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Intersection Graphs of Paths in a Tree

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The intersection graph for a family of sets is obtained by associating each set with a vertex of the graph and joining two vertices by an edge exactly when their corresponding sets have a nonempty intersection. Intersection graphs arise naturally in many contexts, such as scheduling conflicting events, and have been widely studied.

We present a unified framework for studying several classes of intersection graphs arising from families of paths in a tree. Four distinct classes of graphs arise by considering paths to be the sets of vertices or the edges making up the path, and by allowing the underlying tree to be undirected or directed; in the latter case only directed paths are allowed. Two further classes are obtained by requiring the directed tree to be rooted. We introduce other classes of graphs as well. The rooted directed vertex path graphs have been studied by Gavril; the vertex path graphs have been studied by Gavril and Renz; the edge path graphs have been studied by Golumbic and Jamison, Lobb, Syslo, and Tarjan.

The main results are a characterization of these graphs in terms of their "clique tree"representations and a unified recognition algorithm. The algorithm repeatedly separates an arbitrary graph by a (maximal) clique separator, checks the form of the resultant nondecomposable "atoms," and finally checks that each separation step is valid. In all cases, the first two steps can be performed in polynomial time. In all but one case, the final step is based on a certain two-coloring condition and so can be done efficiently; in the other case the recognition problem can be shown to be NP-complete since a certain three-coloring condition is needed.

The strength of this unified approach is that it clarifies and unifies virtually all of the important known results for these graphs and provides substantial new results as well. For example, the exact intersecting relationships among these graphs, and between these graphs and chordal and perfect graphs fall out easily as corollaries. A number of other results, such as bounds on the number of (maximal) cliques, related optimization problems on these graphs, etc., are presented along with open problems.

1. INTRODUCTION

Throughout this paper we let $G=(V, E)$ denote a finite connected undirected graph, where V is a set of *vertices* and E is a collection of pairs of vertices called *edges*. Two vertices joined by an edge are called *adjacent*.

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For any subset W of V, the *subgraph* $G(W)$ of G induced by W is the graph whose vertices are given by W with two vertices adjacent exactly when they are adjacent in G. A *completely connected set of* G is a subset of vertices with each pair adjacent in G. A *clique* of G is a maximal, completely connected set of G. We let $\mathbf{C}$ denote the set of cliques of G, and for every $v \in V$ we let $\mathbf{C}_v = \{C \in \mathbf{C}\colon v \in C\}$. A *clique cover* of G is a subset of cliques of G which includes all vertices. Finally, an *independent set* of G is a subset of vertices each pair of which is not adjacent, and a *coloring* of G is an assignment of labels to the vertices of G so that similarly labeled vertices form an independent set.

Let $\mathbf{P}$ be a finite family of nonempty sets. The intersection graph of $\mathbf{P}$ is obtained by associating each set in $\mathbf{P}$ with a vertex and connecting two vertices with an edge exactly when their corresponding sets have a nonempty intersection. A number of papers and books [8–10, 14, 17, 20, 21] have been written about various classes of intersection graphs. Problems and applications of these graphs to areas such as scheduling, genetics, archeology, ecology, and relational database systems can be found in [17, 21].

Often the family of sets is obtained by taking subsets from a given underlying topological structure such as connected subsets of points on a line or on a circle. In these cases the intersection graphs are interval graphs [5, 13] and circular arc graphs [28], respectively.

We consider a tree as the underlying structure and let the family of sets correspond to paths in the tree. The tree can be thought of as a computer communication network with paths representing messages between users. This gives rise to several different classes of intersection graphs. For example, taking a path to be the set of edges making up the path leads to a class of graphs quite different from taking a path to be the set of corresponding vertices; the former case has transmission links as the bottleneck in the computer network example, and the latter has switching nodes as the bottleneck. Another variation is to direct the edges in the tree and allow only directed paths in the directed tree; the directed tree may be required to be rooted. A directed tree is a *rooted tree* if it has exactly one vertex (the *root*) with in-degree zero.

The intersection graph of a family of undirected (directed) vertex paths in a undirected (directed) tree is called a *undirected* (*directed*) *vertex path graph*, or *UV*(*DV*) *graph* for short. A DV graph with a rooted tree representation is called a *rooted directed vertex path graph*, or *RDV graph*. The RDV graphs are characterized in [9] where an efficient algorithm (i.e., polynomial time) is given for recognizing them and constructing a representation. UV graphs are characterized in [10, 20] and an efficient recognition algorithm is given in [10]. It is well known [8, 10] that UV and DV graphs are contained in the class of *chordal graphs* [3, 8]. A *chor-*

dal graph is defined to be a graph which contains no chordless cycle of length exceeding three; chordal graphs can be recognized efficiently [5]. The minimum coloring, maximum clique, minimum clique cover, and maximum independent set problems can be solved efficiently [7] for chordal graphs.

The intersection graph of a family of edge paths in a tree is called an *undirected edge path graph*, or *UE graph* for short. A number of properties of UE graphs are derived in [15, 16, 19, 22–24]. It has been shown that recognizing UE graphs is NP-complete [16]. (We refer the reader to [6] for an excellent introduction to the theory of NP-completeness.) The minimum coloring and minimum clique cover problems are shown to be NP-hard [15] for UE graphs even given their UE representations. The maximum weight clique problem [15] and the maximum weight independent set problem [25] can both be solved in polynomial time.

We now introduce a natural subclass of UE graphs. The intersection graph of a family of directed edge paths in a directed tree is called a *directed edge path graph*, or *DE graph* for short. As before, a DE graph with a rooted tree representation is called a *rooted directed edge path graph* or *RDE graph*. These six classes of graphs differ only by interchanging the words "rooted," "directed," and "undirected," and interchanging "vertex" and "edge."

Given the close correspondence between these classes of graphs, one would suspect that a unified approach could be used to study these graphs. We propose to show that this is largely true.

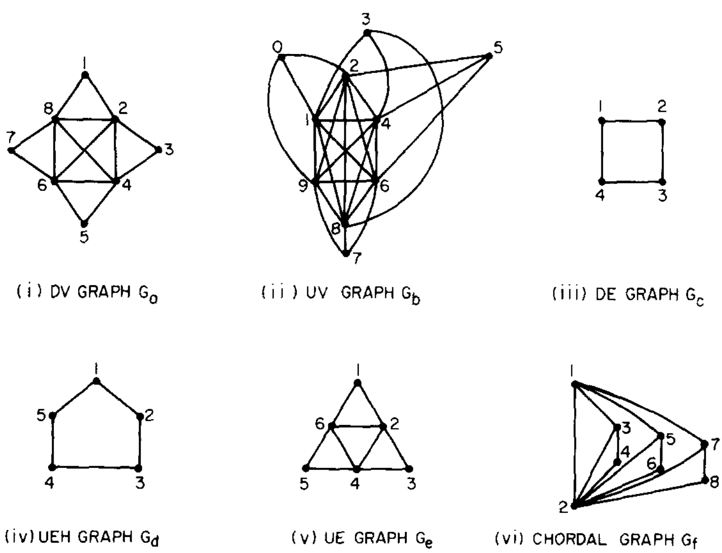


FIG. 1. Examples of path graphs.

