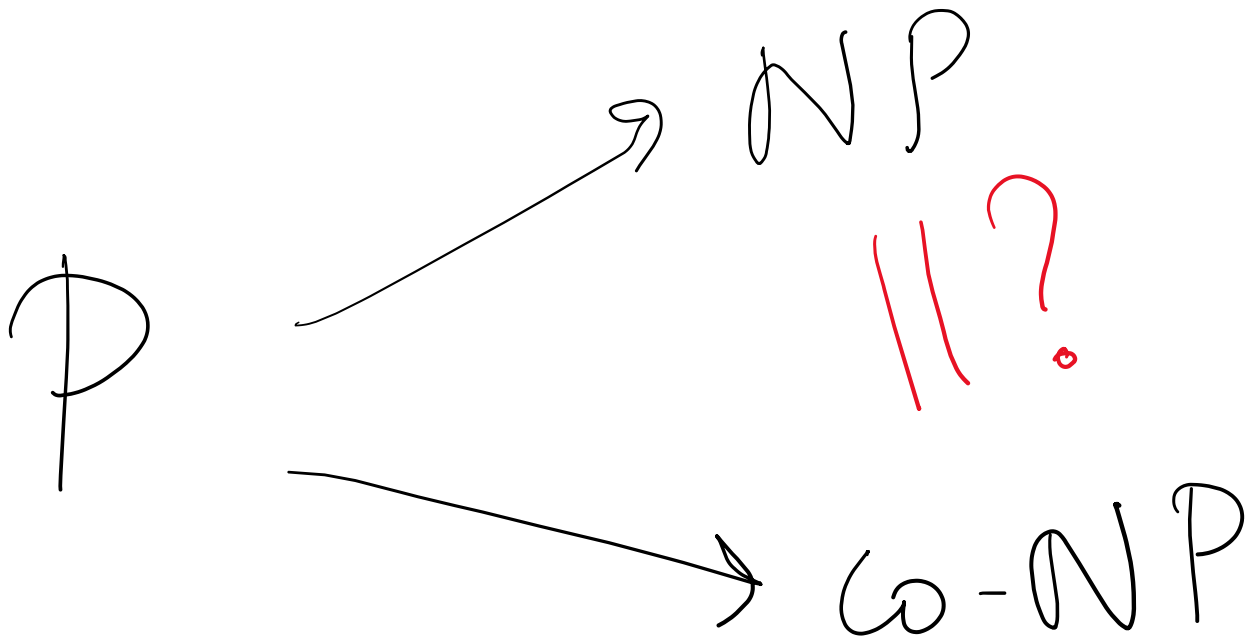
$\log CFL / SAC^1$ is closed under complementation

4/3/21

Complexity Classes



> "Complementation" an important theme in the study of non-det. complexity classes

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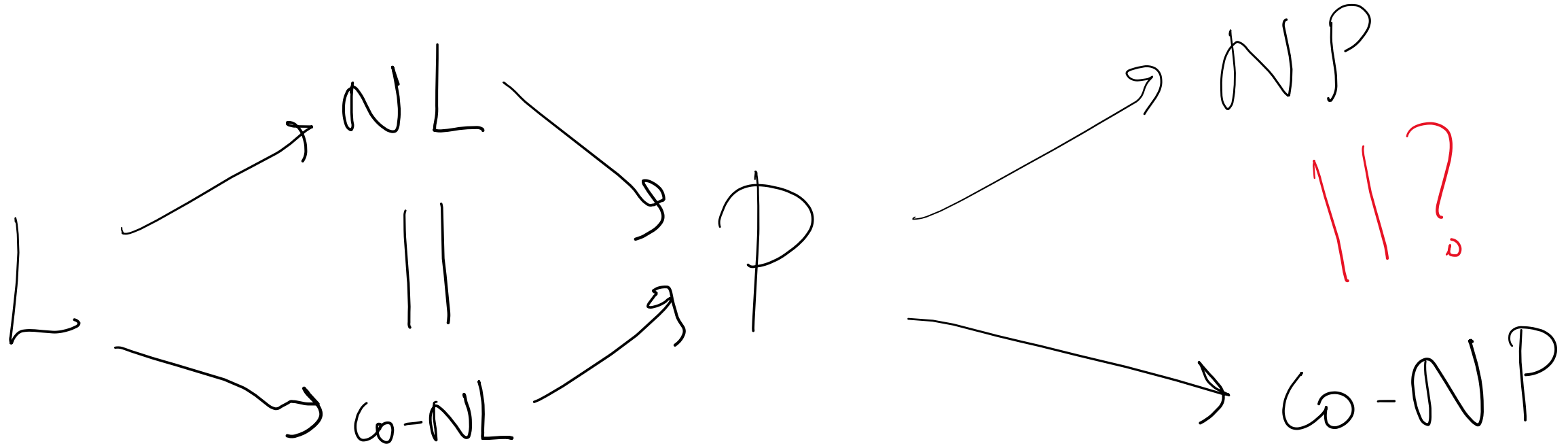

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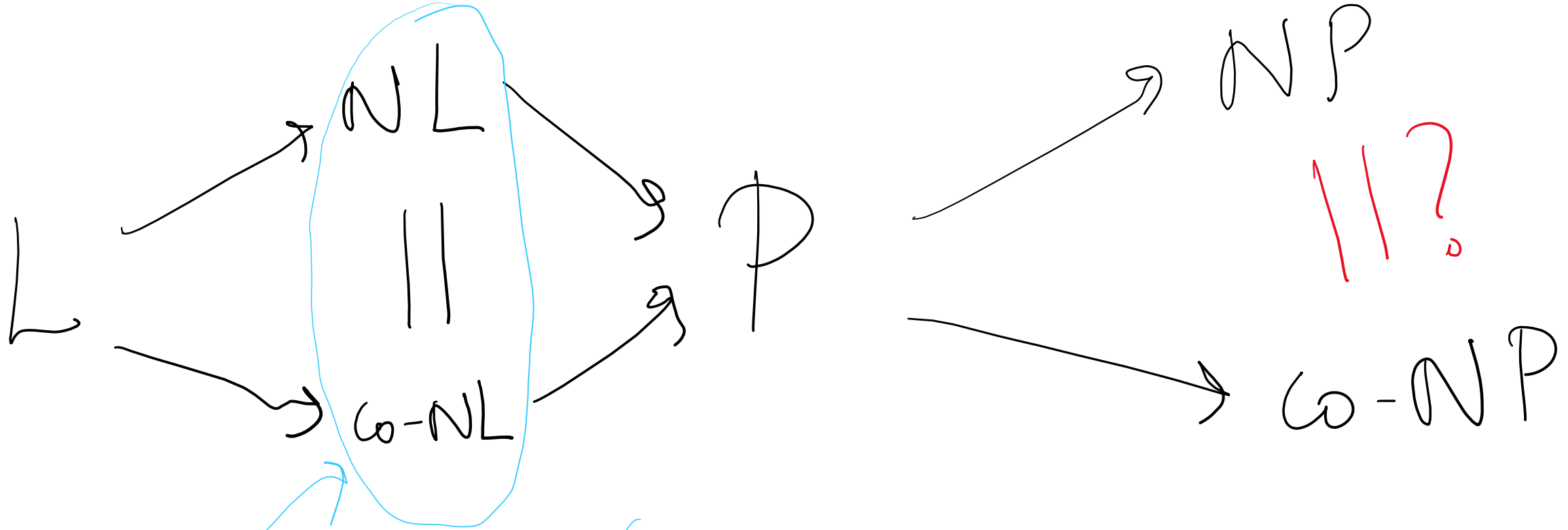

Surprising result (Immerman '88, Szelepcsényi '87)

(... detour) Circuits.

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> Unlike TMs, combinatorial models of comp.
> inherently 'non-uniform' (one for each n)
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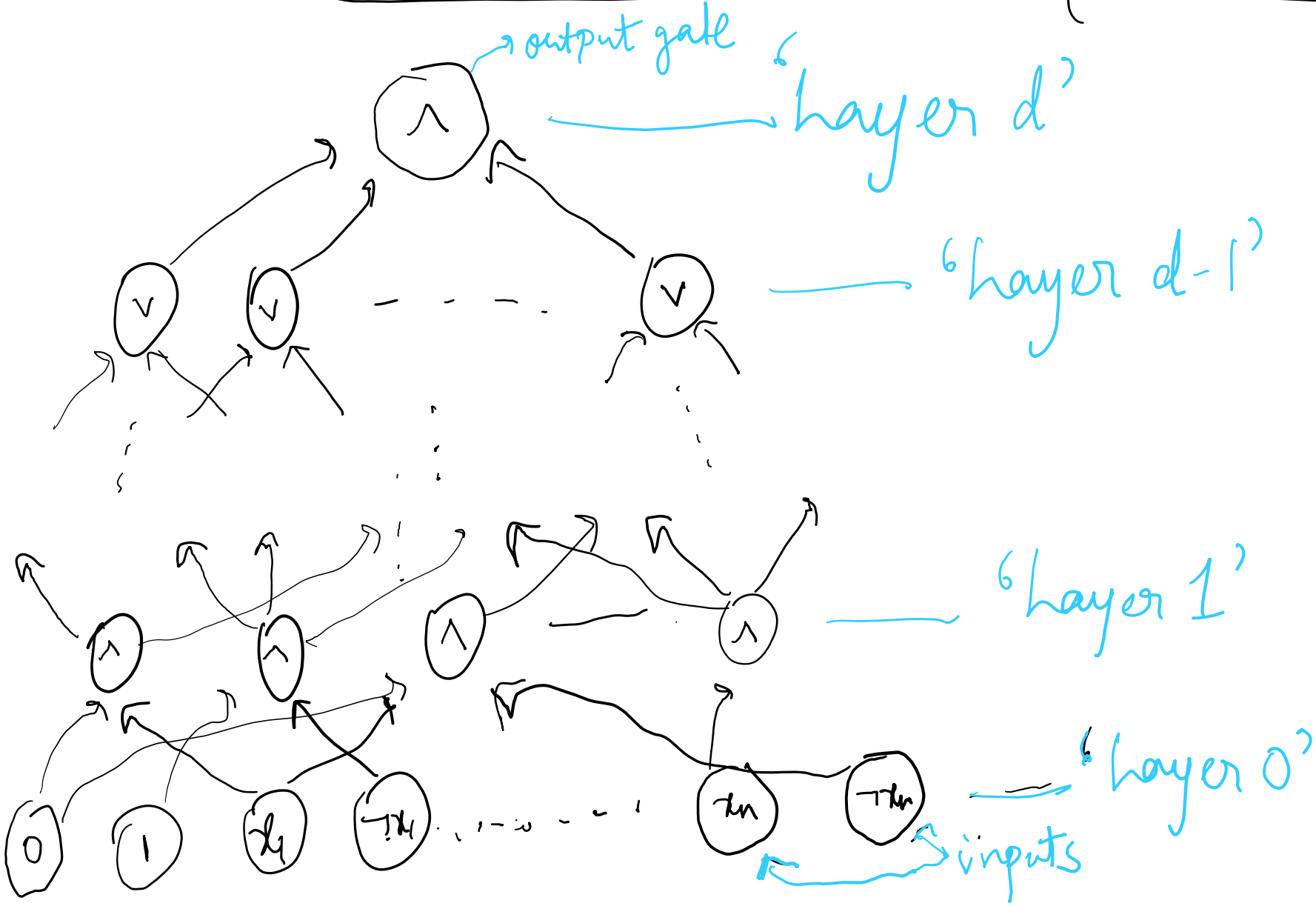
> so makes sense to 'restrict' them in some way and see how their power compares with our beloved TM classes

