Determinant Bound for Discrepancy $\boxed{\overset{a}{2}}$

[Jiang, Reis '21]

$A \in \{0,1\}^{m \times n}$, $m = \mathrm{poly}(n)$

? ? ?

$$\frac{\mathrm{herdisc}(A)}{\mathrm{detlb}(A)} = \underbrace{\Theta\left(\sqrt{\log m \log n}\right)}_{?}$$

?

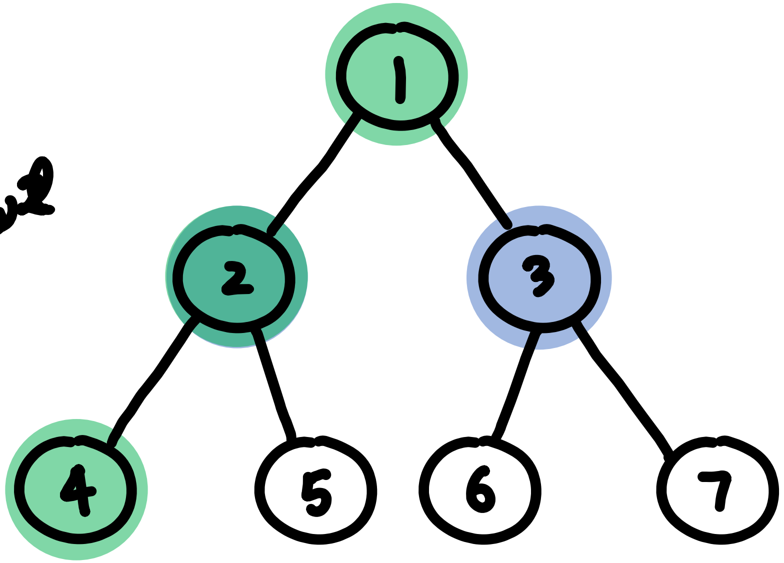
$|X| = n \quad |\mathcal{F}| = m$

set system $(X, \mathcal{F})$ $\qquad A_{\mathcal{F}} \in \{0,1\}^{m \times n}$

$$\mathrm{disc}(\mathcal{F}) = \min_{x \in \{-1,+1\}^n} \|A_{\mathcal{F}} x\|_\infty$$

Hoffman Ex.

$\mathcal{H}_{B_3,2}$



$H_{3,2}$

	1	2	3	4	5	6	7
S_1		1	1				
S_2				1	1		
S_3						1	1
P_1	1	1		1			
P_2	1	1			1		
P_3	1		1			1	
P_4	1		1				1

$\mathcal{H}_k := \mathcal{H}_{k,k} \quad \mathrm{disc}(\mathcal{H}_k) = \Omega(k)$

generally, $A \in \mathbb{R}^{m \times n}$ $\quad \mathrm{disc}(A) = \min_{x \in \{-1,+1\}^n} \|Ax\|_\infty$