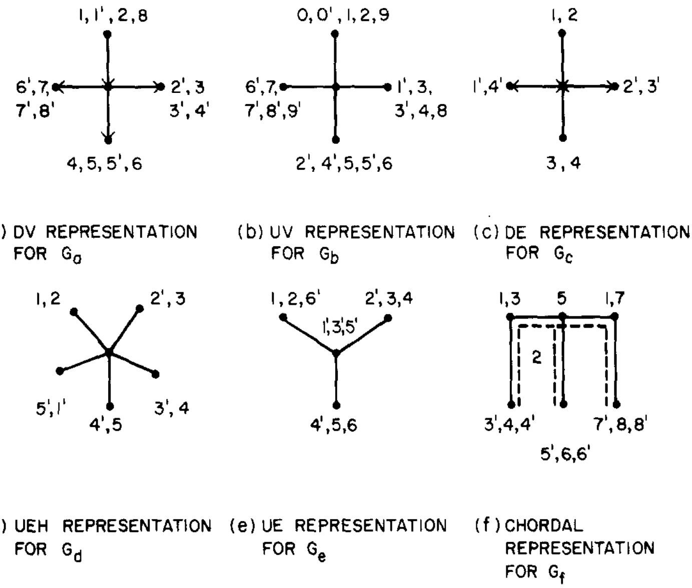


FIG. 2. Representations for the path graphs of Fig. 1. (Endpoints of path i are labeled i and i'.)

In order to obtain completely parallel results we need to define another subclass of UE graphs. We call a graph a *UEH graph* if it can be represented as the intersection graph of a collection of edge paths $\mathbf{P}$ in a tree, where $\mathbf{P}$ satisfies the *Helly property*; namely for any subcollection $\mathbf{P}'$ of $\mathbf{P}$, $P_i \cap P_j \neq \emptyset$ for all $P_i, P_j \in \mathbf{P}'$ implies that $\bigcap_{P_i \in \mathbf{P}'} P_i \neq \emptyset$. It is well known [14] that vertex paths satisfy the Helly property; we show in Section 2 that directed edge paths also satisfy the Helly property. It is easy to see that edge paths need not satisfy the Helly property.

See Fig. 1 for examples of these various path graphs. Figure 2 contains the representations for the graphs in Fig. 1. (Figure 2e shows edge paths 2, 4, and 6, which do not satisfy the Helly property.)

In Section 2 we show that all these graphs (except UE) can be characterized in terms of their "clique tree" representations. These characterizations are completely parallel differing only by interchanging the words "rooted," "directed," and "undirected," and interchanging "vertex" and "edge." We also show that RDV = RDE $\subset$ DV $\subset$ UV $\subset$ CHORDAL and DV $\subset$ DE $\subset$ UEH $\subset$ UE; these inclusions are all strict. (Note that the letters "DV" stand for "directed vertex path" when used in "DV graphs," and they also denote the entire class of such graphs when used as sets such as in the above sentence. The letters "UV," "UEH," "CHORDAL," etc., are used in a similar manner.)

It is well known that a chordal graph $G = (V, E)$ has at most $|V|$ cliques. In Section 3 we obtain polynomial (in $|V|$) bounds on the number of cliques in DE, UEH, and UE graphs. These bounds imply that known algorithms for generating all cliques in a general graph [27] require only polynomial time for these graphs. This results in a polynomial-time algorithm for finding a maximum weight clique in a UE graph. We also show that this implies a polynomial-time bound for decomposing a UE graph by its "clique separators." Finally, we characterize the DE, UEH, and UE atoms in a unified Atom Theorem; an *atom* is a graph which cannot be further decomposed by clique separators. A consequence is to show that DE graphs are "perfect" and that UE graphs satisfy the "strong perfect graph conjecture"; the latter result was previously shown in [15]. Another consequence is that the minimum coloring problem can be solved in polynomial-time for DE graphs; this problem is NP-hard for UE graphs [15].

In Section 4 further structural properties of decomposition by clique separators are developed in a Separator Theorem for RDV, DV, UV, RDE, DE, UEH, and UE $\cap$ CHORDAL graphs. These results are all very similar and lead to a unified recognition algorithm for these graphs. This algorithm can be implemented efficiently for DV, DE, and UEH graphs based on certain two-coloring conditions. The Separator Theorem implies the NP-completeness of recognizing UE $\cap$ CHORDAL graphs since a three-coloring condition is required at each separation step; this result was previously shown in [15] by an ad hoc method. The complexities of recognizing RDV graphs and UV graphs under the framework of the Separator Theorem are left as an open problem; Gavril [9, 10] has presented algorithms for recognizing these two classes of graphs without using separators. A consequence of the Separator Theorem is that DV = CHORDAL $\cap$ UEH, DE = PERFECT $\cap$ UEH, and UE $\cap$ CHORDAL = UE $\cap$ UV.

The proof of the Separator Theorem is presented in Section 5. Concluding remarks and open problems are given in Section 6.

We note that the intersection relationships of paths in a tree can also be represented as a hypergraph instead of a graph. This yields a different set of interesting problems. In this case, clique tree characterizations and separator-type theorems can also be obtained. This subject has been treated for edge path hypergraphs in [4, 11, 14] and for vertex hypergraphs in [15]. In fact, [15] contains separator-type theorems for UE and UV hypergraphs.

2. Clique Tree Representation

We will show that RDV, DV, UV, RDE, DE, and UEH graphs can all be characterized in terms of their "clique tree" representations. These results are entirely symmetric, differing only by interchanging the words "rooted," "directed," and "undirected," and interchanging "vertex" and "edge." A number of interrelationships among these graphs, as well as chordal and UE graphs, will also be discussed. The following theorem sets the stage for these results. Recall that **C** denotes the set of all cliques of a graph $G=(V, E)$ and $\mathbf{C}_v$ is the set of cliques containing a particular vertex $v \in V$. In the following theorem, we let $T(\mathbf{C}_v)$ denote the subgraph of a tree T with vertices or edges corresponding to $\mathbf{C}_v$.