(Boolean) Alternating Circuits



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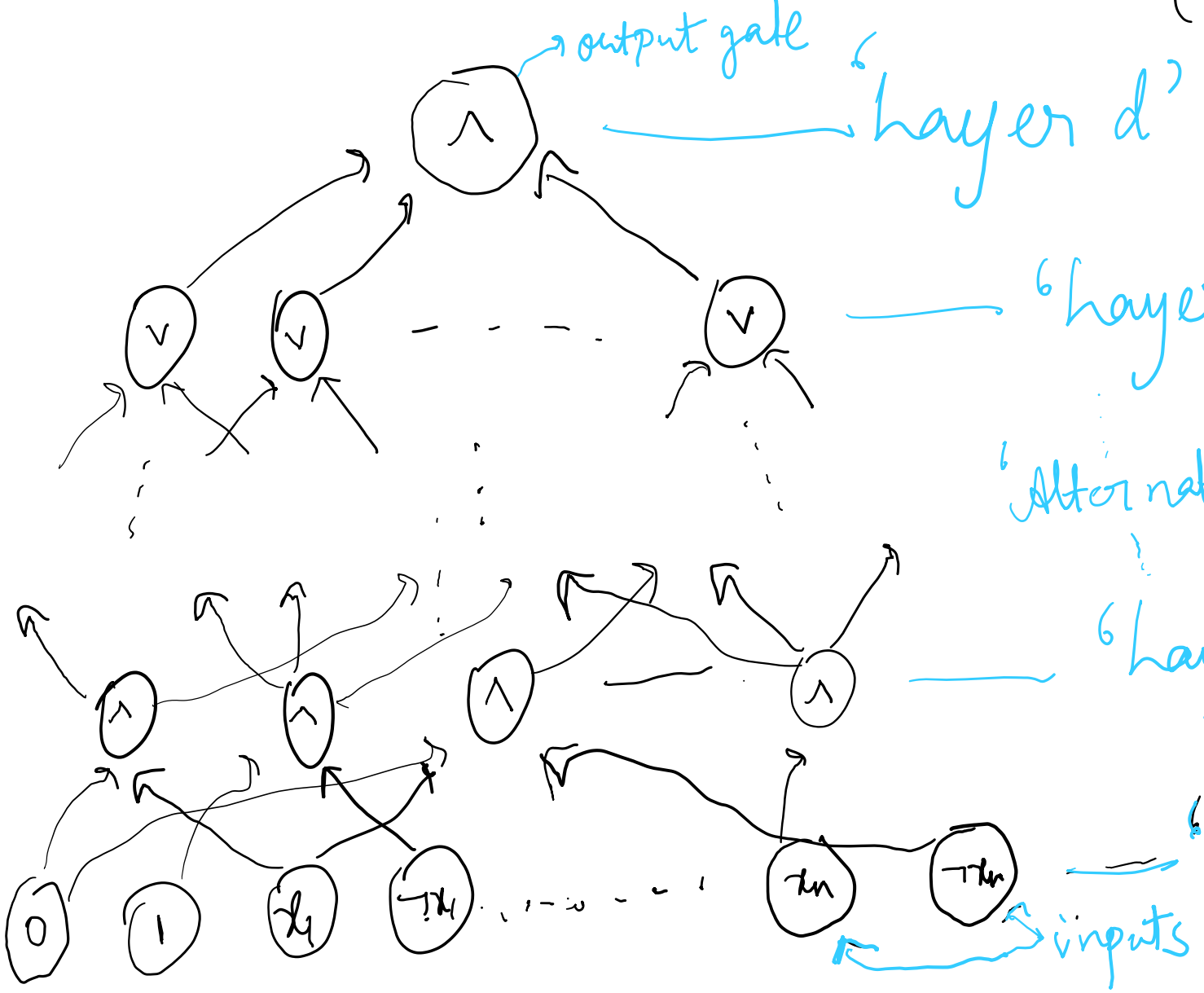

- Each internal node labelled by $\wedge$ / $\vee$
- wires go b/w successive layers
- Two imp. parameters
  - size s = # nodes
  - depth d

$s = s(n)$, $d = d(n)$

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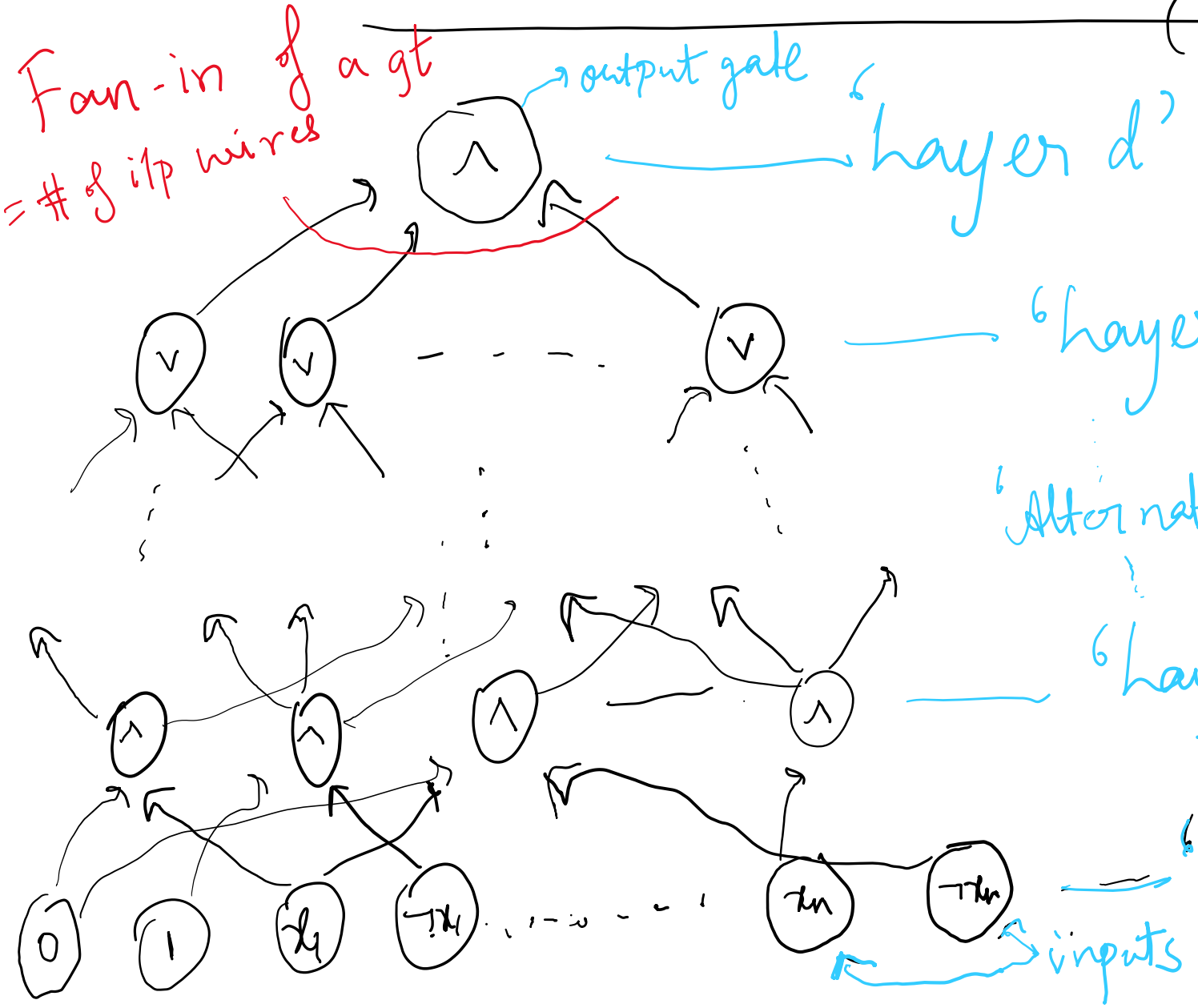

> Each internal node labelled by ∧ / ∨

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> ~~Two~~ Three imp. parameters

