Abstract

In this talk, I will present random graphs, highlighting a classic result of Erdős and Rényi known as the giant component threshold. The result says that if a graph is sampled by including each edge independently with probability c/n, if $c > 1$, a.a.s. there is a giant component of size $\Theta(n)$, and if $c < 1$, a.a.s. largest component is of size $O(\log n)$

I will also talk about random graphs with fixed degree sequences, how to sample such graphs, some of their properties, and outline a giant component threshold in this model, which Molloy and Reed found in 1995.

Random Graphs, Giant Components, and Fixed Degree Sequences

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$\mathcal{G}_{n,d}$ Phase Transition

Random Graphs Models

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Erdős-Rényi Models

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In $\mathcal{G}_{n,p}$ the expected number of edges is $\binom{n}{2}p$. Intuitively, the two models should be similar when $m = \binom{n}{2}p$.

Generating Random Regular Graphs

Random regular graphs are nice and useful! For example, a random regular graph is a good expander (w.h.p), and expanders are useful for many things like derandomization and coding theory.

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Notes:

- This is close to optimal, the lower bound is $2\sqrt{d-1} - o(1)$ [Alo86].

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Notes:

- This is close to optimal, the lower bound is $2\sqrt{d-1} - o(1)$ [Alo86].
- There are explicit construction that meet $2\sqrt{d-1}$ but as far as I know, they require d to be a prime power - 1. [LPS88]

Configuration Model

To generate a random d-regular graph on n vertices.

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- For each vertex v, create d half-edges, $v_1, \dots, v_d$.
- Take a random matching of all the half-edges
- Each edge $\{u_i, v_j\}$ in the matching corresponds to the edge $\{u, v\}$.

Example

Problem

- What if $\{u_i, u_j\}$ appears in the matching?
- What if $\{u_i, v_j\}$, $\{u_k, v_\ell\}$ both appear in the matching?

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There's some probability that you get a non-simple graph in this model.

Solutions

Theorem

Let G be any (simple) d-regular graph. Then

$$\Pr[\textit{Configuration model yields } G] = \frac{(d!)^n}{(nd-1)!!}$$

Solutions

Theorem

$$\Pr[\textit{Configuration model yields a simple graph}] \sim e^{\frac{1-d^2}{4}}$$

Thus for fixed d, and any property $\mathcal{P}$, $\Pr[\mathcal{P}(\mathcal{G}_{n,d})] = \frac{\Pr[\mathcal{P}(C_{n,d})]}{const}$.
So if something holds for a random configuration with probability $o(1)$, it also holds for the the uniform model with probability $o(1)$.

Same thing for $1 - o(1)$.

A good reference for this kind of stuff: *Introduction to Random Graphs by Frieze and Karoński* [FK15] Chapter 11.

$G_{n,\mathbf{d}}$

Everything in the previous couple of slides can be generalized to arbitrary degree sequences (instead of every vertex having degree d). A degree sequences looks like this

$$\mathbf{d} = (d_1, ..., d_n),$$

where d_i is the degree of vertex i.

Random Graphs Models

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$\mathcal{G}_{n,p}$ Phase Transition

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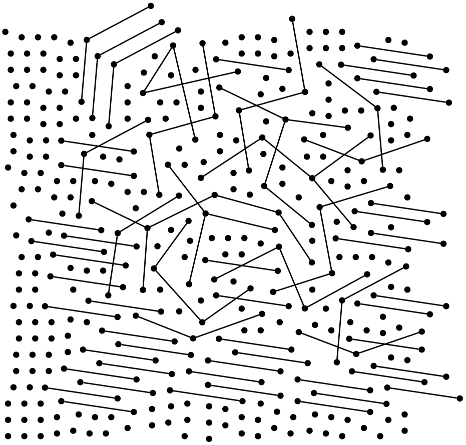
Phase Transitions (Diagram)

