisfy $\mathrm{Var}(g) \le 1$, $\mathrm{Var}(h) \le 1$. G: regular undirected graph on $\{0,1\}^{n/2}$, sample a uar vertex X in G, sample a uar neighbour Y of X. Then $\forall \varepsilon > 0$,

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Proof: Think of fns. from $\{0,1\}^{n/2} \to \mathbb{R}$ as long vectors with the inner product $\langle g, h \rangle = \mathbb{E}_{v \sim U_{n/2}} [g(v) \cdot h(v)]$ & the norm $\|g\| = \sqrt{\langle g, g \rangle}$.

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Moreover, $\mathrm{Var}[\bar{g}] = \mathrm{Var}[g] \le 1$ & $\mathrm{Var}[\bar{h}] = \mathrm{Var}[h] \le 1$, so,

$$|\mathbb{E}[f(X,Y) - \mathbb{E}[f]]| = |\mathbb{E}[\bar{g}(X) \cdot \bar{h}(Y)]| \le \varepsilon.$$

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Cor: $\exists$ explicit ε-PRG for combinatorial rectangle on n bits, with seed length $\frac{n}{2} + O(\log(1/\varepsilon))$

Proof of PRGs for 2-party Communication Protocols

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Idea: Any $f(x,y)$ computed by a cost m protocol, can be "broken up" into 2^m - many combinatorial rectangles. Then just use prev. result and the Δ-inequality for PRGs.

Combinatorial Rectangles in Communication

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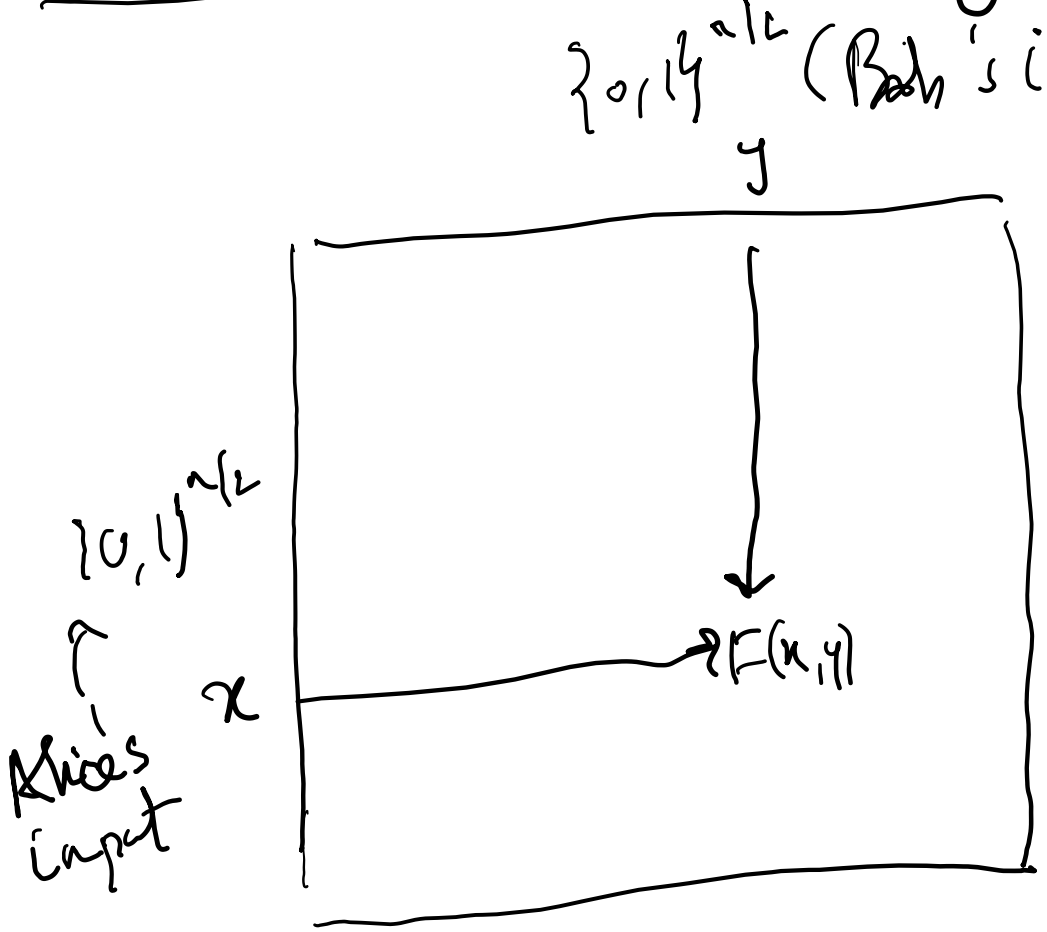

