1 List Coloring in Bipartite Graphs

Let $G = (V, E)$ be a graph. We say that G is k-list-colorable if for every assignment of k colors $S(v)$ to every $v \in V$, there exists a valid coloring χ such that $\chi(v) \in S(v)$[1]. The list-coloring number (aka correspondence number) of G, denoted $\chi_\ell(G)$ is the minimum k for which G is k-list-colorable.

In the following, we explore a collection of list-coloring results pertaining to bipartite graphs.

1.1 Dinitz Conjecture

Conjecture 1. (Dinitz Conjecture.) *For every $n \times n$ grid, if every cell (i, j) was assigned n colors $S(i, j)$, does there exists a valid coloring χ such that $\chi(i, j) \in S(i, j)$ with every color appearing at most once in every row or column.*

The quite ingenious proof that such a coloring exists is due to Galvin (1995). He made use of two previously known results: (1) a list-colorability result due to Janssen (1992) and (2) the Gale-Shapely stable-matching algorithm.

First let us translate the statement using the language of list-coloring. Observe that the $n \times n$ grid has associated adjacency matrix $K_{n,n}$. Next the color of each cell, i.e. edge e in $K_{n,n}$, is restricted by those cells in the same row and column, i.e. all edges incident to same endpoints of e. Let L_G be the line graph of G. The vertices of L_G are the edges of G and there exists an edge between (i, j) and (i', j') in L_G iff $i = i'$ or $j = j'$. Thus Conjuncture 1 asks: $\chi_\ell(L_{K_{n,n}}) \leq n$? Observe that $\chi_\ell(L_{K_{n,n}}) \geq \chi(L_{K_{n,n}}) = n$ so in-fact $\chi_\ell(L_{K_{n,n}}) = \chi(L_{K_{n,n}}) = n$.

First some definitions which will be useful in Lemma 2. For every subset $A \subset V$, let G_A be the induced subgraph of A in G. Let $\overrightarrow{G}$ be a directed graph whose underlying undirected graph is G. For every vertex v, let $\deg^+(v)$ and $\deg^-(v)$ be the out- and in-degrees of v in $\overrightarrow{G}$ respectively. A *kernel* of $\overrightarrow{G}$ is a subset $K \subset V$ such that (1) K is an independent set and (2) for every vertex $v \in V \backslash K$, there exists a $u \in K$ such that vu is a directed edges.

Lemma 2. (Kernel of Induced Subgraphs and List Colorability.) *If every vertex v of $\overrightarrow{G}$ is assigned a set of colors $S(v)$ such that $|S(v)| \geq \deg^+(v) + 1$ and every induced subgraph of $\overrightarrow{G}$ has a kernel, then G there exists a list-coloring with color-list $S(v)$ for each v.*

Proof. The proof is by induction on the number of vertices in G. There is nothing to show when $|V| = 1$. Suppose the lemma is true for all graphs with $|V| = k$. Pick a graph on $k + 1$ vertices. Let c be a color in $S(v)$ for some v. Further let $A_c = \{v : c \in S(v)\}$ i.e. the set of vertices with color set containing c. By assumption, G_{A_c} has a kernel K. Color every vertex of K color c. This is possible since K is an independence set. Note that for all vertices $v \in A_c \backslash K$, $S(v) \backslash c \geq \deg^+(v)$.

[1]You might wonder why we do not specify the total number of colors C available. This is because C is implicitly taken to be arbitrarily large. Note however that there should be a valid coloring for *every* assignment of $S(v)$, thus $|C|$ can be taken small enough so that the k-list-coloring is realized.

Further the number of colors available on every $u \notin A_c$ did not change so the condition holds for all vertices not in K. Thus we can remove the vertices of K and apply the induction hypothesis on the remaining vertices. □

The road-map for proving Dinitz' Conjecture is clear: find an orientation of $L_{K_{n,n}}$ such that $\deg^+(v) \le n-1$ for all $v \in V$ and every induced subgraph has a kernel.

The orientation $L_{K_{n,n}}$ will be inspired by an $n \times n$ Latin square; these are $n \times n$ matrices such that the numbers 1 through n appears exactly once in every row and column (think Sudoku without the block constraints). Every $n \times n$ matrix has a corresponding Latin square; simply let each row be a cyclic permutation of $(1, ..., n)$. Now we extract an orientation as follows. We have directed edge $(i, j) \to (i, j')$ if $j < j'$ and directed edge $(i, j) \to (i', j)$ if $i > i'$ (edges are directed from small to large along the rows and large to small along the columns).