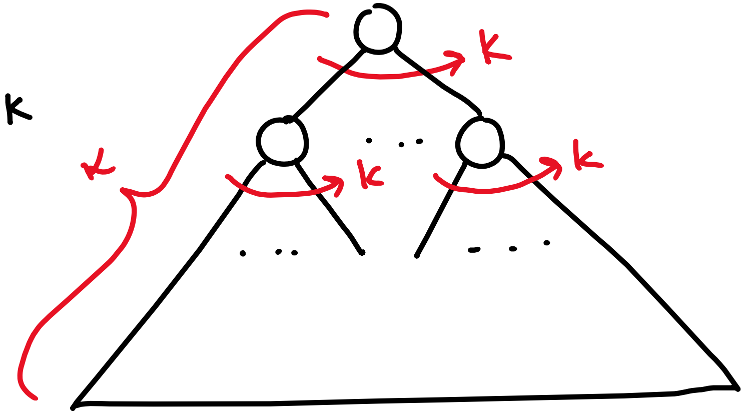
 $\mathrm{disc}([A \;\; A]) = 0$

$\mathrm{herdisc}(A) = \max_{B \in A} \mathrm{disc}(B)$ any submatrix of A

$\mathrm{herdisc}(A) \geq \mathrm{disc}(A)$

$\mathrm{herdisc}(H_k) \geq \Omega(k)$

$\mathcal{H}_k := \mathcal{H}_{k,k}$



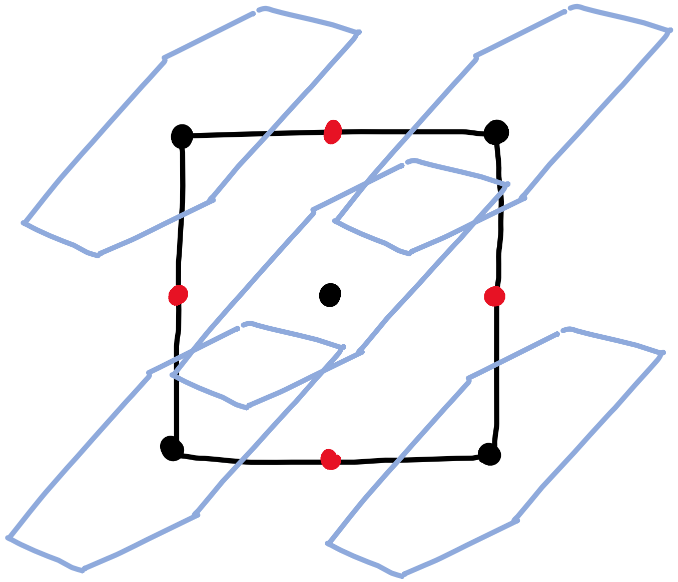
[LOVÁSZ, SPENCER, VESTERGOMBI, '86]

$$\mathrm{disc}(A) = \min_{x \in \{-1,+1\}^n} \|Ax\|_\infty$$

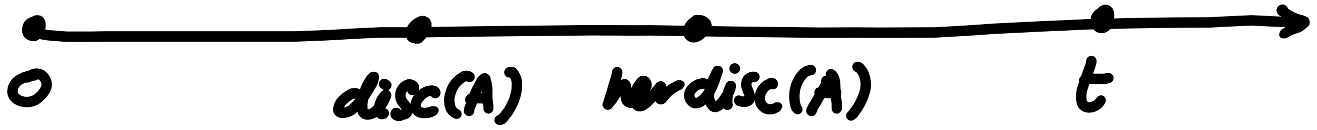
$$\mathrm{herdisc}(A) \geq \tfrac{1}{2}\,\mathrm{detlb}(A)$$

$$\mathrm{herdisc}(A) = \max_{B \in A} \mathrm{disc}(B)$$

$$\max_{k} \max_{[B]_{k\times k} \in A} |\det(B)|^{1/k}$$



$$U_A = \{u \in \mathbb{R}^n : \|Au\|_\infty \leq 1\}$$



$$\underset{(1)}{\mathrm{detlb}(A) \leq t} \underset{(2)}{\leq} 2\,\mathrm{herdisc}(A)$$

[LOVÁSZ, SPENCER, VESTERGOMBI, '86]

$\mathrm{disc}(A) = \min_{x \in \{-1,+1\}^n} \|Ax\|_\infty$

$\mathrm{herdisc}(A) = \max_{B \in A} \mathrm{disc}(B)$

$\mathrm{detlb}(A) = \max_{\substack{k,\, B\in A \\ [B]_{k\times k}}} |\det(B)|^{1/k}$

$\mathrm{herdisc}(A) \geq \frac{1}{2}\,\mathrm{detlb}(A)$

scaling α so that $\bigcup_{x\in\{-1,+1\}^n} \alpha\mathcal{U}_A + x$ covers all points $\{-1,0,1\}^n$

$$\mathrm{detlb}(A) \overset{(1)}{\leq} t \overset{(2)}{\leq} 2\,\mathrm{herdisc}(A)$$