Phase Transitions

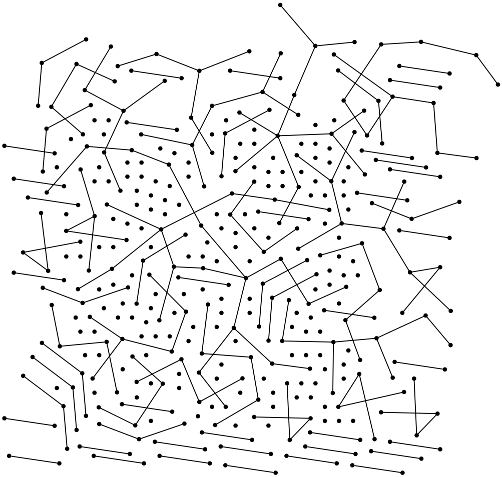
Fun fact: It turns out, physicists are really interested in random graphs since they model physical phenomena very closely.

Phase Transition in $\mathcal{G}_{n,p}$



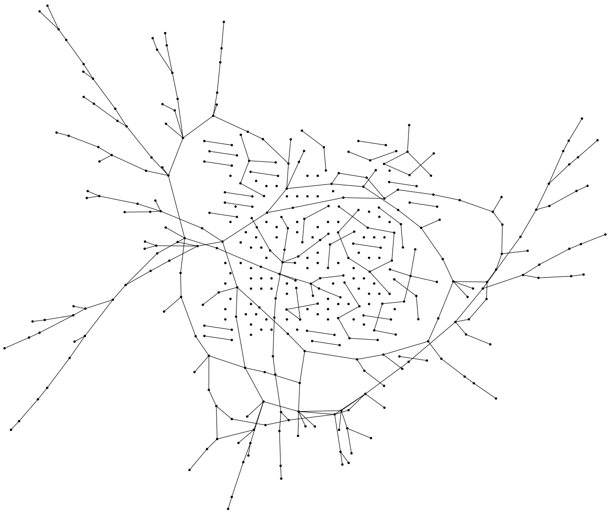
$n = 400$, $p = 0.5/n$. Largest component: 6, second largest component: 5

Phase Transition in $\mathcal{G}_{n,p}$



$n = 400$, $p = 1/n$. Largest component: 24, second largest component: 12

Phase Transition in $\mathcal{G}_{n,p}$



$n = 400$, $p = 1.5/n$. Largest component: 223, second largest component: 9

Phase Transition in $\mathcal{G}_{n,p}$

Let $p = c/n$

- Subcritical Phase $c < 1$: The largest component has size $\Theta(\log n)$.

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This was originally studied by Erdős and Rényi in 1960 [ER60].

$p = 1/n$ is called the 'giant component threshold'.

The proof (of the first two bullet points) I will present follows [JŁR00] Chapter 5, and [AS16] Chapter 11.

Chernoff Bound

If X is the sum of independent 0/1 with mean μ, the probability that X deviates from μ by at least a multiplicative factor[1] is at most

$$e^{-\mathsf{const}\cdot\mu}$$

[1] $\Pr[X \geq c\mu]$ or $\Pr[X \leq c\mu]$, note $c \neq 1$.

Sampling $\mathcal{G}_{n,p}$ component by component

The key is the consider for any vertex, v, what is the probability that it is in a large component?

Since each edge is considered independently from the others, we get to pick an order to consider them in. We'll pick an order that reveals the graph component by component. The method follows a breadth first search starting from v.

- Set $q = [v]$, and $seen = [v]$
- While q is not empty:
  - $u = q.deque()$
  - sample the edges in $\{\{u, w\} : w \in V \setminus seen\}$. If $u \sim w$, enqueue w, and add w to $seen$

Picture

Subcritical Phase, $c < 1$

Let X_i be the number of newly enqueued vertices on the ith step of the algorithm. (i starts at 1)

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- If the component size is greater than k, the queue better not be empty at iteration k! $\implies \sum_{i=1}^{k} X_i \geq k$

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- Thus, $\sum_{i=1}^{k} X_i$ is dominated by $\mathrm{Bin}(kn, c/n)$.
- The mean of $\mathrm{Bin}(kn, c/n)$ is ck. So by Chernoff, $\Pr[\mathrm{Bin}(kn, c/n) \geq k] \leq \exp(-\mathrm{const} \cdot k)$. Picking $k = c_1 \ln(n)$ for an appropriate constant c_1, we can make this probability $o(1/n)$.