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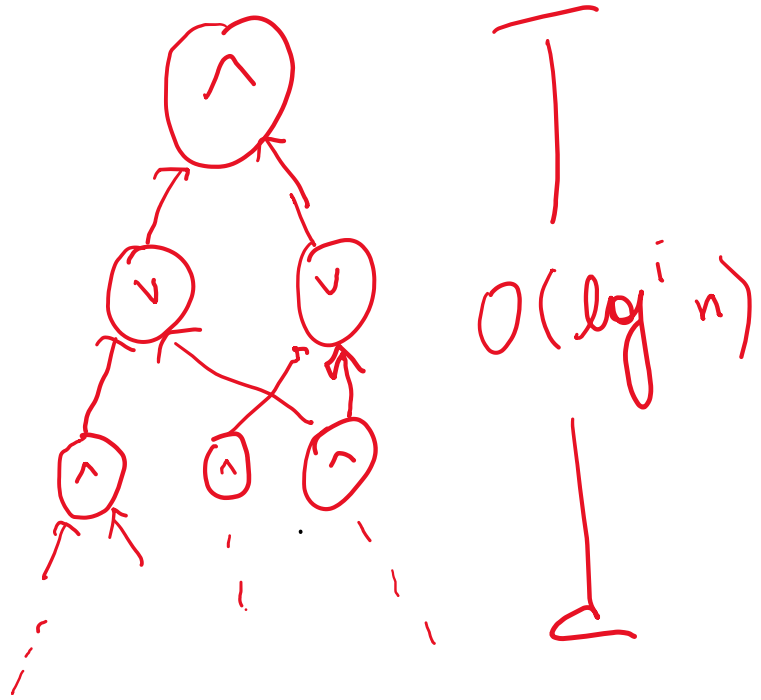
$s = s(n)$, $d = d(n)$

Circuit Classes

NC^i = Class of languages computed by circuits of poly(n) size, $O(\log^i n)$ depth, fan-in = 2.



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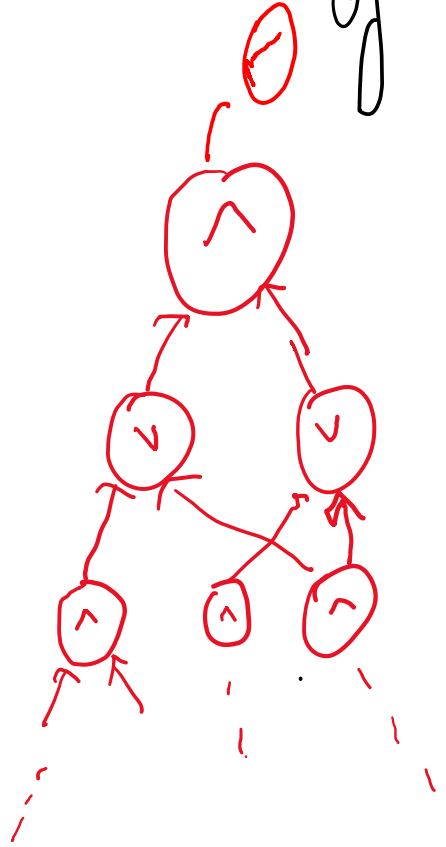

$O(\log^i n)$

We have

$$NC^0 \subset NC^1 \subset \cdots \subset NC = \bigcup NC^i$$

$$\cap$$

$$P$$

Circuit Classes

NC^i = Class of languages computed by circuits of $poly(n)$ size, $O(\log^i n)$ depth, fan-in = 2.

AC^i = Class of languages computed by circuits of $poly(n)$ size, $O(\log^i n)$ depth, unbounded fan-in

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A very rich landscape of classes beneath P.

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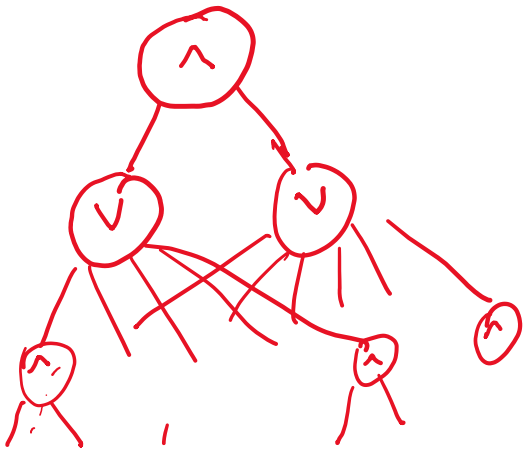

poly(n) size

T

$O(\log n)$

$\wedge$ gates - fan-in = 2

$\vee$ gates - unrestricted

The Class $\log CFL$, finally

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$$\log CFL = SAC^1 = NAuxPDA(n^{O(1)}, \log n)$$

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A certain non-det. class.

$\Rightarrow$ question of complementation is interesting!

The Class $\log CFL$, finally

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> Equivalence (Venkateswaran '87, Sudborough '78):

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Moreover, this class fits as follows:

[Borodin, Cook, Dymond, Ruzzo, Tompa '88]

$$SAC^1 = co\text{-}SAC^1$$

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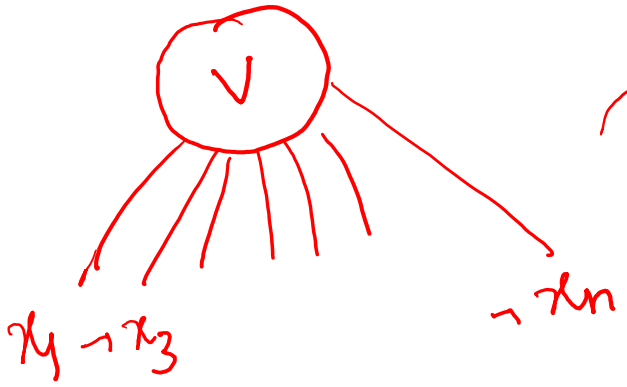
$$SAC^1 = co\text{-}SAC^1$$

Given a semi-unbounded circuit C on n input vars, of poly-size, log-depth, construct another SAC^1 circuit D that computes $\neg C$.

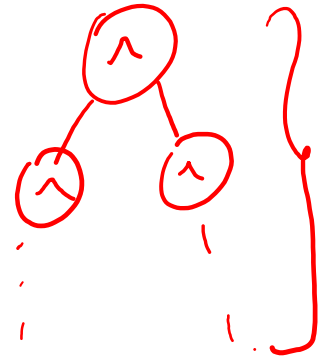
Note: this is not trivial! The simple application of De-Morgan Laws that works for AC^0 doesn't work here.



NOT of this

via DM becomes a large AND.

with only fan-in 2 AND gates.



requires log depth

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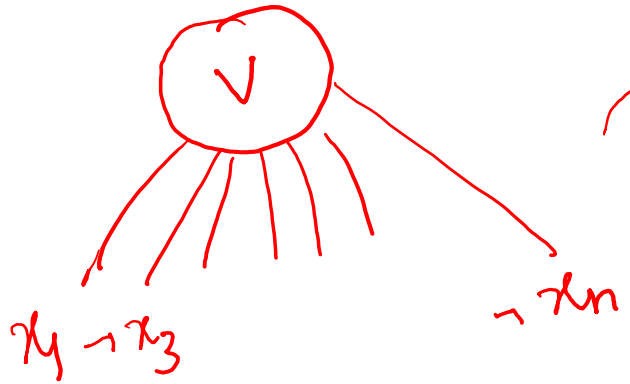
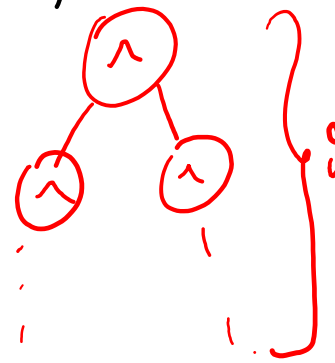


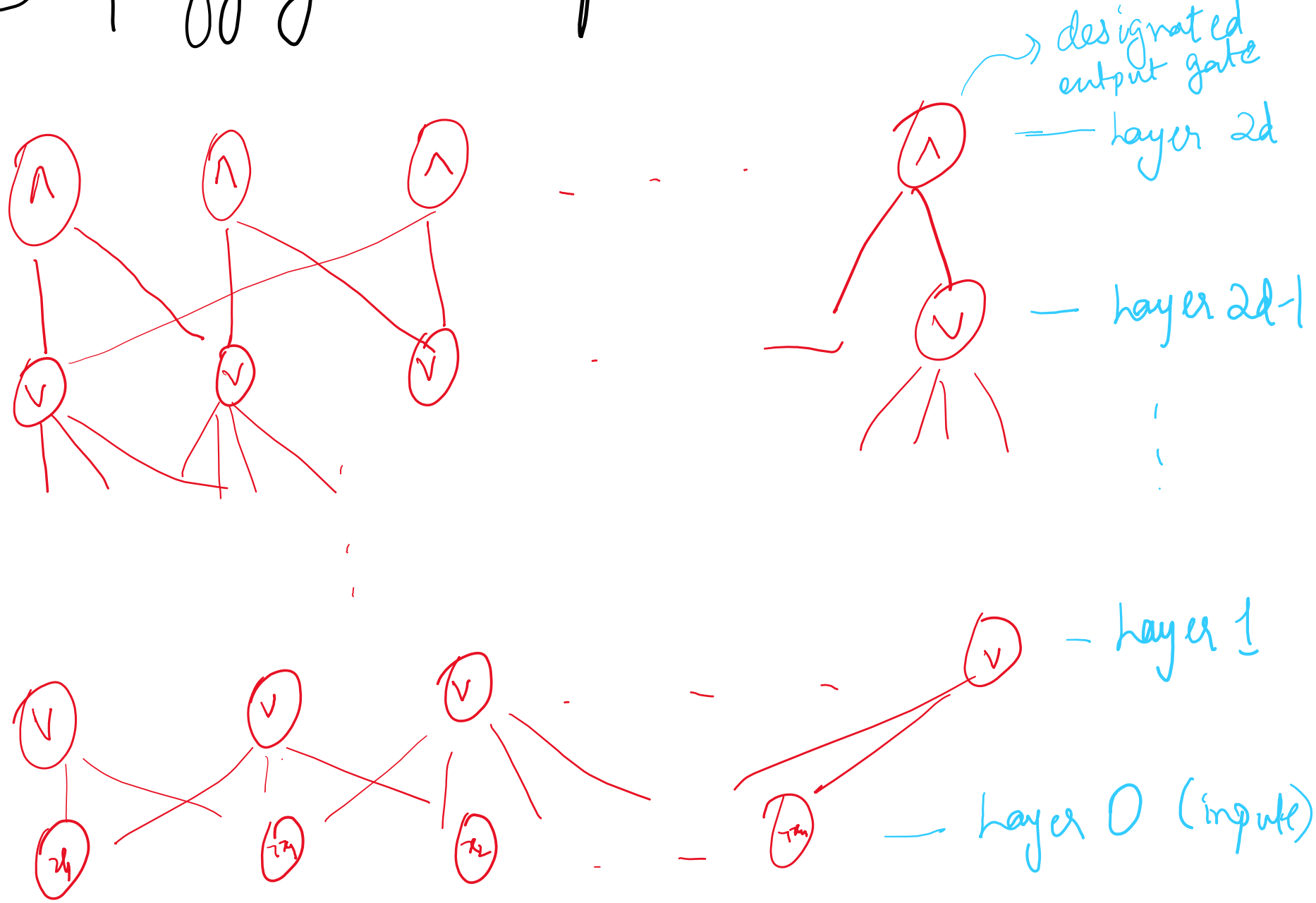
NOT of this via DM becomes a large AND.