$[A]_{n\times n}$

$t =$ scaling so that $\bigcup_{x\in\{-1,+1\}^n} t\mathcal{U}_A + x$ covers $[-1,1]^n$

$\mathcal{U}_A = \{u \in \mathbb{R}^n : \|Au\|_\infty \leq 1\}$

(1) $A\mathcal{U}_A = [-1,1]^n \Rightarrow \mathcal{U}_A = A^{-1}[-1,1]^n$

$$\begin{aligned}\mathrm{vol}(t\mathcal{U}_A) &= t^n \mathrm{vol}(\mathcal{U}_A) \\ &= t^n \mathrm{vol}(A^{-1}[-1,1]^n) = t^n 2^n |\det(A)|^{-1} \\ &\geq \mathrm{vol}([-1,1]^n) = 2^n \quad \Rightarrow t \geq |\det(A)|^{1/n}\end{aligned}$$

[LOVÁSZ, SPENCER, VESTERGOMBI, '86]

$\mathrm{disc}(A) = \min_{x \in \{-1,+1\}^n} \|Ax\|_\infty$

$\mathrm{herdisc}(A) = \max_{B \in A} \mathrm{disc}(B)$

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scaling α so that $\bigcup_{x \in \{-1,+1\}^n} \alpha U_A + x$ covers all points $\{-1,0,1\}^n$

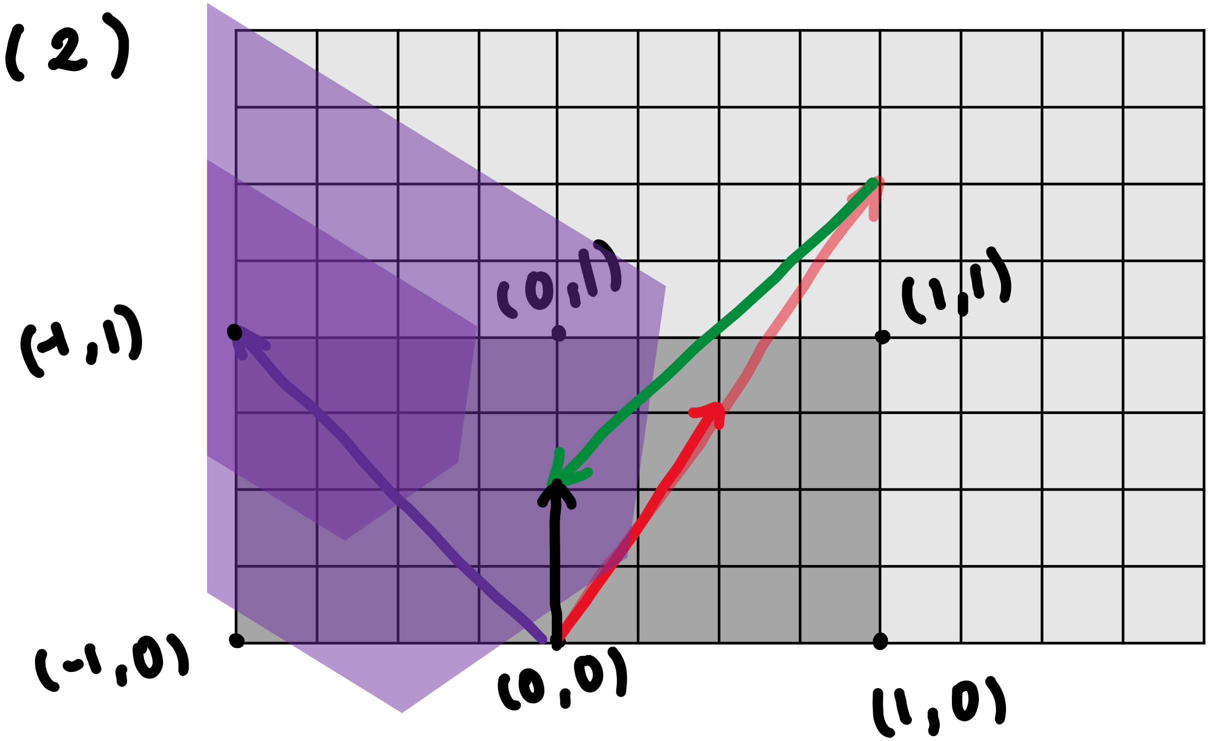
$\mathrm{herdisc}(A) \geq \frac{1}{2}\, \mathrm{detlb}(A)$