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- Union bound over all vertices to show that no vertex is in a component of greater than $k = O(\ln(n))$.

Supercritical Phase, $c > 1$.

Call a vertex...

- Small if it lies in a component of size $< K \ln(n)$
- Large if it lies in a component of size $(y \pm \delta)n$.
- Awkward, otherwise (if it is between $K \ln(n)$, and $(y - \delta)n$, or larger than $(y + \delta)n$).

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Outline of the proof:

1. Show that there are no awkward vertices.
2. Count the number of small vertices (and hence the number of large vertices).
3. Show there's a unique large component.

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- By induction, $N_i \sim \mathrm{Bin}(n-1, (1-p)^i)$
- If the process terminates at time step k (and thus the component it uncovers has size k), it had better be the case that $N_i = n - k$. Thus

$$\begin{aligned}
\Pr(v \text{ is in a component of size } k) &\leq \Pr(\mathrm{Bin}(n-1,(1-p)^k) = n-k). \\
&= \Pr(\mathrm{Bin}(n-1, 1-(1-p)^k) = k-1) \\
&\approx \Pr(\mathrm{Bin}(n, 1-(1-p)^k) = k)
\end{aligned}$$

No Awkward Vertices

A vertex is awkward if it lies in a component of size between $K\ln(n)$, and $(y-\delta)n$, or larger than $(y+\delta)n$.

The probability that a vertex is in a component of size exactly k is at most $\Pr(\mathrm{Bin}(n, 1-(1-p)^k) = k)$.

Let $Y = \mathrm{Bin}(n, 1-(1-c/n)^k)$, and let μ be the mean of Y.

- Case $k = o(n)$. Approximate $1-(1-c/n)^k$ with ck/n. $\mu = ck$. Asking Y to be k is asking it to deviate from it's mean by a constant factor $\rightarrow$ apply Chernoff and pick K to make this probability $o(n^{-10})$.

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- Case $k = xn$. In this case, $1-(1-c/n)^k \approx 1-e^{-cx}$, so $\mu = (1-e^{-cx})n$. Whenever $x \neq (1-e^{-cx})$, we are again asking the RV to deviate from it's mean by a constant factor, this probability is exponentially small in n. (Set y to be the solution to $x = 1-e^{-cx}$).

Number of Large Vertices

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The probability that v is not small is the sandwiched by uniform processes with number of vertices discovered at each step distributed according to $\mathrm{Bin}(n - S, c/n)$, and $\mathrm{Bin}(n, c/n)$. Since for both of these distributions, the limit of the means is c, they are both asympotic to $\mathrm{Po}(c)$.

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Since $S \to \infty$, and c is constant, the probability that the process parameterized by $\mathrm{Po}(c)$ yields a component of size at least S tends towards the probability that the component is infinite.

The Poisson Process

Let z be the probability that the Poisson Process terminates after a finite number of steps this can be written recursively as

$$z = \sum_{k=0}^{\infty} \frac{e^{-c} c^k}{k!} z^i = e^{c(z-1)}$$

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$$z = \sum_{k=0}^{\infty} \frac{e^{-c}c^k}{k!} z^i = e^{c(z-1)}$$

Then, $y = 1 - z$ is the probability the component is infinite is, written in terms of y, the recursion is $y = 1 - e^{-cy}$.

Hey! That's the same constant we found before!

Unique Large component

The sprinkling argument. “Sprinkle” a couple of edges in not enough to mess up any of the analysis that we did, but enough so that two large components are joined w.h.p.

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- Sprinkle: Let $p' = n^{-3/2}$, for any edge not included in the original random graph, include it with probability $n^{-3/2}$. Note the resulting graph is the same as sampling edges from the start with probability $p + p' - pp' \approx p$

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- Since there are $\Omega(n^2)$ edges between the distinct components, the components are joined by sprinking with probability $1 - o(1)$, creating a component of size at least $2(y - \delta)n$, which is awkward!