THEOREM 1. (Clique Tree Theorem). (a) *A graph $G=(V, E)$ is RDV if and only if there exists a rooted tree T with vertex set* **C**, *such that for each $v \in V$, $T(\mathbf{C}_v)$ is a directed path in T.*

(b) *A graph $G=(V, E)$ is DV if and only if there exists a directed tree T with vertex set* **C**, *such that for every $v \in V$, $T(\mathbf{C}_v)$ is a directed path in T.*

(c) *A graph $G=(V, E)$ is UV if and only if there exists a tree T with vertex set* **C**, *such that for every $v \in V$, $T(\mathbf{C}_v)$ is a path in T.*

(d) *A graph $G=(V, E)$ is chordal if and only if there exists a tree T with vertex set* **C**, *such that for every $v \in V$, $T(\mathbf{C}_v)$ is a subtree in T.*

(e) *A graph $G=(V, E)$ is RDE if and only if there exists a rooted tree T with directed edge set* **C**, *such that for every $v \in V$, $T(\mathbf{C}_v)$ is a directed path in T.*

(f) *A graph $G=(V, E)$ is DE if and only if there exists a directed tree T with directed edge set* **C**, *such that for every $v \in V$, $T(\mathbf{C}_v)$ is a directed path in T.*

(g) *A graph $G=(V, E)$ is UEH if and only if there exists a tree T with edge set* **C**, *such that for every $v \in V$, $T(\mathbf{C}_v)$ is a path in T.*

A tree satisfying Theorem 1 is called a *clique tree* for the graph it characterizes. See Fig. 2 for an illustration of the clique trees for the graphs in Fig. 1. Note that a clique tree has the minimum number of edges for a representation tree. This property will be used later. The Clique Tree Theorem for RDV, UV, and CHORDAL graphs has been obtained by Gavril [8–10]. In the following, we prove the Clique Tree Theorem for DV, RDE, DE, and UEH graphs only.

Proof for DV Graphs. Given a tree T satisfying the conditions of the theorem we can construct paths $\mathbf{P}=T(\mathbf{C}_v)$ for all $v \in V$ to obtain a DE representation. Conversely, let G be a DV graph. Let $(T, \mathbf{P})$ be a DV representation for G where T has the smallest possible number of nodes. Note that every node of T corresponds to a completely connected set in G. We claim that there is a one-to-one correspondence between the nodes of T and the cliques of G. On one hand, by the Helly property of vertex paths [14], there is a node in T corresponding to every clique in G. On the other hand, assume there are two distinct nodes n_1 and n_2 in T which correspond to completely connected sets S_1 and S_2 in G, with $S_1 \subseteq S_2$. Let n_3 be the node next to n_1 on the path connecting n_1 and n_2 in T, and S_3 be the completely connected set in G corresponding to n_3. It is easy to see that $S_1 \subseteq S_3$. Obtain a new tree T' from T by coalescing n_1 and n_3 and eliminate the edge between them. Any path in $\mathbf{P}$ that contains n_1 necessarily contains n_3. Therefore T' is a DV representation tree for G. But T has the fewest possible nodes, a contradiction. Therefore the nodes of T correspond to distinct cliques and T is a clique tree. ∎

Proof for UEH Graphs. Given a tree T satisfying the conditions of the theorem, we can construct paths $P_v = T(\mathbf{C}_v)$ for all $v \in V$. Now $P_v \cap P_w \neq \emptyset$ exactly when v and w are contained in a common clique which occurs exactly when $[v, w] \in E$. Hence the paths together with T give a UE representation for G. Consider any subcollection $\mathbf{P}'$ of paths, with $P_i \cap P_j \neq \emptyset$ for all $P_i, P_j \in \mathbf{P}'$. The vertices corresponding to P' form a completely connected set of vertices C' in G. Let C be a clique containing C', then $C \in \mathbf{C}_v$ for all $v \in C$. So every path $P_v \in P'$ pass through the edge corresponding to the clique C and the Helly property holds. Therefore, G is UEH.

Conversely, consider a UEH representation for G consisting of a tree T and a collection of paths $\mathbf{P}$ which satisfy the Helly property. Every edge e in the tree corresponds to a completely connected set of vertices (in G) whose paths contain this edge. The Helly property implies that every clique C in G has a corresponding edge in T given by an edge in the nonempty intersection $\bigcap_{v \in C} P_v$.

Consider an edge e in the tree with a corresponding completely connected set C_e. Contracting the edge e in the tree and in the paths containing e results in another UEH representation $(T', \mathbf{P}')$ for a graph $G' = (V, E')$, where $E' = E - \{[v, w]\colon P_v \cap P_w = \{e\}\}$. So given two different edges e and f in T with corresponding completely connected sets C_e and C_f with $C_e \subseteq C_f$ we may contract edge e in T and produce another UEH representation for G. Repeating this process results in a tree whose edges correspond in a one-to-one fashion with the cliques of G and $P_v = T(\mathbf{C}_v)$.

This completes the proof of the Clique Tree Theorem for UEH graphs. ∎

Proof for DE Graphs. We claim that directed edge paths in a directed tree satisfy the Helly property. The proof is by induction on the number k

of paths. The lemma is clearly true for $k=2$ paths. Consider a collection $\mathbf{P}=\{P_1, P_2, \ldots, P_k\}$ of $k>2$ paths with $P_i \cap P_j \neq \emptyset$ for $1 \leqslant i, j \leqslant k$ and assume, by induction, that $\bar{P}=\bigcap_{i=1}^{k-1} P_i$ forms a directed path in T containing at least one edge. Now if $P_i \subseteq \bar{P}$ for any i, $1 \leqslant i \leqslant k-1$, then $P_k \cap \bar{P} \supseteq P_k \cap P_i \neq \emptyset$ and we are done. Therefore, each path P_i, $1 \leqslant i \leqslant k-1$, meets the subpath $\bar{P}$ together with an edge directed into the entering end of $\bar{P}$ or an edge directed out of the leaving end of $\bar{P}$ in T, or both. Also, since $k>2$, there are at least two such edges. However, any path P in T with $P \cap \bar{P}=\emptyset$ can meet at most one edge directed into the entering end of $\bar{P}$ or one edge directed out of the leaving end of $\bar{P}$, and so $P \cap P_i=\emptyset$ for some $1 \leqslant i \leqslant k-1$. Since $P_k \cap P_i \neq \emptyset$ for $1 \leqslant i \leqslant k-1$ it follows that $P_k \cap \bar{P} \neq \emptyset$. This completes the proof of the claim.

