A COUNTEREXAMPLE TO BORSUK'S CONJECTURE

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ABSTRACT. Let $f(d)$ be the smallest number so that every set in R^d of diameter 1 can be partitioned into $f(d)$ sets of diameter smaller than 1. Borsuk's conjecture was that $f(d) = d+1$. We prove that $f(d) \geq (1.2)^{\sqrt{d}}$ for large d.

1. INTRODUCTION

Sixty years ago Borsuk [2] raised the following question.

Problem 1 (Borsuk). Is it true that every set of diameter one in R^d can be partitioned into $d + 1$ closed sets of diameter smaller than one? The conjecture that this is true has come to be called Borsuk's conjecture.

Let $f(d)$ be the smallest number so that every set in R^d of diameter 1 can be partitioned into $f(d)$ sets of diameter smaller than 1. The vertices of the regular simplex in R^d show that $f(d) \geq d + 1$. (Another example showing this is, by the Borsuk-Ulam theorem, the d-dimensional Euclidean ball.) The assertion of Borsuk's conjecture was proved in dimensions 2 and 3 and in all dimensions for centrally symmetric convex bodies and smooth convex bodies. See [9, 1, 4] and references cited there. Lassak [14] proved that $f(d) \leq 2^{d-1} + 1$, and Schramm [16] showed that for every ϵ, if d is sufficiently large, $f(d) \leq (\sqrt{(3/2)} + \epsilon)^d$. A different proof of Schramm's bound was given by Bourgain and Lindenstrauss [3]. See [9, 1, 4] for surveys and many references on Borsuk's problem.

Borsuk's conjecture seems to have been believed generally, and various stronger conjectures have been proposed. The possibility of a counterexample based on combinatorial configurations was suggested by Erdős [6], Larman [15], and perhaps others. In 1965 Danzer [5] showed that the finite set $K \subseteq R^d$ consisting of all $\{0, 1\}$-vectors of an appropriate weight cannot be covered by $(1.003)^d$ *balls* of smaller diameter. Larman [13] observed that, for sets consisting of 0-1 vectors with constant weight, Borsuk's conjecture reduces to:

Conjecture 1. Let K be a family of k-subsets of $\{1, 2, ..., n\}$ such that every two members of K have t elements in common. Then K can be partitioned into n parts so that in each part every two members have $(t + 1)$ elements in common.

Here we prove

Theorem 1. *For large enough d, $f(d) \geq (1.2)^{\sqrt{d}}$ by constructing an appropriate family of sets.*

We need the following result of Frankl and Wilson [8].

Theorem 2 (Frankl and Wilson). *Let k be a prime power and $n = 4k$. Let K be a family of $n/2$-subsets of $\{1, 2, ..., n\}$, so that no two sets in the family have intersection of size $n/4$. Then*

$$|K| \leq 2 \cdot \binom{n-1}{n/4-1}.$$

This settled, in particular, a (much weaker) conjecture of Larman and Rogers [12] and implies that, if $g(d)$ is the smallest number so that R^d can be colored by $g(d)$ colors such that no two points of the same color are distance one apart, then $g(d) \geq (1.2)^d$.

Let us also mention the following related result conjectured by Erdős and proved by Frankl and Rödl [7].

Theorem 3 (Frankl and Rödl). *Let n be a positive integer divisible by four. Let K be a family of $n/2$-subsets of $\{1, 2, ..., n\}$ such that no two sets in the family have intersection of size $n/4$. Then $|K| \leq (1.99)^n$.*

2. The construction

However contracted, that definition is the result of expanded meditation.

—Herman Melville, *Moby Dick*

Let $V = \{1, 2, ..., m\}$, and $m = 4k$, and k is a prime power. Let W be the set of pairs of elements in V. For every partition $P = \{A, B\}$ of V let $S(A, B)$ be the sets of all pairs which contain one element from A and one element from B. Let K be the family of all sets of pairs which correspond to partitions of V into two *equal* parts, i.e., $K = \{S(A, B) : |A| = 2k\}$. Thus, K is a family of $(m^2/4)$-subsets of an $m(m-1)/2$-set. The smallest intersection between $S(A, B)$ and $S(C, D)$ occurs when $|A \cap C| = k$, and by the Frankl-Wilson theorem every subfamily of more than $2 \cdot \binom{m-1}{m/4-1}$ sets in K contains two sets which realize the minimal distance. Thus, K cannot be partitioned into fewer than

$$\frac{\frac{1}{2}\binom{m}{m/2}}{2 \cdot \binom{m-1}{m/4-1}}$$

parts so that the minimal intersection is not realized in any of the parts. This expression is greater than $(1.203)^{\sqrt{d}}$ for sufficiently large $d = \binom{m}{2} - 1$, and Theorem 1 for general (large) d follows via the prime number theorem.