$$\mathrm{detlb}(A) \overset{(1)}{\leq} t \overset{(2)}{\leq} 2\, \mathrm{herdisc}(A)$$

$t =$ scaling so that $\bigcup_{x \in \{-1,+1\}^n} tU_A + x$ covers $[-1,1]^n$

$U_A = \{u \in \mathbb{R}^n : \|Au\|_\infty \leq 1\}$

(2)



Induct on k

Show $v \in 2U + x$

$v = a/2^k$ $\quad b \quad a \quad c \in \{\pm 1\}^n$

$2v - b \in 2U + \cancel{\{\pm1\}^n}\, a \Rightarrow v \in U + \frac{a+b}{2}$

$\Rightarrow v \in U + (U + c) \in 2U + c$

[Matoušek '12]

$$\frac{\mathrm{herdisc}(A)}{\mathrm{detlb}(A)} \leq O\left(\log mn \sqrt{\log n}\right)$$

$$\mathrm{disc}(A) = \min_{x \in \{-1,+1\}^n} \|Ax\|_\infty$$

$$\mathrm{herdisc}(A) = \max_{B \subseteq A} \mathrm{disc}(B)$$

$$\mathrm{detlb}(A) = \max_{\substack{k,\, B \subseteq A \\ [B]_{k \times k}}} |\det(B)|^{1/k}$$

(1) $\mathrm{herdisc}(A) \lesssim \log mn \cdot \mathrm{her}\underbrace{\mathrm{vecdisc}}(A)$ (Bansal '10)

$$\max_{\substack{k,\, B \subseteq A \\ [B]_{k \times k}}} \min_{\substack{x_1, \ldots, x_n \in \mathbb{R}^n \\ \|x_i\|^2 = 1}} \max_{j \in [m]} \left\| \sum_{i=1}^n B_{ji} x_i \right\|_2$$

(2) $\mathrm{hervecdisc}(A) \lesssim \sqrt{\log n}\ \mathrm{detlb}(A)$

$A \in \{0,1\}^{m\times n}$, $m = \mathrm{poly}(n)$

[LSV '86] $$\frac{\mathrm{herdisc}(A)}{\mathrm{detlb}(A)} \geq 1/2$$

[M '12] $$\frac{\mathrm{herdisc}(A)}{\mathrm{detlb}(A)} = O\left(\log mn \sqrt{\log n}\right)$$

$$\mathrm{herdisc}(A) \leq \cancel{\log mn}\ \sqrt{\log m \log n} \cdot \text{part}\ \mathrm{herveedisc}(A)$$

$$\text{part}\ \mathrm{hervecdisc}(A) \leq \cancel{\sqrt{\log n}}\ 1\ \mathrm{detlb}(A)$$

[Jiang, Reis '21] $$\frac{\mathrm{herdisc}(A)}{\mathrm{detlb}(A)} = \Theta\left(\sqrt{\log m \log n}\right)$$