The rest of the proof parallels that for UEH graphs. ∎

Proof for RDE Graphs. Since edge-contraction maintains a rooted tree, this proof parallels those for UEH graphs and DE graphs. ∎

The following theorem summarizes the inclusion relationships between these various class of graphs. We show in Section 4 that DV $=$ CHORDAL $\cap$ UEH, DE $=$ PERFECT $\cap$ UEH, and that UE $\cap$ CHORDAL $=$ UE $\cap$ UV.

THEOREM 2. (a) *RDV = RDE ⊂ DV ⊂ UV ⊂ CHORDAL.*

(b) *DV ⊂ DE ⊂ UEH ⊂ UE.*

Proof. RDV ⊂ DV ⊂ UV ⊂ CHORDAL follows from the Clique Tree Theorem. RDE ⊂ DE ⊂ UEH follows from the Clique Tree Theorem in the same way. UEH ⊂ UE is trivial.

To see that DV ⊂ DE we consider a clique tree for a DV graph G consisting of a directed tree T and a collection $\mathbf{P}$ of directed vertex paths. We construct a new directed tree T' by expanding each vertex v in T to a directed edge (v, v'); every edge in T directed into v remains so in T' and every edge directed out of v in T is directed out of v' in T'. The collection of directed edge paths $\mathbf{P}'$ for T' consist of the directed paths in $\mathbf{P}$ where any path in T passing through vertex v in T passes through the new directed edge in T'. It is easy to see that two paths in $\mathbf{P}$ meet in a vertex in T if and only if their corresponding paths in P' meet in an edge in T'. So $(T', \mathbf{P}')$ is a DE representation for G.

Similarly, we can show that RDV ⊆ RDE. To see that RDE ⊆ RDV, consider an RDE representation tree T for a (connected) graph G. Construct a directed tree T' by associating a vertex for each edge in T. Direct vertex u to vertex v in T' whenever edge u points to edge v in T. Clearly, T' is an RDV representation tree. ∎

3. Decomposition by Clique Separators

In this section we show that the cliques of a UE graph can be generated in polynomial-time (in $|V|$) and can be used to decompose a UE graph efficiently by "clique separators." We also characterize the DE, UEH, and UE atoms, i.e., those graphs which cannot be decomposed by clique separators.

A clique C is a *separator* if $G(V \setminus C)$ is not connected. An *atom* is a connected graph with no separator. Any graph G which has a separator C can have its vertices uniquely partitioned into C, V_1, $V_2, \ldots, V_s$, with $s \geqslant 2$, so that each $G(V_i)$ is a connected subgraph and no vertex in V_i is adjacent to any vertex in V_j $(i \neq j)$. We say that C *separates* G into $G(V_i \cup C)$ for $1 \leqslant i \leqslant s$. By repeating this process we obtain a *clique decomposition* of G. This process can be represented by a *clique decomposition tree* with each leaf node being associated with an atom of G and each internal node being associated with a clique separator of G. The original graph can be reconstructed by composing subgraphs in the decomposition tree.

This decomposition scheme differs from that of Tarjan [25] and Whitesides [29] in that they decompose by completely connected sets (not necessarily maximal). In a private communication, Tarjan [25] has extended his algorithm to do decomposition by maximal clique separators. The following lemmas are easily verified.

Lemma 3. *Let G be separated by a clique C into $G_i = G(V_i \cup C)$, $1 \leqslant i \leqslant s$, then the cliques of G are exactly the union of cliques of G_i, $1 \leqslant i \leqslant s$. Furthermore, C is a clique in every G_i and every other clique in G is in exactly one G_i.*

Lemma 4. *Let G be separated by a clique C into $G_i = G(V_i \cup C)$, $1 \leqslant i \leqslant s$. If a clique C' is not a separator of G then it is not a separator of any G_i. Also, C is not a separator of any G_i.*

The algorithm of [27] generates all cliques in a general graph in $O(|V|\,|E|\,|\mathbf{C}|)$ time. By Lemmas 3 and 4, this collection of cliques can be used to decompose a graph by clique separators, in $O((|V| + |E|)\,|\mathbf{C}|)$ time. This follows since each clique $C \in \mathbf{C}$ can be examined, in turn, to see if it separates G. If C does not separate G, then by Lemma 4 it will not separate any later subgraph and so need not be considered further. On the other hand, if G is separated into subgraphs by C, then by Lemma 3 every other clique will be contained in exactly one of the subgraphs and the process can be repeated for the remaining cliques; also by Lemma 4, C need not be considered further. Therefore, each clique is examined exactly once. For each clique C, the connected components obtained by deleting C can be found in at most $O(|V| + |E|)$ time.

It is well known that a chordal graph $G=(V, E)$ has at most $|V|$ cliques [5]. We now provide polynomial (in $|V|$) bounds on the number of cliques in DE, UEH, and UE graphs. This implies that the clique generation algorithm and the clique decomposition algorithm can be implemented in polynomial-time. It is easy to see [15, Theorem 11] that a clique in a UE graph corresponds either to a set of paths containing a common edge in the tree (an *edge clique*) or to a set of paths each containing two out of three edges intersecting in a vertex (a *claw clique*). We let $\mathbf{C}_e$ and $\mathbf{C}_c$ be the partition of $\mathbf{C}$ according to the edge cliques and claw cliques, respectively. Clearly, for UEH graphs $\mathbf{C}_c$ is empty.

THEOREM 5. (a) *For any UEH graph* $G=(V, E)$ *with* $|V| \geqslant 4$ *we have* $|\mathbf{C}| \leqslant \lfloor (3|V|-4)/2 \rfloor$. *Furthermore, this bound can be achieved by a DE graph* $G=(V, E)$ *for any value of* $|V| \geqslant 4$.

(b) *For any UE graph* $G=(V, E)$ *with* $|V| \geqslant 3$ *we have* $|\mathbf{C}_e| \leqslant 2|V|-3$ *and* $|\mathbf{C}_c| \leqslant |V|(|V|-1)/2$. *Furthermore, a UE graph* $G=(V, E)$ *can attain* $\Omega(|V|^{3/2})$ *claw cliques for any value of* $|V|$.

Proof of (a). Let $G=(V, E)$ be a UEH graph and let T be a clique tree. It suffices to show that the number of edges in T is bounded by $(3|V|-4)/2$ or equivalently that the number of vertices in T is bounded by $(3|V|-2)/2$; this follows since each edge in T corresponds to a clique and a tree has one fewer edges than vertices. Let t_1, t_2, t_3, and t_4 denote the number of vertices in T of degree 1, 2, 3, and 4 or more, respectively.