But original circuit already has log-depth, this'll blow-up depth to $\log^2 n$.

with only fan-in 2 AND gates.



Simplifying Assumption: C looks as follows:



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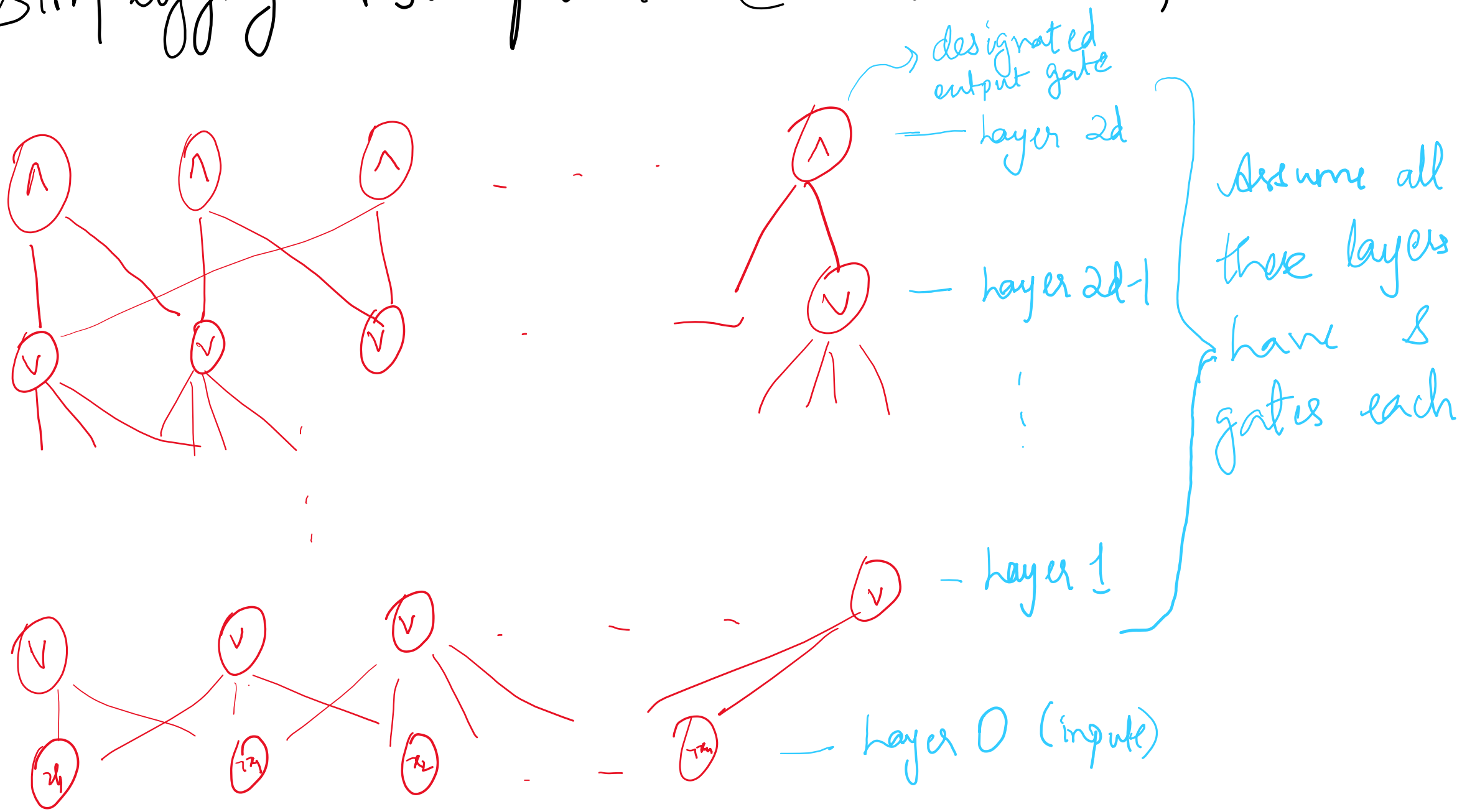

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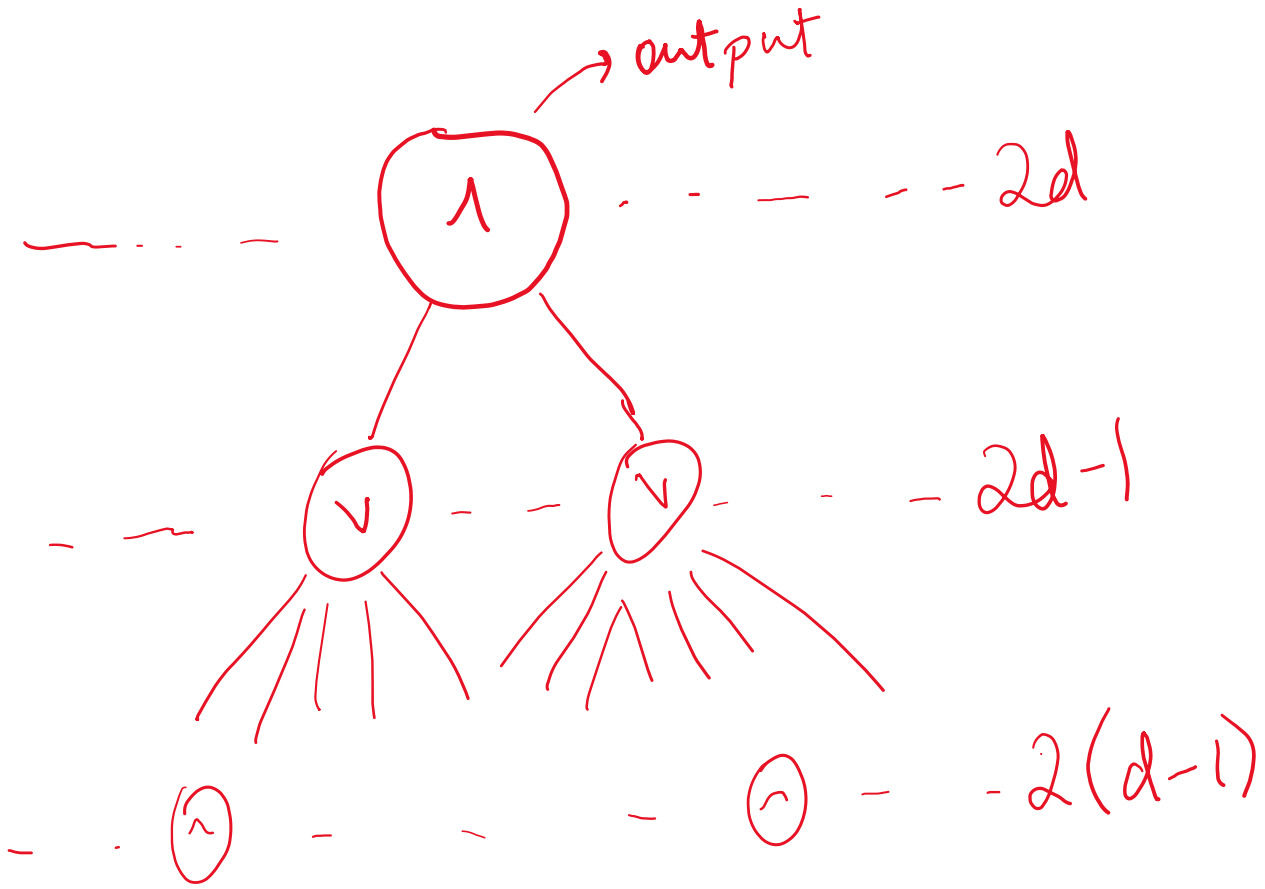
Here, $1 \leq c \leq s$, $1 \leq k \leq d$.

What else is needed?