First we observe that the out-degree of each vertex is exactly $n-1$. In particular if entry (i, j) of the Latin square is k then the out-degree of (i, j) to the vertices in the same row is exactly $n-k$ and in the same column is exactly $k-1$.

Thus it remains to show that every induced subgraph of this orientation of $L_{K_{n,n}}$ has a kernel. For this, recall that every bipartite graph has a stable matching by the Gale-Shapely algorithm. That is a matching M such that for every $uv \notin M$, either $M(u) > v$ or $M(v) > u$ in the preference list of u and v respectively. Consider an induced subgraph in our orientation of $L_{K_{n,n}}$. We claim that a stable matching in the underlying undirected graph is a kernel in the induced subgraph. We define the preference list for each vertex in the natural way: in every row (resp. column) the larger (resp. smaller) numbers in the corresponding entries of the Latin Square is more preferable. To see that the stable matching is indeed a kernel, note that (1) the edges in a matching have distinct endpoints and (2) for every uv not in the matching, it must be the case that there exists an edge $u'v$ or uv' where u' and v' are endpoints of edges in the matching.

This proof can be extended to show that the line graph of any bipartite graph G satisfies $\chi_\ell(L_G) = \chi_\ell(L_G)$. A well know open problem asks if this is true for general graphs G.

1.2 Lowerbound on List Coloring Number as a Function of Degree

This does not really pertain only to bipartite graphs, but it is suspected to be tight for bipartite graphs since it is known that $\chi_\ell(K_{d,d}) = (1 + o(1)) \log d$.

Theorem 3. (Lowerbound on List Coloring Base on Degree.) *For graph G with minimum degree at least d, the list-coloring number satisfies $\chi_\ell(G) > s$ if*

$$d > s^6 2^{2s}. \tag{1}$$

Corollary 4. *For a simple graph G with minimum degree d, $\chi_\ell(G) \ge (1/2 + o(1)) \log_2 d$ with constant of multiplicity off by at most 2.*

Proof of Theorem 3. We are going to assign color-lists of size s to each vertices in G from among $S = \{1, ..., s^2\}$ different colors. We will pay particular attention to two sets of vertices A and B. Each vertex of G will be added to B with probability $1/\sqrt{d}$. For every vertex $b \in B$, we will assign it a color-list $S(b)$ from among the $\binom{s^2}{s}$ sets of size s from S uniformly at random. A vertex $v \in A$ if: (1) $v \notin B$ and (2) for every subset $T \subset S$ of size $\lceil s^2/2 \rceil$ there exists some neighbour $b \in B$ of v such that $S(b) \subset T$.

Probability that $v \notin A$: Condition on whether or not $v \in B$. If $v \in B$, then $v \notin A$. This occurs with probability $1/\sqrt{d}$. Conversely, if $v \notin B$, then $v \notin A$ if for every one of the $\binom{s^2}{\lceil s^2/2 \rceil}$ sets T, it must be the case that $u \in N(a)$ is either not in B or in B and has set $S(u) \not\subset T$. Thus we have

$$\mathbb{P}[v \notin A] = \frac{1}{\sqrt{d}} + \left(1 - \frac{1}{\sqrt{d}}\right) \binom{s^2}{\lceil s^2/2 \rceil} \left(1 - \frac{1}{\sqrt{d}} \frac{\lceil s^2/2 \rceil \cdot \left(\lceil s^2/2 \rceil - 1\right) \cdot \dots \cdot \left(\lceil s^2/2 \rceil - s + 1\right)}{s^2 \cdot (s^2 - 1) \cdots (s^2 - s + 1)}\right)^d.$$

By Stirling's approximation, we have

$$\binom{s^2}{\lceil s^2/2 \rceil} \leq \frac{2^{s^2}}{\sqrt{\lceil s^2/2 \rceil}} \leq \frac{2^{s^2}}{4}.$$