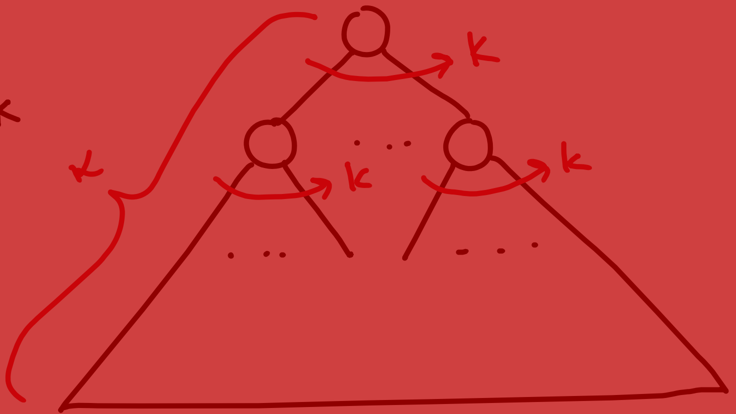
$$\mathrm{disc}(A) = \min_{x \in \{-1,+1\}^n} \|Ax\|_\infty$$

$$\mathrm{herdisc}(A) = \max_{B \in A} \mathrm{disc}(B)$$

$$\mathrm{detlb}(A) = \max_{\substack{k,\, B \in A \\ [B]_{k\times k}}} |\det(B)|^{1/k}$$

$$\mathrm{vecdisc}(A) = \min_{\substack{x_1,\dots,x_n \in \mathbb{R}^t \\ \|x_i\| = 1}} \max_{j \in [m]} \left\|\sum_{i=1}^{n} A_{ji} x_i\right\|_2$$

$\mathcal{H}_k := \mathcal{H}_{k,k}$



$H_k \in \{0,1\}^{m \times n}$

$m = n = k^k - 1 \approx k$

$\text{herdisc}(H_k) = \Omega($ ... $= O(1)$

To Be Continued...

$$\frac{\text{herdisc}(H_k)}{\text{detlb}(H_k)} = \Omega\left(\log n / \log\log n\right)$$

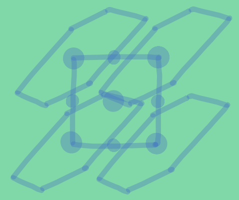
Other examples achieve tight bound

[Jiang, Reis '21] $\frac{\text{herdisc}(A)}{\text{detlb}(A)} = \Theta\left(\sqrt{\log m \log n}\right)$

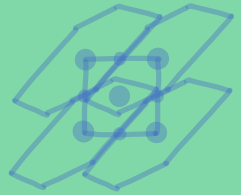
$A \in \{0,1\}^{m \times n}$, $m = \text{poly}(n)$

What happens when $m >> n$?

$A \in \{0,1\}^{m \times n}$, $m = \mathrm{poly}(n)$



$$\mathrm{disc}(A) = \min_{x \in \{-1,+1\}^n} \|Ax\|_\infty$$



$$\mathrm{herdisc}(A) = \max_{B \in A} \mathrm{disc}(B)$$

$$\mathrm{detlb}(A) = \max_{\substack{k,\, B \in A \\ [B]_{k \times k}}} |\det(B)|^{1/k}$$

$$\mathrm{vecdisc}(A) = \min_{\substack{x_1,\dots,x_n \in \mathbb{R}^t \\ \|x_i\| = 1}} \max_{j \in [m]} \left\| \sum_{i=1}^{n} A_{ji} x_i \right\|_2$$

[LSV '86] $\dfrac{\mathrm{herdisc}(A)}{\mathrm{detlb}(A)} \geq 1/2$

[M '12] $\dfrac{\mathrm{herdisc}(A)}{\mathrm{detlb}(A)}$

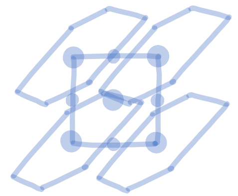
Last Time...