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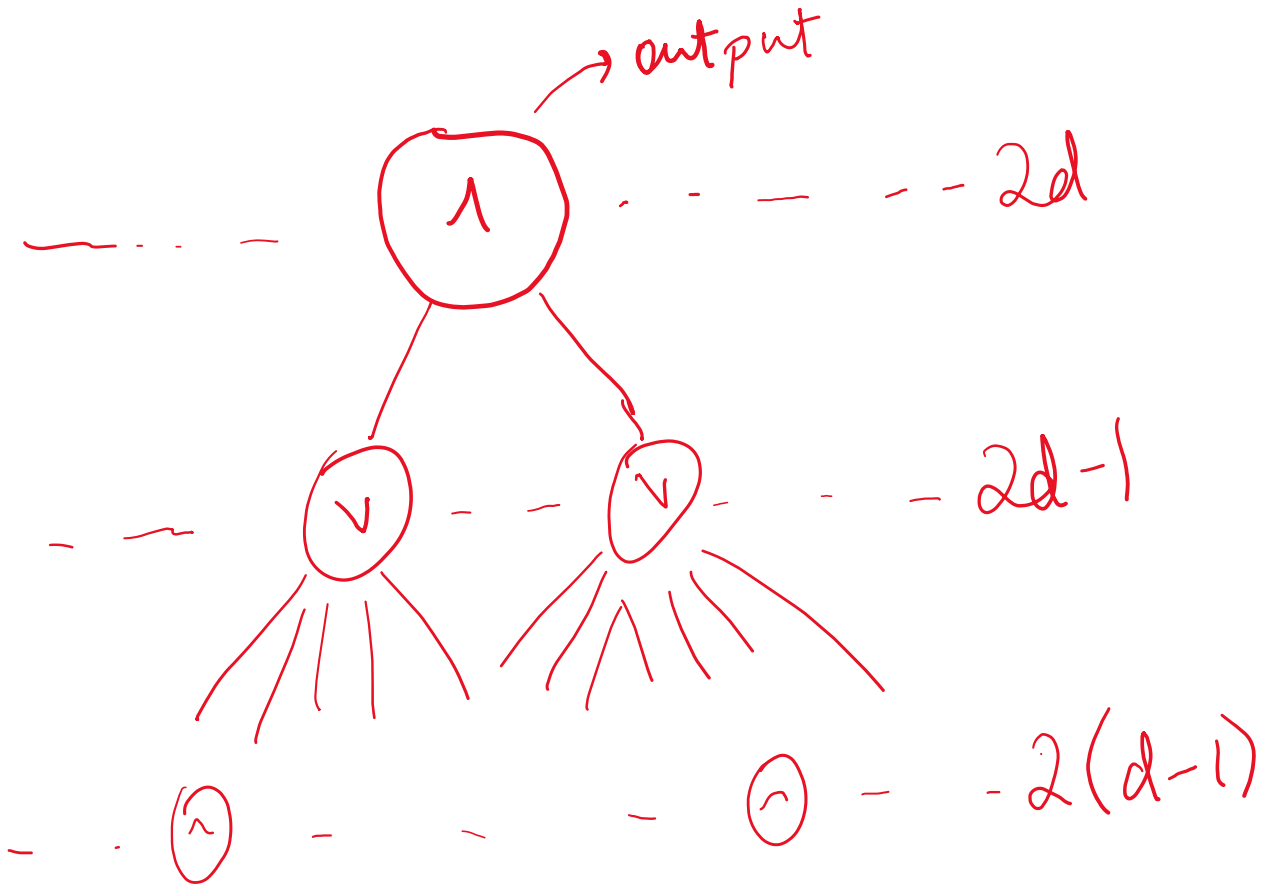
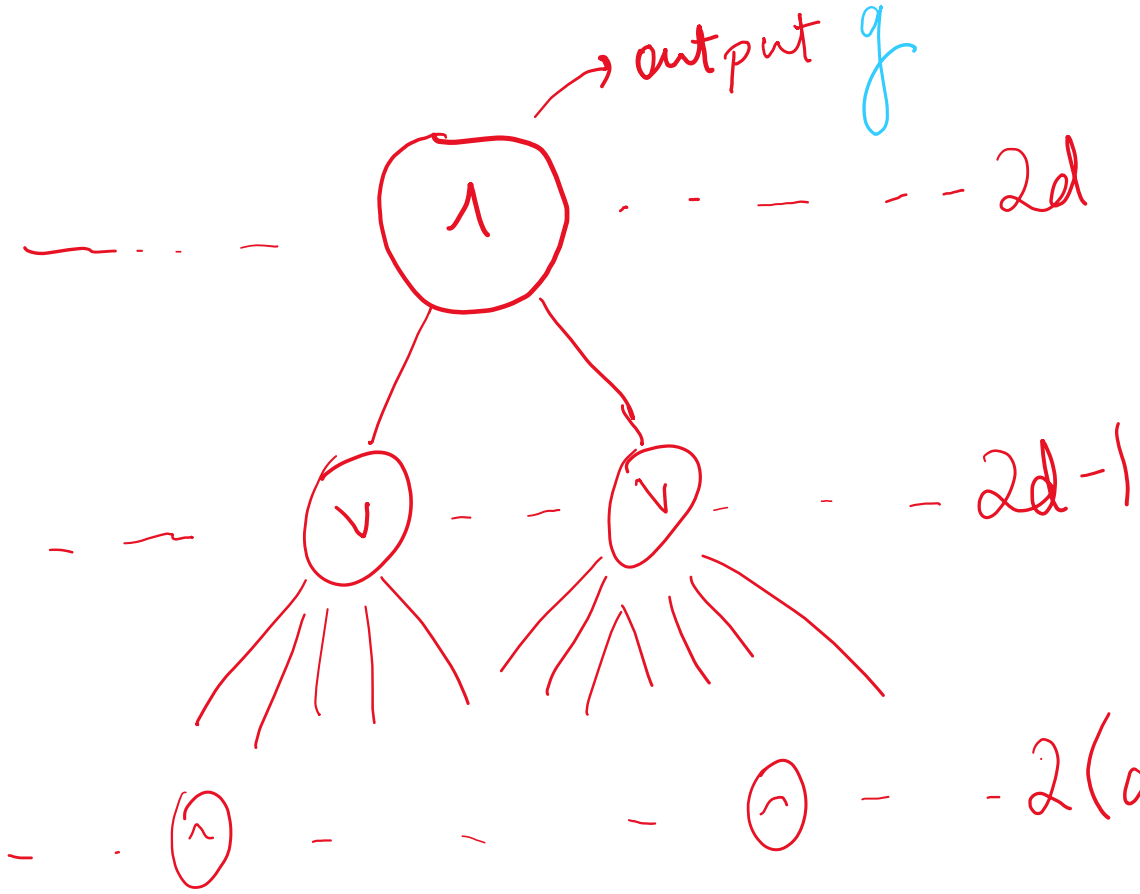


Know P_{d-1} = the only c for which $\text{count}(c, d-1) = 1$

What else is needed?



Simply the complement of g contingent on the fact that $P_{k-1} = C$

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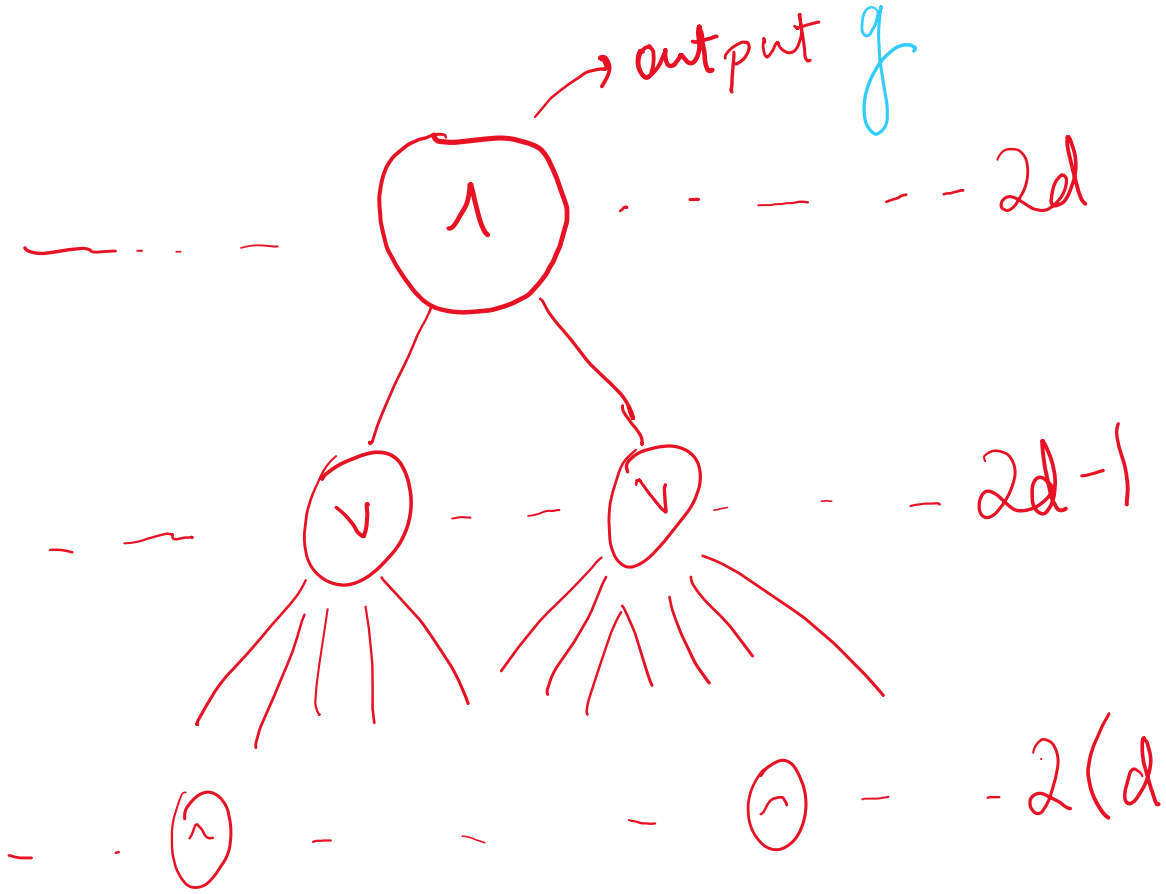