We can bound the probability that $S(b) \subset T$ by

$$\begin{aligned}
\mathbb{P}[S(b) \subset T] &= \frac{\lceil s^2/2 \rceil \cdot \left(\lceil s^2/2 \rceil - 1\right) \cdot \dots \cdot \left(\lceil s^2/2 \rceil - s + 1\right)}{s^2 \cdot (s^2 - 1) \cdots (s^2 - s + 1)} \\
&\geq \frac{1}{2^s} \prod_{i=0}^{s-1} \frac{s^2 - 2i}{s^2 - i} \\
&= \frac{1}{2^s} \prod_{i=0}^{s-1} \left(1 - \frac{i}{s^2 - i}\right) \\
&\geq \frac{1}{2^s} \left(1 - \frac{\sum_{i=0}^{s-1}}{s^2 - s}\right) \\
&\geq \frac{1}{2^{s+1}}
\end{aligned}$$

where the first inequality follows by removing the ceilings, and the second inequality on (line 4) can be seen by consider the coefficient of x in $\prod_{i=0}^{s-1} \left(1 - \frac{ix}{s^2 - i}\right)$. Thus $\mathbb{P}[v \notin A]$ can be bounded as

$$\mathbb{P}[v \in A] \leq \frac{1}{\sqrt{d}} + \frac{2^{s^2}}{4} \left(1 - \frac{1}{\sqrt{d} 2^{s+1}}\right)^d \leq \frac{1}{\sqrt{d}} + \frac{2^{s^2}}{4} \left(\exp \sqrt{d}/2^{s+1}\right) < \frac{1}{2}$$

by our choice of s in 1.

Finding a Set of Color-List Which has no Proper Coloring. Let X_A and X_B be the random variables counting the number of vertices in A and B respectively. Since the $\mathbb{P}[v \notin A] \leq 1/4$, $\mathbb{E}[n - |X_A|] < n/4$. By Markov inequality,

$$\mathbb{P}[n - |X_A| > n/2] < \frac{1}{2} \text{ and thus } \mathbb{P}[|X_A| > n/2] > 1/2.$$

Similarly, $\mathbb{E}[|X_B|] = n/\sqrt{d}$ so

$$\mathbb{P}[|X_B| > 2n/\sqrt{d}] < \frac{1}{2} \text{ and thus } \mathbb{P}[|X_B| \leq 2n/\sqrt{d}] > \frac{1}{2}.$$

Together, there exists some random choice of B and list-colors $S(b)$ for $b \in B$ such that simultaneously $|X_A| > n/2$ and $|X_B| \leq 2n/\sqrt{d}$. In the following, fix such a set B and list-colors $S(b)$.

We randomly assign color-lists $S(a)$ to vertices in A and show that there exists a random assignment such that vertices in $A \cup B$ cannot be properly list-colored. Consider any coloring $c(b)$ for the vertices in B. There are $s^|B|$ such colorings. For an $a \in A$, let $T_a = \cup_{b \in N(a), b \in B} c(b)$ be the set of colors on the neighbours of a in B. If $S(a) \subset T_a$ then a cannot be properly colored. In order for this to happen, we will show that T_a is large. Remember that $a \in A$ since for every $T \subset S$ of size $\lceil s^2/2 \rceil$ there exists some $b \in N(a) \cap B$ such that $S(b) \subset T$. Since $S(b) \subset T$, $c(b) \in T$. Thus there cannot be a set T of $\lceil s^2/2 \rceil$ colors such that no $c(b) \in T$. This also means that $|T_a| \geq \lceil s^2/2 \rceil$.

Finally let us calculate the probability $a \in A$ can be properly colored. Note that

$$\mathbb{P}[S(a) \subset T_a] = \frac{\lceil s^2/2 \rceil \cdot \left(\lceil s^2/2 \rceil - 1\right) \cdot \dots \cdot \left(\lceil s^2/2 \rceil - s + 1\right)}{s^2 \cdot (s^2 - 1) \cdots (s^2 - s + 1)} \geq \frac{1}{2^{s+1}}$$

as above. Since there are at least $n/2$ vertices in A and they are all independent, the expected number colorings which results in a valid coloring for all $a \in A$ (denoted Y_A) is

$$\mathbb{E}[Y_A] \leq e^{|B|} \left(1 - \frac{1}{2^{s+1}}\right)^{n/2} \leq e^{\frac{2n}{\sqrt{d}} - \frac{n}{2^{s+1}}} < 1$$

by our choice of s from s in 1. Thus there exists some random choice of B and the color-lists $S(b)$ such that there exists some $a \in A$ such that $S(b) \subset T_a$ for every coloring $c(b)$ with $c(b) \in S(b)$. $\square$