$\mathrm{herdisc}(A) \leq \sqrt{\log m \log n} \cdot \mathrm{part\ hervecdisc}(A)$

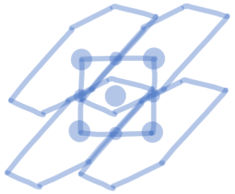
$\mathrm{part\ hervecdisc}(A) \leq \mathrm{detlb}(A)$

[Jiang, Reis '21] $\dfrac{\mathrm{herdisc}(A)}{\mathrm{detlb}(A)} = \Theta\left(\sqrt{\log m \log n}\right)$

$A \in \{0,1\}^{m\times n}$, $m = \text{poly}(n)$



$$\text{disc}(A) = \min_{x\in\{-1,+1\}^n} \|Ax\|_\infty$$



$$\text{herdisc}(A) = \max_{B\in A} \text{disc}(B)$$

$$\text{detlb}(A) = \max_{\substack{k,\, B\in A \\ [B]_{k\times k}}} |\det(B)|^{1/k}$$

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[M '12] $$\frac{\text{herdisc}(A)}{\text{detlb}(A)} = O\left(\log mn \sqrt{\log n}\right)$$

$$\text{herdisc}(A) \leq \sqrt{\log m \log n} \cdot \text{part hervecdisc}(A)$$

$$\text{part hervecdisc}(A) \leq \text{detlb}(A)$$

[Jiang, Reis '21] $$\frac{\text{herdisc}(A)}{\text{detlb}(A)} = \Theta\left(\sqrt{\log m \log n}\right)$$

Goal: $[A]_{m \times n}$ $m \gg n$

$$\frac{\text{herdisc}(A)}{\text{detlb}(A)} = \Omega\left(\sqrt{\log m \log n}\right)$$

$$\text{disc}(A) = \min_{x \in \{-1,+1\}^n} \|Ax\|_\infty$$

$$\text{herdisc}(A) = \max_{B \in A} \text{disc}(B)$$

$$\text{detlb}(A) = \max_{\substack{k,\, B \in A \\ [B]_{k \times k}}} |\det(B)|^{1/k}$$

$$A = P \otimes A'$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes [A']_{q \times q} = \begin{bmatrix} A' & A' \\ A' & 0 \\ 0 & A' \\ 0 & 0 \end{bmatrix}$$

① detlb amplification

$$\text{detlb}(A) \lesssim \sqrt{N}\, \text{detlb}(A')$$

② disc. amplification

$$\text{disc}(A) \gtrsim N \underbrace{\text{disc}_1(A')}_{\min_{x \in \{\pm 1\}^n} \|A'x\|_1 / m}$$

Haar Basis (discrete) [Kunisky '21]

$$A_0 = \begin{bmatrix} 1 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad A_2 = \begin{array}{c} \begin{matrix} W & X & Y & Z \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \end{bmatrix} \end{array} \qquad A_k = \begin{bmatrix} A_{k-1} & I \\ A_{k-1} & -I \end{bmatrix}$$