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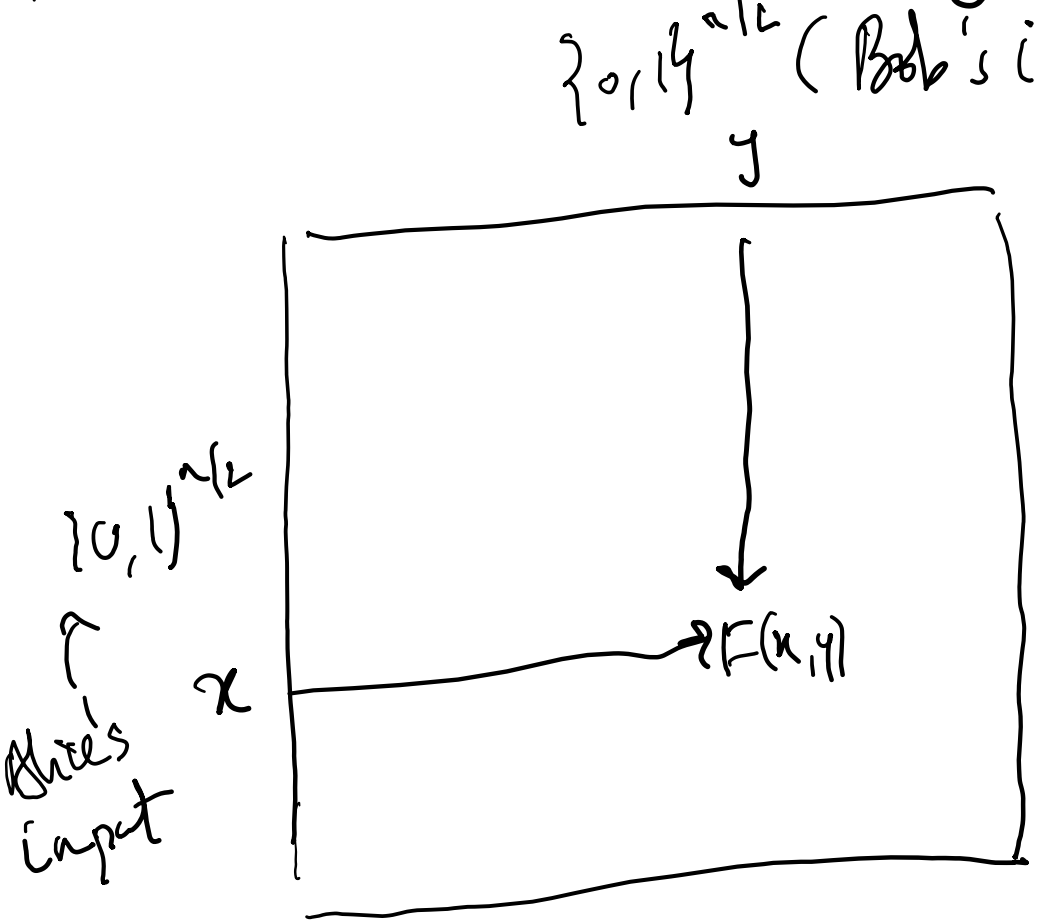
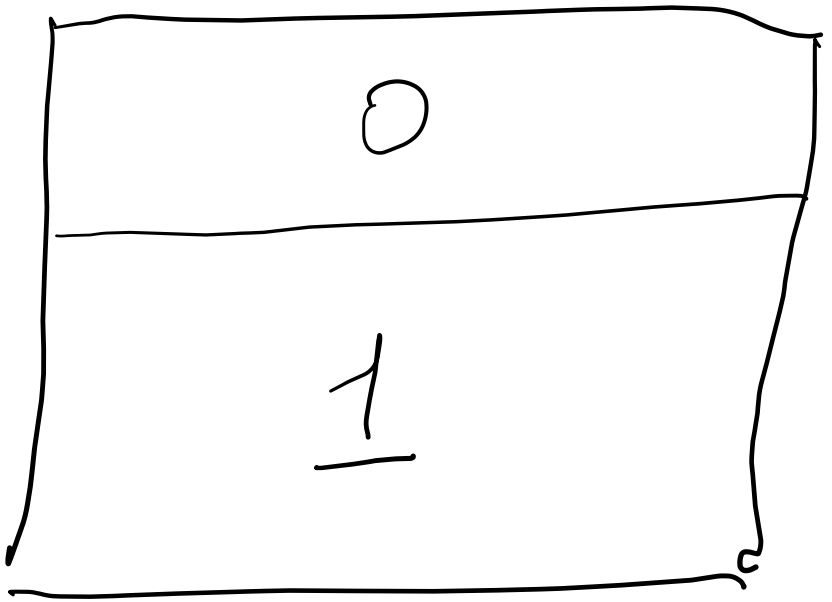
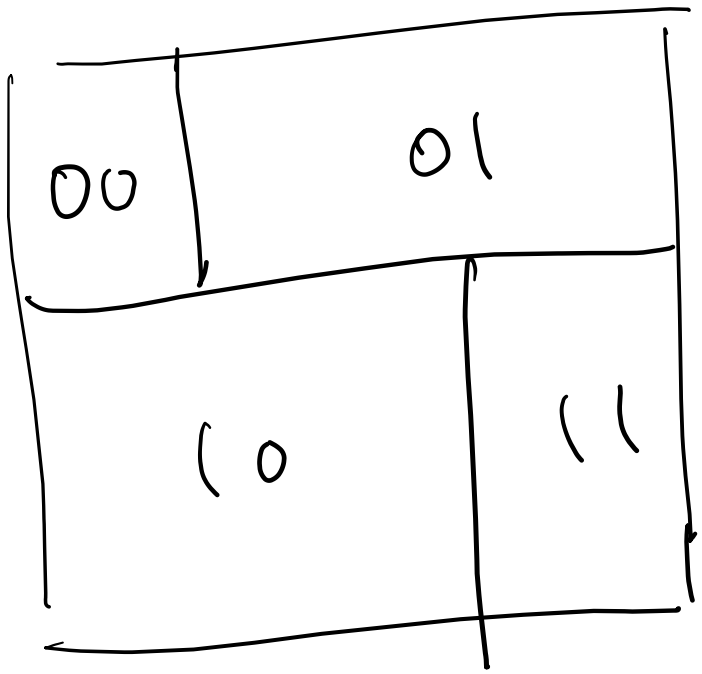


Now, if Π halts after 1 rd, then necessarily:



If TT stops after two rds, then.



and so on...

Thank You!