Since G is connected, each clique is of size two or greater. Therefore, each vertex a of degree 1 adjacent to a vertex b is the endpoint of at least two paths in $\mathbf{P}$ containing edge $[a, b]$ in T. Since cliques are incomparable, each vertex a of degree 2 adjacent to vertices b and c must be the endpoint of at least one path in $\mathbf{P}$ containing the edge $[a, b]$ but not $[a, c]$, and one path containing the edge $[a, c]$ but not $[a, b]$. Now consider a vertex a of degree 3 adjacent to vertices b, c, and d. Since the cliques corresponding to the three edges $[a, b]$, $[a, c]$, and $[a, d]$ are incomparable and using the Helly property, at least two paths have vertex a as an endpoint. Therefore, $t_1+t_2+t_3 \leqslant |V|$.

We complete the proof by showing that $t_4 \leqslant (t_1-2)/2$ using induction. Every tree has $t_1 \geqslant 2$; therefore the induction holds for $t_4=0$. Let $t_4>0$ and suppose that the result is true for smaller values. Let a be a vertex in T of degree $\delta \geqslant 4$. Let $T_1, T_2, \ldots$ and T_δ be a partition of T into δ trees all rooted at a. Let $t_1^{(i)}$ and $t_4^{(i)}$ denote the number of vertices of degree 1 and 4 or more in each tree T_i, $1 \leqslant i \leqslant \delta$. By induction, $t_4^{(i)} \leqslant (t_1^{(i)}-2)/2$. Summing over all i, $1 \leqslant i \leqslant \delta$, and noting that $\sum_{i=1}^{\delta} t_4^{(i)} = t_4-1$ and $\sum_{i=1}^{\delta} t_1^{(i)} \leqslant t_1+\delta$ yields the desired result.

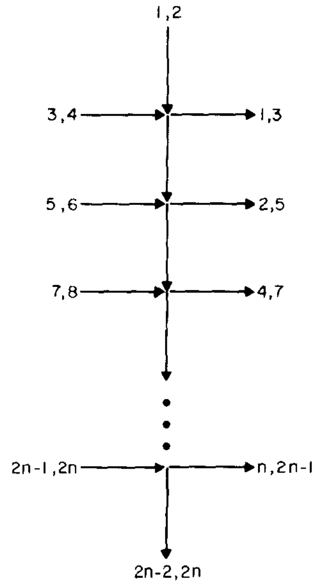


FIG. 3. Example of a DV graph attaining the bound in Theorem 5.

This bound is attained for any $n \geqslant 4$ by the DE representations depicted in Fig. 3. ∎

Proof of (b). Let $G=(V, E)$ be a UE graph with a representation $(T, \mathbf{P})$ where the number of edges in T is minimum. We will show that T has at most $2|V|-2$ vertices and so $|\mathbf{C}_e| \leqslant 2|V|-3$. We let t_1, t_2, and t_3 denote the number of vertices of degree 1, degree 2, and degree 3 or more in T, respectively. The completely connected sets $\mathbf{C}_e$ of G associated with edges e in T are incomparable. (If not, edge e can be shrunk, contradicting the minimality of T.) Therefore, using the same reasoning as in (a), $t_1+t_2 \leqslant |V|$. It is easy to show by induction, as in part (a), that $t_3 \leqslant t_1-2$ to complete the first part.

To see the second part we note that any pair of paths P_i and P_j which appear in a clique must share an edge and so can appear in at most two claw cliques which must be at different vertices in the tree. Also, for any pair of paths which form two claw cliques there are at least two pairs of paths which do not meet. Therefore, $|\mathbf{C}_c| \leqslant \binom{|V|}{2}$.

To see that $|\mathbf{C}_c|$ can attain $\Omega(|V|^{3/2})$ consider a tree T where all edges share a common vertex with k spokes. Let $\mathbf{P}$ be the collection of all $\binom{k}{2}$ paths. Then $\mathbf{C}_c$ consists of one claw clique for each of the $\binom{k}{3}$ claws. ∎

A by-product of Theorem 5 is that a maximum-weight clique can be found in polynomial-time for UE graphs.

For the independent set problem on a UE graph, Tarjan [25] describes an efficient algorithm for recursively constructing a maximum weight independent set from repeated decomposition by completely connected sets. Note that although the algorithm in [24] can be used to generate the entire collection **L** of maximal independent sets for a general graph in $\Omega(|V|\,|E|\,|\mathbf{L}|)$ time, there may be exponentially many independent sets in a UE graph. For example, the chordless path of length n is a DV graph and it has exponentially many independent sets.

We next describe the relationships between the atoms of DE, UEH, and UE graphs and certain classes of line graphs. This provides a number of interesting result for these graphs. We first require a few definitions.

A *multigraph* is a graph with (possibly) multiple edges between pairs of vertices. A *line graph* $L(M)$ of a multigraph M has a vertex for every edge of M with two vertices adjacent in $L(M)$ exactly when the corresponding edges in M are adjacent. When $H = L(M)$ we say that $M = L^{-1}(H)$. A multigraph M is *bipartite* if the vertices can be partitioned into two parts so that no two vertices in the same part are adjacent. A multigraph M is *triangle free* if it does not contain three mutually adjacent vertices. A *star tree* is a connected bipartite graph G with one part consisting of a single vertex. We can draw a star tree with a center vertex and edges surrounding it like *spokes* of a wheel.

THEOREM 6. (Atom Theorem). (a) *The line graphs of multigraphs are exactly the UE graphs with star tree representations. Furthermore, every UE atom has a star tree representation.*

(b) *The line graphs of triangle-free multigraphs are exactly the UEH graphs with star tree representations. Furthermore, every UEH atom has a star tree representation.*

(c) *The line graphs of bipartite multigraphs are exactly the DE graphs with star tree representations. Furthermore, every DE atom has a star tree representation.*

(d) *A graph consisting of a single clique or two intersecting cliques is an RDV graph. Furthermore, a chordal atom consists of either a single clique or two intersecting cliques.*

Proof for UE Graphs. Consider a multigraph $M = (V, E)$ with line graph $H = L(M)$. Construct a star tree T with a spoke i for every vertex $i \in V$. Construct a collection **P** of paths with a path P_{ij} consisting of spokes i and j for every $[i, j] \in E$. Multiple edges produce multiple paths. Now two vertices in H are adjacent exactly when the corresponding edges in M are adjacent, which is exactly when the corresponding paths in **P** share an edge in T. Therefore, $(T, \mathbf{P})$ is a UE representation of $L(M)$.