$$\mathrm{disc}(A) = \min_{x \in \{\pm 1\}^n} \|Ax\|_\infty$$

$$\mathrm{disc}_1(A) = \min_{x \in \{\pm 1\}^n} \|Ax\|_1 / m$$

$$\mathrm{herdisc}(A) = \max_{B \subseteq A} \mathrm{disc}(B)$$

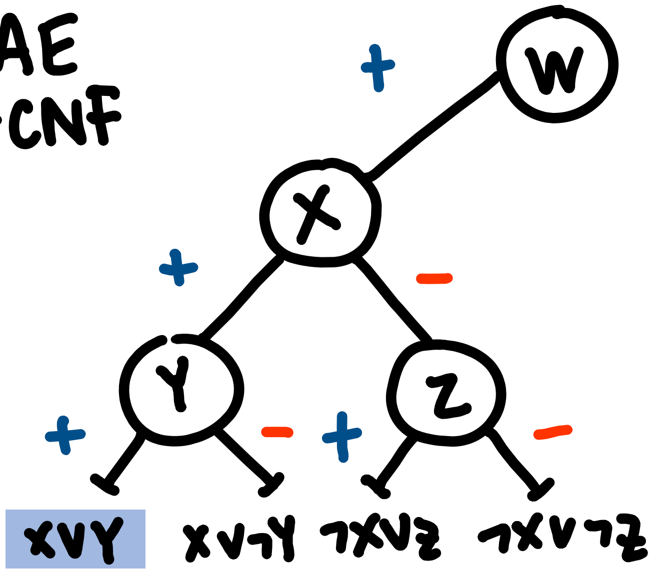
$$\mathrm{detlb}(A) = \max_{\substack{k,\, B \subseteq A \\ [B]_{k \times k}}} |\det(B)|^{1/k}$$

$A = P_N \otimes A'$ $\quad [A]_{m \times n}$ $\quad m \gg n$

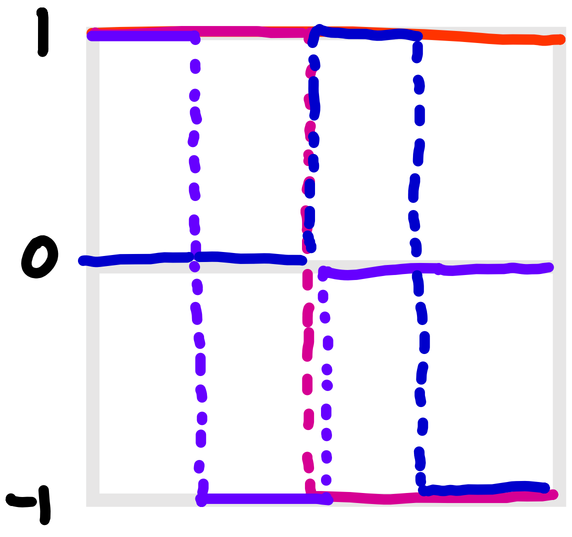
1. $\mathrm{detlb}(A) \lesssim \sqrt{N}\, \mathrm{detlb}(A')$
2. $\mathrm{disc}(A) \gtrsim N\, \mathrm{disc}_1(A')$

GOAL: $\dfrac{\mathrm{herdisc}(A)}{\mathrm{detlb}(A)} = \Omega\left(\sqrt{\log m \log n}\right)$

NAE
k-CNF



Haar Wavelet



1. $\mathrm{detlb}(A_K) \lesssim 1 \Rightarrow \mathrm{detlb}(A) \lesssim \sqrt{N}$

2. $\mathrm{disc}_1(A_K) \gtrsim \sqrt{K} \Rightarrow \mathrm{disc}(A) \gtrsim N\sqrt{K}$

$$\frac{\mathrm{herdisc}(A)}{\mathrm{detlb}(A)} \geq \frac{\mathrm{disc}(A)}{\mathrm{detlb}(A)} \gtrsim \sqrt{NK}$$

$$\gtrsim \sqrt{\log m \log n}$$

$[A_K]_{2^K \times 2^K} \quad [A]_{m \times n}$

$m = 2^{N+K} \qquad n = N 2^K$

$\log m = N + K \qquad \log n = \log N + K$

SET $N = 2^K$ $\qquad = \Theta(N) \qquad = \Theta(K)$

$\mathrm{disc}(A) = \min_{x \in \{\pm 1\}^n} \|Ax\|_\infty$

$\mathrm{disc}_1(A) = \min_{x \in \{\pm 1\}^n} \|Ax\|_1 / m$

$\mathrm{herdisc}(A) = \max_{B \in A} \mathrm{disc}(B)$

$\mathrm{detlb}(A) = \max_{\substack{k,\, B \in A \\ [B]_{k \times k}}} |\det(B)|^{1/k}$