3. Remarks

1. In view of Theorem 1, the upper bounds on $f(d)$ cited earlier seem much more reasonable than formerly. It would be of considerable interest to have a better understanding of the asymptotic behavior of $\log f(d)$. At the moment, we cannot distinguish the asymptotic behavior of $f(d)$ from that of $g(d)$. Also of interest would be counterexamples in small dimensions. Our construction shows that Borsuk's conjecture is false for $d = 1,325$ and for every $d > 2,014$.

2. Larman's conjecture for $t = 1$ is open and still quite interesting, in part because of its similarity to the Erdős-Faber-Lovász conjecture. See [11, 10] for some discussion and related results.

3. Intersection properties of edge-sets of graphs were first studied by Sós; see [17] and references quoted therein.

References

1. V. Boltjansky and I. Gohberg, *Results and problems in combinatorial geometry*, Cambridge Univ. Press, Cambridge, 1985.
2. K. Borsuk, *Drei Sätze über die n-dimensionale euklidische Sphäre*, Fund. Math. **20** (1933), 177–190.
3. J. Bourgain and J. Lindenstrauss, *On covering a set in R^d by balls of the same diameter*, Geometric Aspects of Functional Analysis (J. Lindenstrauss and V. Milman, eds.), Lecture Notes in Math., vol. 1469, Springer-Verlag, Berlin, 1991, pp. 138–144.
4. H. Croft, K. Falconer, and R. Guy, *Unsolved problems in geometry,* Springer-Verlag, New York, 1991, pp. 123–125.
5. L. Danzer, *On the k-th diameter in E^d and a problem of Grünbaum*, Proc. Colloq. on Convexity 1965 (W. Fenchel, ed.), Københavns Univ. Math. Inst., 1967.
6. P. Erdős, *My Scottish book "problems"*, The Scottish Book, Mathematics from the Scottish Café (R. D. Mauldin, ed.), Birkhäuser, 1981, pp. 35–43.
7. P. Frankl and V. Rödl, *Forbidden intersections*, Trans. Amer. Math. Soc. **300** (1987), 259–286.
8. P. Frankl and R. Wilson, *Intersection theorems with geometric consequences*, Combinatorica **1** (1981), 357–368.
9. B. Grünbaum, *Borsuk's problem and related questions*, Proc. Sympos. Pure Math., vol. 7, Amer. Math. Soc, Providence, RI, 1963.
10. J. Kahn and G. Kalai, *A problem of Füredi and Seymour on covering intersecting families by pairs* (to appear).
11. J. Kahn and P. Seymour, *A fractional version of the Erdős-Faber-Lovász conjecture*, Combinatorica **12** (1992), 155–160.
12. D. Larman and C. Rogers, *The realization of distances within sets in Euclidean space*, Mathematika **19** (1972), 1–24.
13. D. Larman, *Open problem* 6, Convexity and Graph Theory (M. Rozenfeld and J. Zaks, eds.), Ann. Discrete Math., vol. 20, North-Holland, Amsterdam and New York, 1984, p. 336.
14. M. Lassak, *An estimate concerning Borsuk's partition problem*, Bull. Acad. Polon. Sci. Ser. Math. **30** (1982), 449–451.
15. C. A. Rogers, *Some problems in the geometry of convex bodies*, The Geometric Vein—The Coxeter Festschrift (C. Davis, B. Grünbaum, and F. A. Sherk, eds.), Springer-Verlag, New York, 1981, pp. 279–284.
16. O. Schramm, *Illuminating sets of constant width*, Mathematika **35** (1988), 180–199.
17. M. Simonovits and V. Sós, *Graph intersection theorems*. II, Combinatorics (A. Hajnal and V. Sós, eds.), North-Holland, Amsterdam, 1978, pp. 1017–1030.

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