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Generalizing to $\mathcal{G}_{n,d}$ [MR95]

There's no longer a single parameter p to set the threshold in.

What should the criteria for having a large threshold be instead?

Asymptotic Degree Sequences

In order to talk about degree sequences with $n \to \infty$, we need to generalize degree sequences to families of degree sequences for growing n.

Definition

$d_1, d_2, \ldots$, are integer valued functions such that $d_i(n)$ gives the number of vertices of degree i for a graph on n vertices. Note these must satisfy

- $d_i(n) = 0$ for all $i \geq n$.
- $\sum_i d_i(n) = n$ for all n.

Asymptotic Degree Sequences

We really only want to consider asymptotic degree sequences that are in some sense similar for growing values of n. In particular, we'll require that there exists constants λ_i such that

$$\lim_{n\to\infty} d_i(n)/n = \lambda_i$$

For example, 3-regular, is $\lambda_3 = 1$, and $\lambda_j = 0$ for $j \neq 0$.

Let D be the maximum degree of any graph and let $\lambda_1, ..., \lambda_D$, be the fraction of vertices of each degree. What should the criteria be?

Drift

In $\mathcal{G}_{n,c/n}$, the expected change in the queue size at any is about $c-1$. Thus, when $c>1$, the drift is positive and when $c<1$, the drift is negative.

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The exposure process of a random configuration

To expose a random configuration component by component (a random matching of the half-edges) do the following

- Set all half-edges to available
- Repeat while there are available half-edges:
  - Pick any available vertex and activate all of its half-edges
  - While there are still active half-edges:
    - Pick any active half-edge u_i
    - Pick any available half-edge v_j
    - If v_j was not already active, set all of v's half-edges to active.
    - Add the edge $\{u, v\}$
    - Set u_i and v_j to be unavailable

Drift

What is the expected increase in the the number of active half-edges?

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$$\sum_{i=1}^{L} \text{Pr[Get a vertex of degree i]}(i-2) \approx \frac{\sum_{i=1}^{L} d_i(n) i(i-2)}{\sum_{j=1}^{L} j d_j(n)}$$
$$= \frac{\sum_{i=1}^{L} \lambda_i i(i-2)}{K}$$

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Set $Q = \sum_{i=1}^{L} \lambda_i i(i-2)$. If $Q < 0$, the process has negative drive corresponding to small components only and if $Q > 0$, there is a giant component

Giant component threshold for $\mathcal{G}_{n,\mathbf{d}}$

Theorem ([MR95][2])

Let $\mathbf{d} = (d_1, ...,)$ be an asymptotic degree sequence with maximum degree D. Furthermore, suppose λ_i, for $i \in [D]$ are such that $\lim_{n\to\infty} d_i(n)/n = \lambda_i$. Let $Q = \sum_{i\in[D]} i(i-2)\lambda_i$ Then

- *If $Q < 0$, the largest component has size at most $O(\log(n))$.*
- *If $Q > 0$, there is one component of size $\Theta(n)$, and all other components have size $O(\log(n))$.*

[2] The original theorem was more general, allowing maximum degree up to $n^{1/4-\epsilon}$. This involves several additional conditions and requires the theorem statements to include the maximum degree as an additional parameter.

Proof Ideas

The proof follows an analysis of the exposure process. Here are some general themes:

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- Concentration inequalities.
- You need to handle some complexities like the drift changing over time, and forming self-loops/multi-edges. Do this similarly to before by bounding the process with a uniform one and saying that asymptotically they don't matter.

More stuff

- Bollobás did a more detailed analysis of what happens very close to $c = 1$ [Bol84]. In fact, with a finer parameterization, you can define 'Barely Subcritical' and 'Barely Supercritical'.
- If D, the maximum degree is not required to be constant there is obviously at least a component of degree D. Recently, it was shown that in the subcritical phase, a tight bound on the size of the largest component is $O(D\log(n))$ [CP21], can this be improved for specific degree sequences?

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