Conversely, let T be a star tree with a collection of paths $\mathbf{P}$. Without loss of generality, we may assume that all paths consist of exactly two spokes of T, by extending every path of length one by adding a new spoke to the star and joining it with the path. Let H be the UE graph represented by T and $\mathbf{P}$. Construct a multigraph $M=(V, E)$ with a vertex $i \in V$ for every spoke of T and an edge $[i, j] \in E$ for every path which uses spokes i and j. Now two edges in G are adjacent exactly when two paths in $\mathbf{P}$ share an edge in T. Therefore, $H=L(M)$.

Now let G be a UE atom with representation $(T, \mathbf{P})$, where T is a representation tree with a minimum number of edges. We propose to show that if T is not a star tree then G contains a separator, thus contradicting G being an atom. If T is not a star tree than it contains a path on four vertices, a, b, c, and d, respectively. Let C_{ab}, C_{bc}, and C_{cd} denote the completely connected set of vertices G whose corresponding paths in T include the edges (a, b), (b, c), and (c, d), respectively. (C_{ab}, C_{bc}, andC_{cd} are incomparable else the edges corresponding to contained sets could be contracted while preserving a UE representation of G, contradicting the minimality of T.) Therefore, there are vertices x and y such that $x \in C_{ab} \backslash C_{bc}$ and $y \in C_{cd} \backslash C_{bc}$. If C_{bc} is a clique then we are done as it separates G. So C_{bc} is strictly contained in a claw clique C. The paths corresponding to the vertices in C must all share either vertex b or c in T but not both. Without loss of generality, we choose vertex c with the paths of C using edge (c, e) in addition to edges (b, c), and (c, d). (Hence, $x \notin C$.) We partition the paths for C into the nonempty sets P_{bd}, P_{be}, and P_{de} corresponding to the path containing edges (b, c) and (c, d), edges (b, c) and (c, e), and edges (c, d) and (c, e), respectively. If the subtree containing vertex c obtained by deleting edge (b, c) in T contains no path P_z for a vertex $z \notin C$ then vertices d and e are leaves in T (by the connectivity of G) and so the edges (c, d) and (c, e) can be deleted from T and deleted from the paths in P_{bd} and P_{be}. The paths in P_{de} can all be replaced by the single edge (b, c) while preserving a UE representation for G. This would contradict the minimality of T. Hence, the subtreee containing vertex c obtained by deleting edge (b, c) in T must contain a path P_z for some vertex $z \notin C$ and so C separates x and z in G. ∎

Proof for UEH Graphs. We use the same reduction as in the proof for UE graphs. Suppose first that H is the line graph of a traingle-free multigraph; then $(T, \mathbf{P})$ is a star tree UE representation for H by (a). Let $P' \subseteq \mathbf{P}$ be a subset of paths all of which pairwise intersect in at least one edge. If all of these paths are identical then we are done. If not, consider two paths P_{ij} and P_{ik} with $j \neq k$. Any other path P in $\mathbf{P}'$ which meets both paths must do so in spoke i else a triangle is present in $L^{-1}(H)$. Therefore, all paths in $\mathbf{P}'$ contain the spoke i and the Helly property holds.

Conversely, suppose that $(T, \mathbf{P})$ is a star UEH representation; then, by (a), H is a line graph of a multigaph. If $L^{-1}(H)$ had a triangle then the corresponding paths in $\mathbf{P}$ would not satisfy the Helly property, a contradiction.

Finally, by the Clique Tree Theorem for UEH graphs, every edge in T corresponds to a clique and removing an edge which disconnects the tree is equivalent to removing a clique to separate the graph. So T must be a star tree. ∎

Proof for DE Graphs. Again we use the same reduction as in the proof for UE graphs. Suppose first that H is the line graph of a bipartite graph $G = (V, E)$ with vertex partition $V = V_1 \cup V_2$; then $(T, \mathbf{P})$ is a star tree UE representation for H. All edges $(i, j) \in E$ are of the form $i \in V_1$ and $j \in V_2$. Therefore we may direct the spokes of T corresponding to vertices in V_1 towards the central vertex in T, and direct the spokes of T corresponding to vertices in V_2 away from the central vertex. So all of the paths $P_{ij} \in \mathbf{P}$ for $i \in V_1$ and $j \in V_2$ are uniformly directed as well.

Conversely, let $(T, \mathbf{P})$ be a star tree DE representation for a graph H; then by (a), H is the line graph of a multigraph $G = (V, E)$. We may partition V into $V_1 \cup V_2$ corresponding to the spokes of T directed towards the central vertex of T and the spokes of T directed away from the central vertex of T. Each path consists of one spoke corresponding to a vertex in V_1 and one spoke corresponding to a vertex in V_2 and so G is bipartite.

Finally, using the Clique Tree Theorem for DE graphs, and the fact that $\mathrm{DE} \subset \mathrm{UEH}$ of Theorem 2, an argument identical to the last part of the proof for UEH graphs shows that a DE atom has a star tree representation. ∎

Proof for RDV Graphs and CHORDAL Graphs. An atom consisting of a single clique has an RDV representation consisting of a tree with a single vertex representing the clique and also representing the degenerate paths for all vertices in the clique. An atom consisting of two intersecting cliques has an RDV representation consisting of a tree with two vertices connected by a directed edge. The directed path for each vertex v in the graph is the vertices in the tree corresponding to the cliques containing the vertex v. By the Clique Tree Theorem for chordal graphs, a chordal atom must consist of a tree with either one vertex or two vertices connected by an edge; each vertex represents a clique; since an atom is connected the two cliques must intersect. ∎

A graph G is called *perfect* if for every induced subgraph H of G, the size of a maximum clique in H is equal to the size of a minimum coloring of H [1, 2]. The Strong Perfect Graph Conjecture (SPGC) [2] states that a graph is perfect if and only if it contains no odd chordless cycles of length five or greater, or the complement of one, as an induced subgraph. It is easy to see that a graph is perfect if and only if the atoms produced by the clique decomposition are perfect, and that the SPGC holds for a class of graphs if and only if it holds for their atoms. It is also known [26] that a line graph is perfect if and only if it contains no odd chordless cycle of length five or greater. This yields the following result as a corollary of Theorem 6. Part (a) of Corollary 7 was previously shown in [12].

COROLLARY 7. (a) *A UE graph is perfect if and only if it contains no odd chordless cycle of length five or greater.*

(b) *DE graphs are perfect.*

COROLLARY 8. (a) *The atoms for CHORDAL, DE, UEH, and UE graphs can be efficiently recognized.*

(b) *The minimum coloring problem can be solved in polynomial-time for DE graphs and CHORDAL graphs, and is NP-complete for UE graphs.*

Proof of Part (a). For CHORDAL graphs the atoms are trivial to recognize. For UE graphs, the atoms are line graphs of multigraphs which can be recognized in linear-time [31].