$A = P_N \otimes A' \quad [A]_{m \times n} \quad m \gg n$

1. $\mathrm{detlb}(A) \lesssim \sqrt{N}\, \mathrm{detlb}(A')$

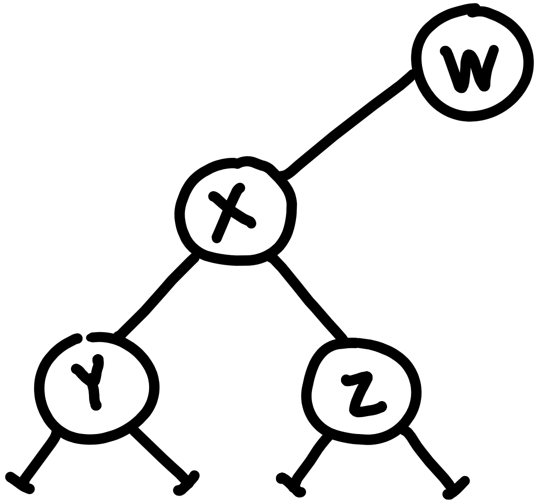
2. $\mathrm{disc}(A) \gtrsim N \, \mathrm{disc}_1(A')$

GOAL: $\frac{\mathrm{herdisc}(A)}{\mathrm{detlb}(A)} = \Omega\left(\sqrt{\log m \log n}\right)$

$$A' := A_k = \begin{bmatrix} A_{k-1} & I \\ A_{k-1} & -I \end{bmatrix}$$

1. $\mathrm{detlb}(A_k) \lesssim 1$

2. $\mathrm{disc}_1(A_k) \gtrsim \sqrt{k}$



$$A_2 \quad \begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \end{bmatrix} \end{array}$$

$$\mathrm{disc}_1(A_k)$$

$$= \min_{x \in \{\pm 1\}^k} \frac{1}{2^k} \|A_k x\|_1$$

$$= \min_x \mathbb{E}_a |a^T x|$$

$$= \frac{1}{2^k} \sum_{\ell=0}^{k} \binom{k}{\ell} |k - 2\ell|$$

$$= \frac{1}{2^k}\left(\sum_{\ell < k/2} \binom{k}{\ell}(k-2\ell) - \sum_{\ell > k/2} \binom{k}{\ell}(k-2\ell) \right)$$

$$= 2/2^k \left(\cdots \sum_{\ell < k/2} \binom{k}{\ell} \ell \right)$$

MATH

$$= 2k/2^k \left(\sum_{\ell > k/2} \binom{k-1}{\ell-1} - \sum_{\ell < k/2} \binom{k-1}{\ell-1} \right)$$

$$= 2k \binom{k-1}{\lfloor k/2 \rfloor} / 2^k \approx \sqrt{k}$$

$$\mathrm{disc}(A) = \min_{x \in \{\pm 1\}^n} \|Ax\|_\infty$$

$$\mathrm{disc}_1(A) = \min_{x \in \{\pm 1\}^n} \|Ax\|_1 / m$$

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1. $\operatorname{detlb}(A_k) \lesssim 1 \Rightarrow \operatorname{detlb}(A) \lesssim \sqrt{N}$

2. $\operatorname{disc}_1(A_k) \gtrsim \sqrt{k} \Rightarrow \operatorname{disc}(A) \gtrsim N\sqrt{k}$

$$\frac{\operatorname{herdisc}(A)}{\operatorname{detlb}(A)} \geq \frac{\operatorname{disc}(A)}{\operatorname{detlb}(A)} \gtrsim \sqrt{Nk}$$

$$\gtrsim \sqrt{\log m \log n}$$

$[A_k]_{2^k \times 2^k}$ $[A]_{m \times n}$

$m = 2^{N+k}$ $n = N 2^k$

$\log m = N + k$ $\log n = \log N + k$

SET $N = 2^k$ $= \Theta(N)$ $= \Theta(k)$

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