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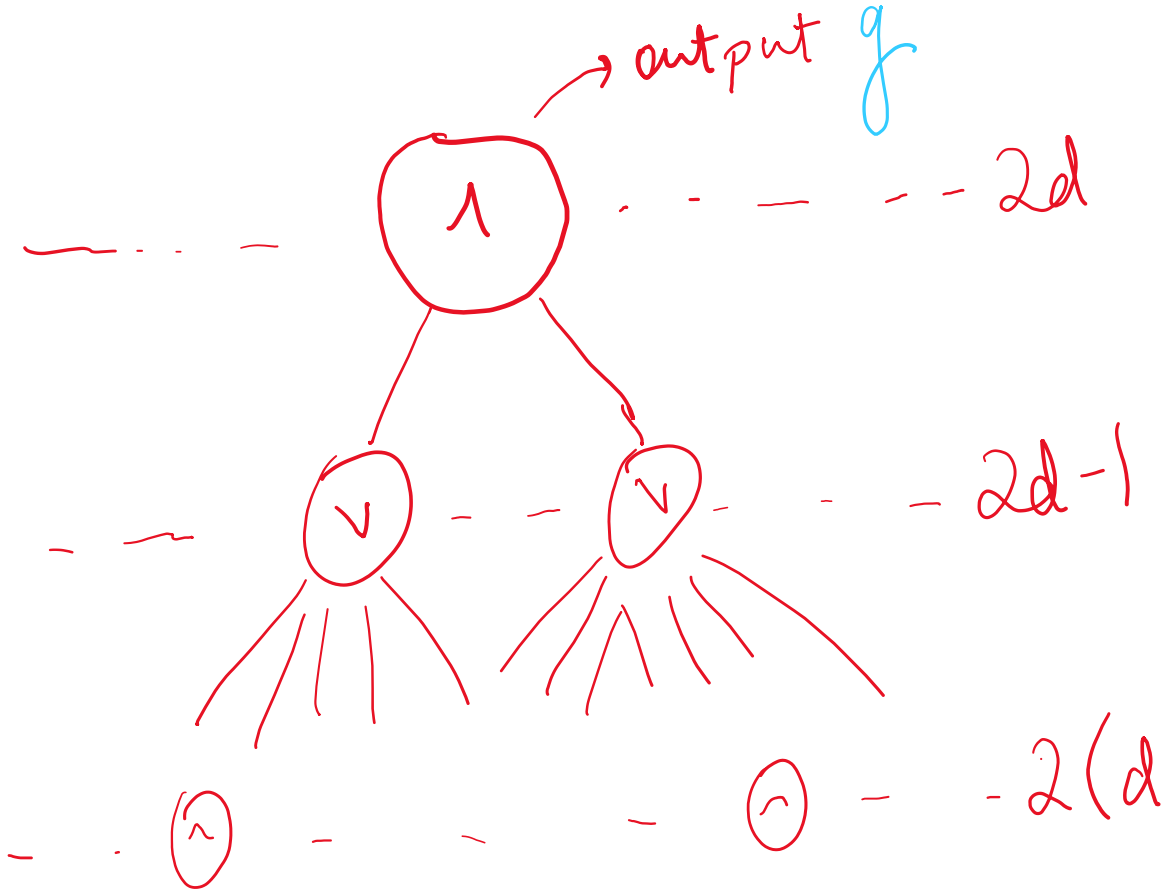

Simply the complement of g contingent on the fact that $P_{k-1} = c$

Know P_{d-1}

Idea is: along with count(c, k), also keep track of $CC(f|c) = \neg f$ given that $P_{k-1} = c$ (for all gates f of C and all $1 \leq c \leq s$)

Okay, but how do you compute count(c, k), CC(flc) inductively?

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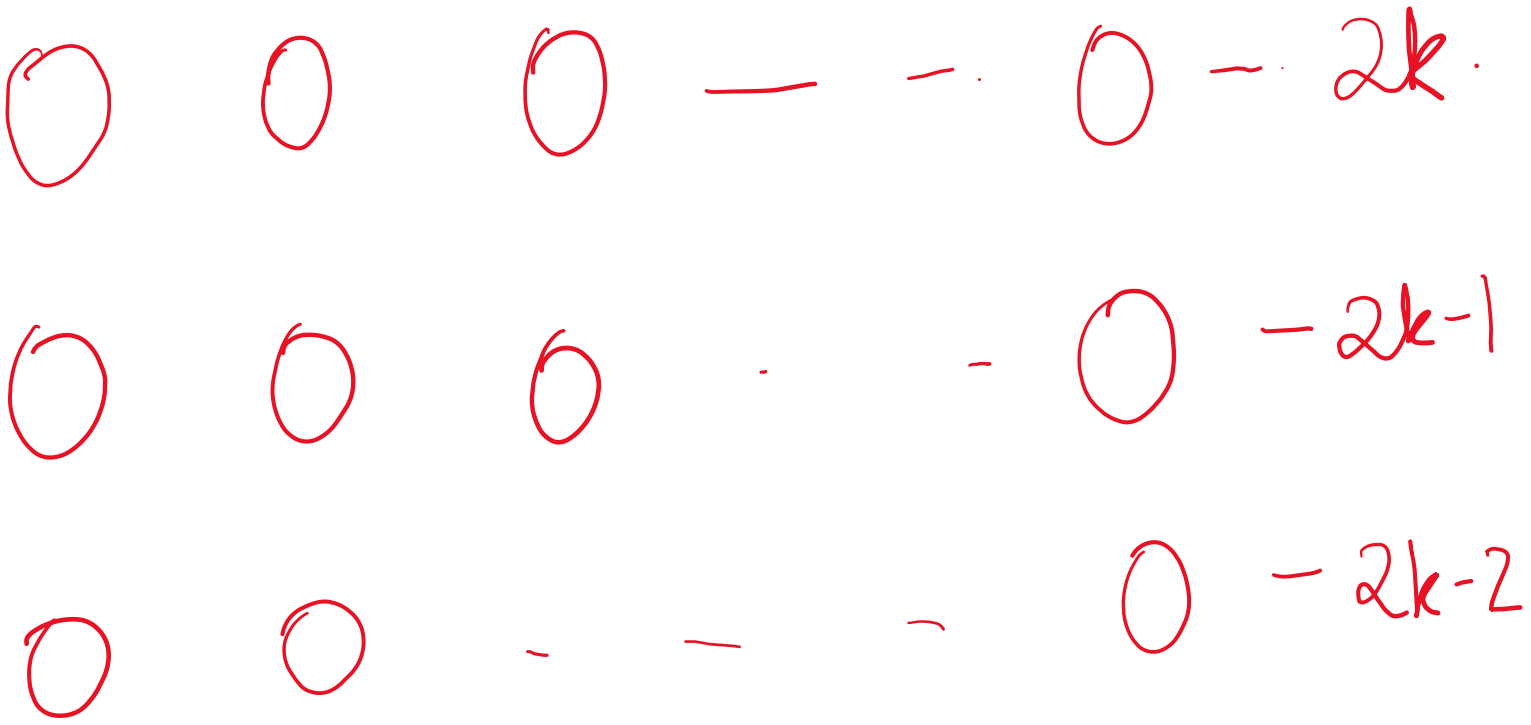

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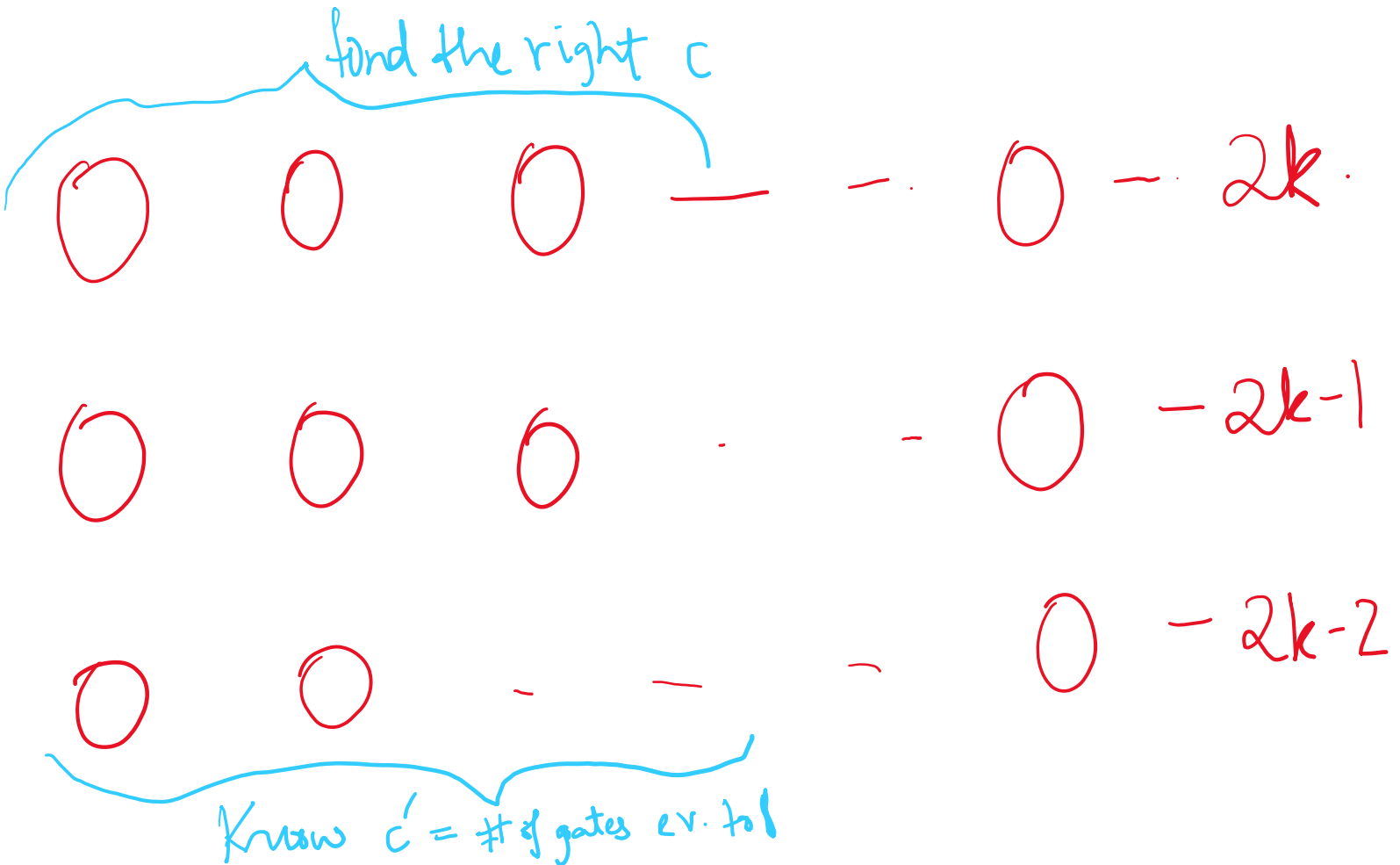