We claim that an atom is UEH if and only if each vertex $v \in V$ is contained in at most two cliques. To see this, consider a UEH atom $G=(V, E)$. By the Atom Theorem, G has a star tree representation T with each spoke representing a clique. Clearly, the path for a vertex v can meet at most two cliques. Conversely, if $G=(V, E)$ satisfies the stated condition, then we can build a star tree with an edge for each clique of G. Let $\pi(v)$ be the spokes (cliques) containing v. This is clearly a clique tree for G, hence G is UEH by the Clique Tree Theorem.

To determine whether a graph G is a DE atom or not we first verify that G is a UEH atom by the procedure described above. Then it is easy to see that G is a DE atom if and only if the auxiliarly graph $H=(U, F)$ is bipartite, where there is a vertex $u_C \in U$ for each clique C of G and $[u_C, u_D] \in F$ when $C \cap D \neq \varnothing$. ∎

Proof of Part (b). We first show how to efficiently color the atoms of a DE graph and then show how to color the entire graph using the same number of colors.

By the Atom Theorem, a DE atom $H=L(G)$ is the line graph of a bipartite multigraph G. Coloring the vertices of H is equivalent to coloring the edges of G. The minimum coloring of the edges of a bipartite multigraph G is easily obtained and is equal in size to the largest vertex degree in G [18]. Now given a coloring for the atoms we obtain a coloring for the entire graph by applying the following procedure recursively. Suppose a graph G is separated by a clique C into $G_i=G(V_i \cup C)$, $1 \leqslant i \leqslant s$. Each component can be colored recursively using, say, at most k colors. We extend this k-coloring to G by renaming the colors in each component consistently. (Recall that no vertex v_i in V_i and vertex v_j in V_j $(i \neq j)$ are adjacent in G.)

It is shown in [6] that the minimum coloring problem is solvable in polynomial time for CHORDAL graphs. It is shown in [32] that the minimum coloring problem is NP-complete for line graphs hence is also NP-complete for UE graphs by the Atom Theorem. ∎

4. Separator Theorem and Recognition Algorithms

In this section, we present further results on the structure of path graphs when decomposed by clique separators. Together with the earlier results on an efficient algorithm for generating a clique decomposition and the characterization of the atoms, these structural theorems yield efficient algorithms for recognizing DV, DE, and UEH graphs and constructing their clique trees based on certain two-coloring criteria. The algorithms for these classes of path graphs are very similar to each other. Within the same framework, we also prove the NP-completeness of recognizing UE graphs, in a manner similar to that of Golumbic and Jamison [16], since we show that the recognition of UE $\cap$ CHORDAL graphs requires a certain three-coloring criterion. Stronger inclusion and intersecting relationships among the classes are also by-products of the Separator Theorem. First some definitions.

Assume clique C separates G into $G_i = G(C \cap V_i)$, $1 \leqslant i \leqslant s$. Cliques which intersect C but are not identical to C are called *relevant*. In the following, only relevant cliques are considered. For example, when we say "some clique" and "every clique," we mean "some relevant clique" and "every relevant clique," unless explicitly specified otherwise.

Two cliques C_1 and C_2 are *unattached*, denoted $C_1 \mid C_2$, if $(C_1 \cap C) \cap (C_2 \cap C) = \varnothing$. We say that C_1 *dominates* C_2, denoted $C_1 \geqslant C_2$, if $C_1 \cap C$ contains $C_2 \cap C$. We say that C_1 *properly dominates* C_2, denoted $C_1 > C_2$, if $C_1 \cap C$ properly contains $C_2 \cap C$. C_1 and C_2 are *congruent*, denoted $C_1 \sim C_2$, if $C_1 \geqslant C_2$ and $C_2 \geqslant C_1$. The two cliques are *antipodal*, denoted $C_1 \leftrightarrow C_2$, if they are attached (i.e., not unattached) and neither dominates the other. The dominance relation on cliques is transitive, i.e., $C_1 \geqslant C_2$ and $C_2 \geqslant C_3$ imply $C_1 \geqslant C_3$. Note that all definitions are with respect to the separator C.

Two separated subgraphs $G_1 = G(C \cup V_1)$ and $G_2 = G(C \cup V_2)$ are *unattached*, denoted $G_1 \mid G_2$, if $C_1 \mid C_2$ for every C_1 in G_1 and every C_2 in G_2. We say that G_1 *dominates* G_2, denoted $G_1 \geqslant G_2$, if they are attached, and for every C_1 in G_1, either $C_1 \geqslant C_2$ for all C_2 in G_2, or $C_1 \mid C_2$ for all C_2 in G_2. We say that G_1 *properly dominates* G_2, denoted $G_1 > G_2$, if $G_1 \geqslant G_2$ but not

$G_2 \geqslant G_1$. They are *congruent*, denoted $G_1 \sim G_2$, if $G_1 \geqslant G_2$ and $G_2 \geqslant G_1$; in this case $C_1 \sim C_2$ for every C_1 in G_1 and every C_2 in G_2. We say that G_1 and G_2 are *antipodal*, denoted $G_1 \leftrightarrow G_2$, if they are attached, and neither dominates the other. A subgraph not properly dominated by any other subgraph is called a *sovereign* subgraph.