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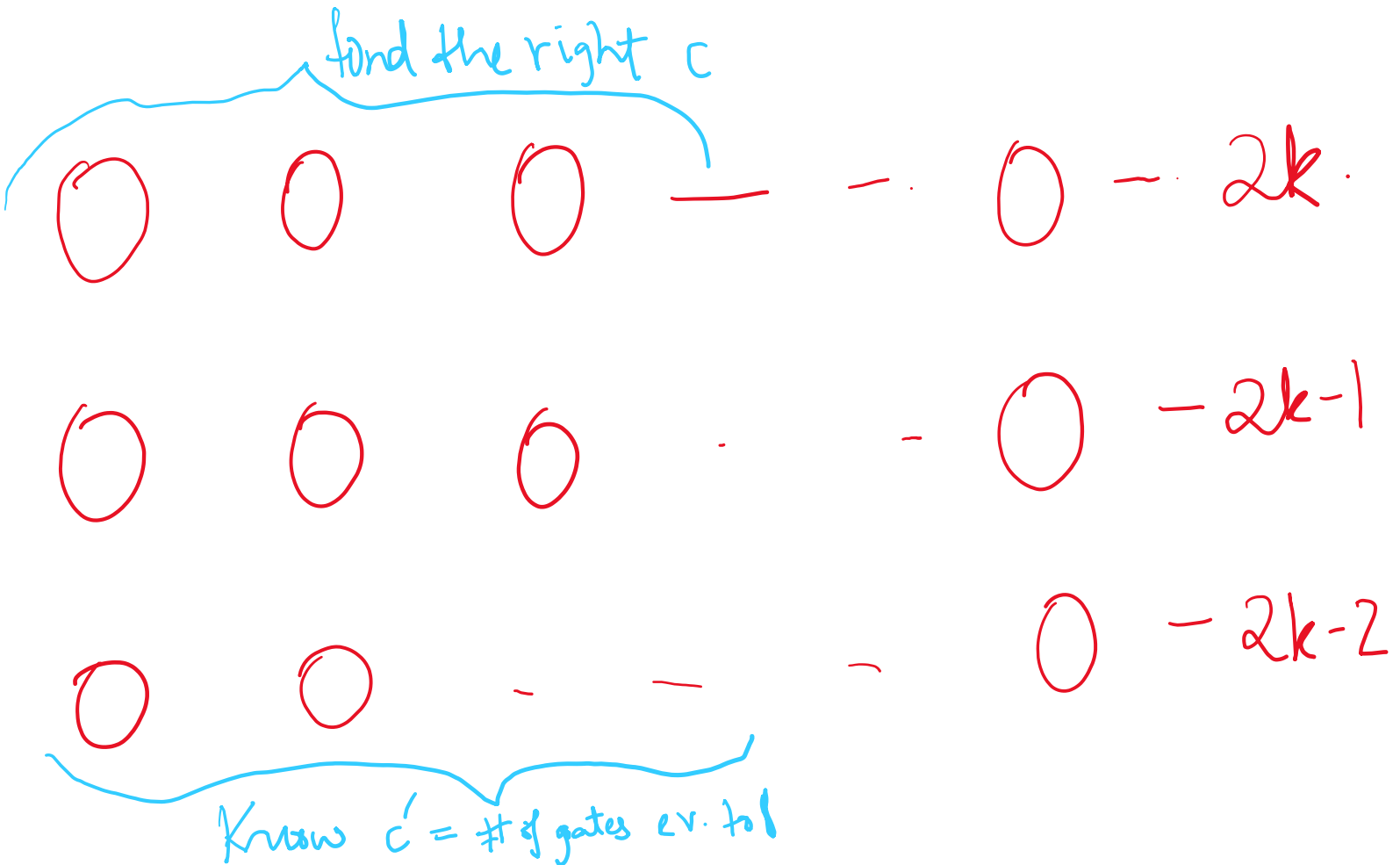

c is the unique $\in \{1, \ldots, s\}$ for which $\geq c$ gates are 1 on layer 2k AND $\geq s-c$ complement gates are 1 on layer 2k.

$$\text{count}(c, k) = \bigvee_{c'=0}^{8} \text{AND}(\text{count}(c', k-1),$$

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Fact: Threshold gates $\in NC^1 \subseteq SAC^1$.

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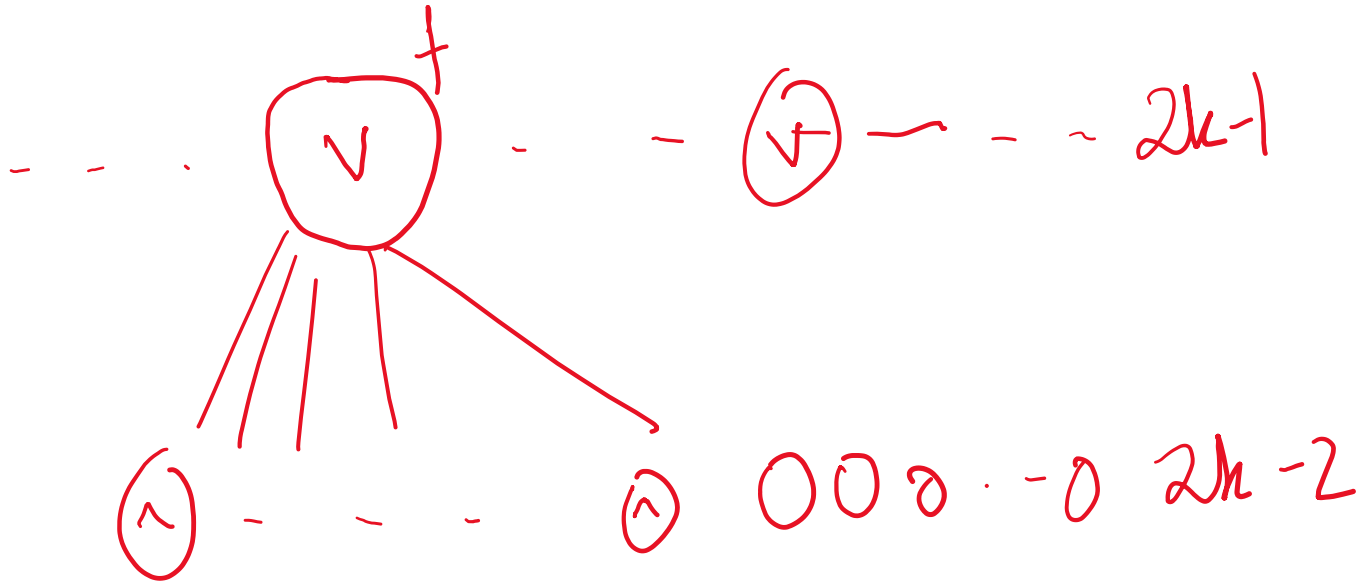

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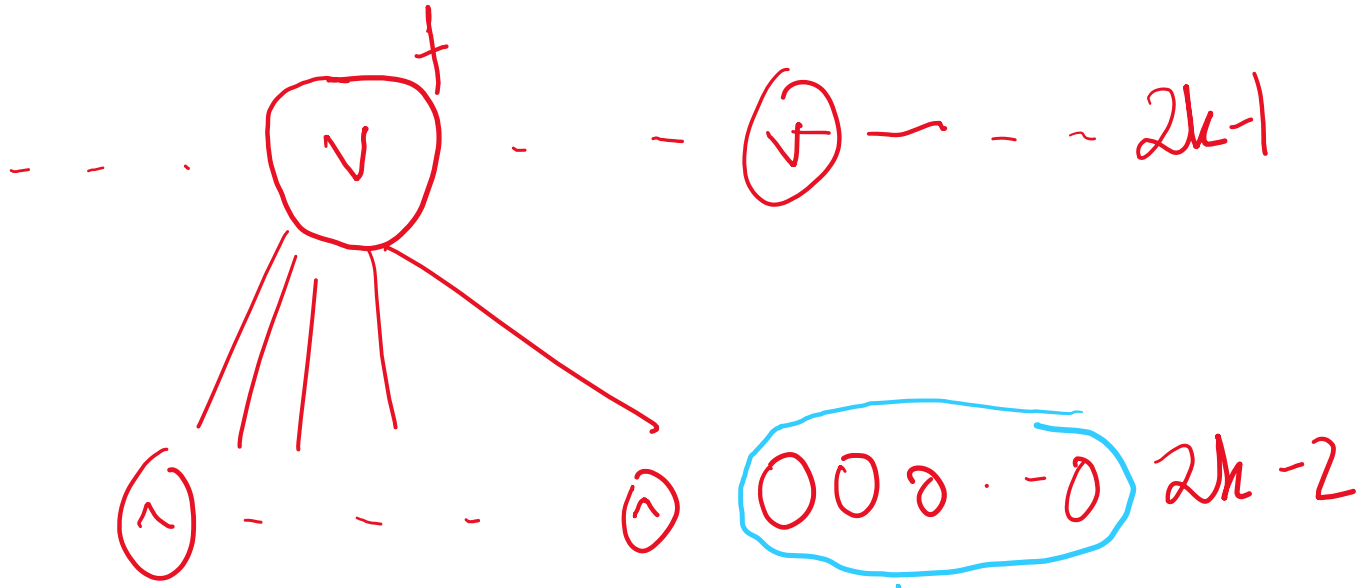

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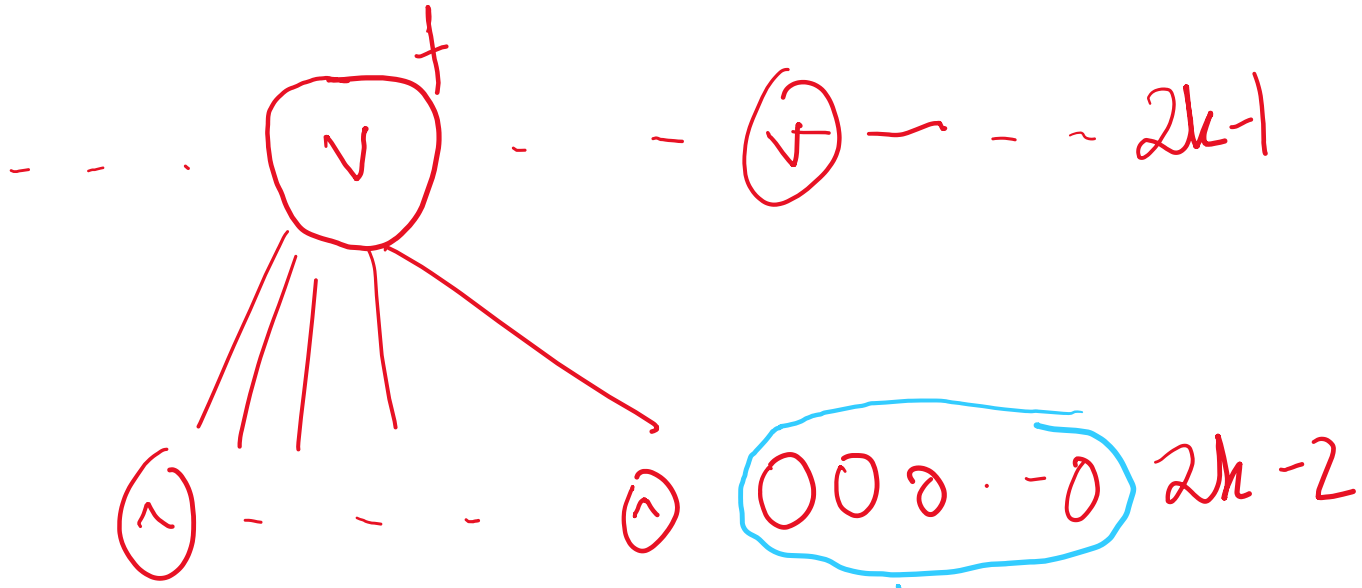

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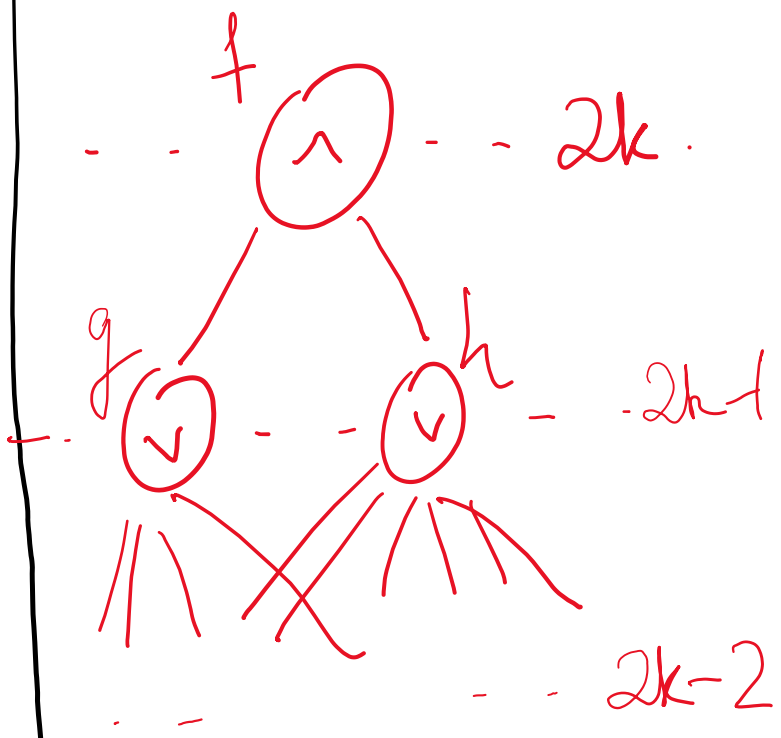

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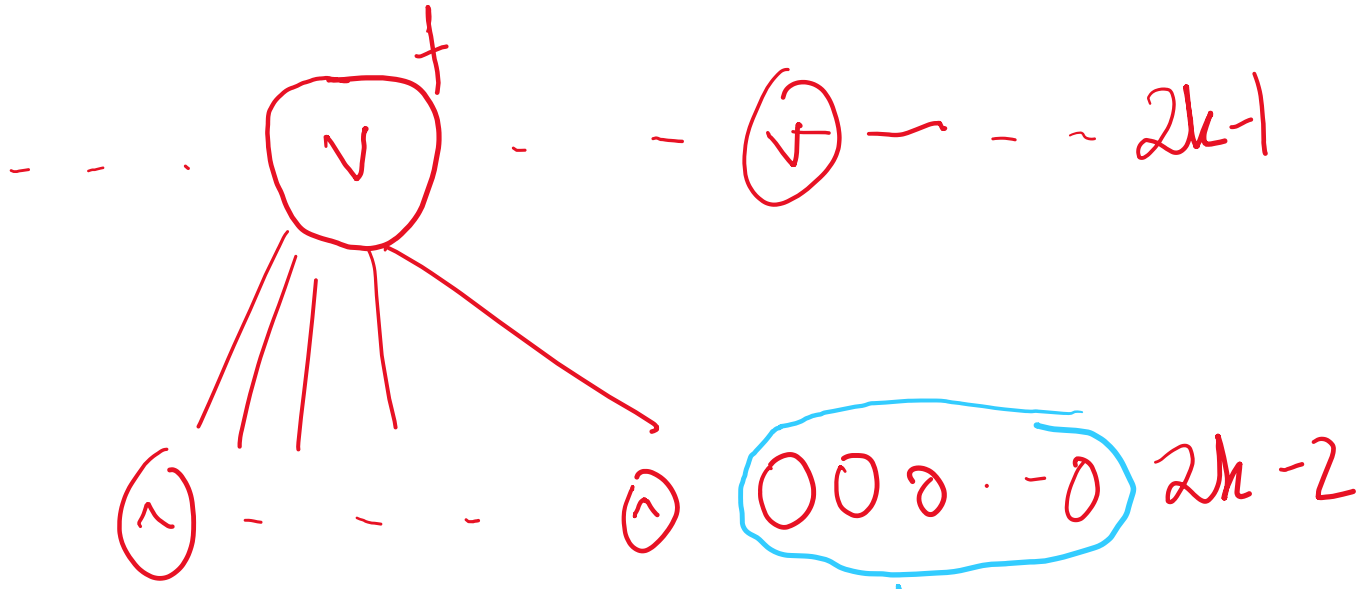

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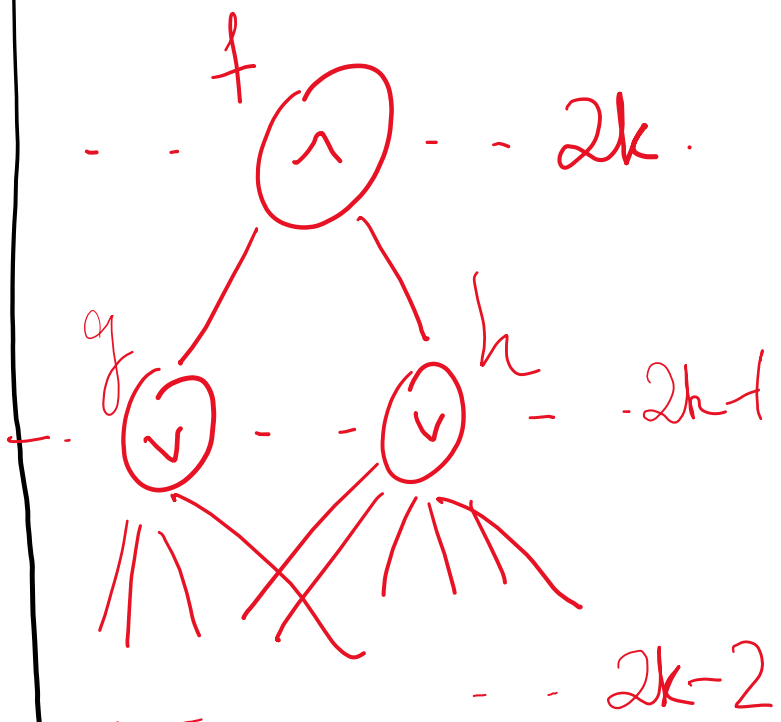

Then $CC(f|c) = CC(g|c) \vee CC(h|c)$

Thank You!