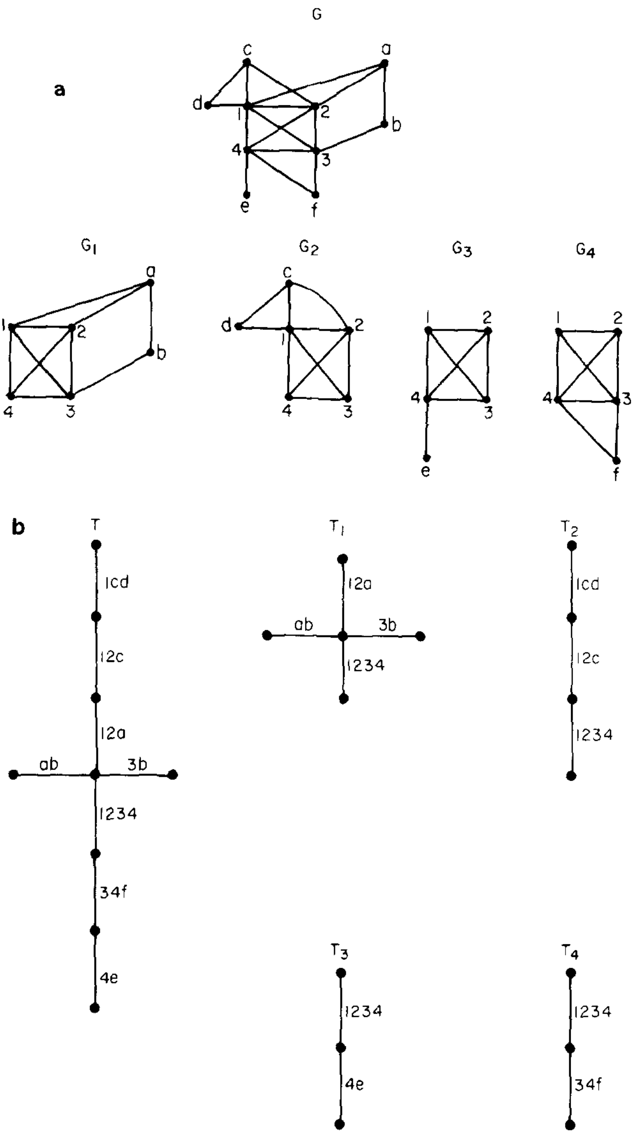


FIG. 4. (a) Example of the separation of G into G_1, G_2, G_3, and G_4 by $C = \{1, 2, 3, 4\}$. (b) The UEH clique trees for the graphs in (a).

We claim that the relation "$\geqslant$" is transitive, i.e., $G_i \geqslant G_j$ and $G_j \geqslant G_k$ implies $G_i \geqslant G_k$ for any i, j and k. Assume $G_i \geqslant G_j$ and $G_j \geqslant G_k$. There exists a clique C_i^* in G_i which dominates every clique C_j in G_j, and there exists a clique C_j^* in G_j which dominates every clique C_k in G_k. So $C_i^* \geqslant C_j^* \geqslant C_k$ for every clique C_k in G_k, and G_i and G_k are attached. Consider any C_i in G_i. If C_i dominates every clique in G_j then $C_i \geqslant C_j^* \geqslant C_k$ for every C_k in G_k. Otherwise, $C_i | C_j^*$ and hence $C_i | C_k$ for every C_k in G_k. This proves that $G_i \geqslant G_k$.

Consider the graph $G=(V, E)$ separated by the clique $C=\{1, 2, 3, 4\}$ into G_1 through G_4 as shown in Fig. 4a. We have $G_1 > G_2$, $G_1 | G_3$, $G_1 \leftrightarrow G_4$, $G_2 | G_3$, $G_2 | G_4$, and $G_3 < G_4$. Incidentally, G is a DE (hence UEH) graph, G_1, G_3, G_4 are atoms, while $\{1, 2, c\}$ separates G_2. The UEH clique trees for these graphs are shown in Fig. 4b. Each edge is labelled by the clique to which it corresponds.

Lemma 9. *Two subgraphs G_1, G_2 are antipodal if and only if*

(1) $C_1 \leftrightarrow C_2$,

(2) $C_1 > C_2$, $C_1' < C_2'$,

(3) $C_1 > C_2$, $C_1' \geqslant C_2'$, $C_1' | C_2''$ (or $C_2 > C_1$, $C_2' \geqslant C_1'$, $C_2' | C_1''$), or

(4) $C_1 \sim C_2$, $C_1 | C_2'$, $C_2'' \sim C_1'$, $C_2'' | C_1''$,

for some C_1, C_1', C_1'' in G_1 and C_2, C_2', C_2'' *in G_2. (These cliques need not all be different).*

Proof. It is straightforward to verify that each condition implies $G_1 \leftrightarrow G_2$. Assume $G_1 \leftrightarrow G_2$ and that no clique in G_1 is antipodal to any clique in G_2. If there are some $C_1 > C_2$, then since G_1 does not dominate G_2, we have some C_1', C_2', C_2'', with either $C_1' < C_2'$, or $C_1' \geqslant C_2'$, $C_1' | C_2''$; i.e., Conditions (2) or (3). On the other hand, if no clique in G_1 properly dominates any clique in G_2 and vice versa, then we must have Condition (4). ∎

Lemma 10. *Let C_1 and C_2 be cliques in G_1 and G_2, respectively, with G_1 and G_2 not antipodal. If $C_1 > C_2$, then $G_1 > G_2$.*

Proof. Straightforward. ∎

Lemma 11. *A collection of pairwise non-antipodal subgraphs G_i of a (general) graph G can be arranged in such a way that $G_i > G_j$ implies $i < j$.*

Proof. First, assume there are no $G_i \sim G_j$, $i \neq j$. Let l be the largest number for which there exists a chain $G_1 > G_2 > \cdots > G_l$. We will use induction on l. If $l=1$ then the G_i's are mutually unattached, and any arrangement will do.

Assume the lemma true for $1, \ldots, l-1$, with $l \geqslant 2$. Let k be the number of sovereign subgraphs $H_1, \ldots, H_k$. Every subgraph G_i is dominated by a unique sovereign subgraph; otherwise we would have two subgraphs antipodal to each other. Assume H_m dominates a_m subgraphs, including itself, where $\sum_{m=1}^{k} a_m = s$, and s is the total number of subgraphs.

Consider the $a_m - 1$ subgraphs properly dominated by H_m. There are no antipodal pairs and the maximum length of a chain is at most $l-1$. By the induction assumption, the $a_m - 1$ subgraphs can be arranged.

Let the $(1+\sum_{t=1}^{m-1} a_t)$th subgraph be H_m. For each m, let the $(2+\sum_{t=1}^{m-1} a_t)$th through $(\sum_{t=1}^{m} a_t)$th subgraphs be those dominated by H_m arranged in the inductive way just described. Let G_i dominate G_j. If G_i is sovereign, then $i = 1+\sum_{t=1}^{m-1} a_t$ for some m and $j > i$ by the construction used. If G_i is not sovereign, then both G_i and G_j are dominated by some H_m, hence $j > i$ by the inductive arrangement.

If there are congruent subgraphs $G_i \sim G_j$, take one subgraph from each congruence class and rearrange. Then insert the other G_i's into the sequence immediately after the subgraph to which it is congruent in an arbitrary order. ∎

We note that a collection of pairwise non-antipodal subgraphs G_i of a (general) graph G (not necessarily a path graph) possesses several interesting properties. Let $H = (U, D)$ be an undirected graph with a vertex $i \in U$ for each subgraph G_i, and an edge $[i, j]$ if G_i dominates or is dominated by G_j. Assume the subgraphs are arranged in such a way that $G_i > G_j$ implies $i < j$. Lemma 11 guarantees such an arrangement. Note that $[i, j] \in D$, $i < j$, imply that $G_i \geqslant G_j$. Orienting an edge $[i, j]$ from i to j if $i < j$, we see that H is transitively orientable. It is also triangulated since $G_i < G_j$ and $G_j < G_k$ imply that $[i, k]$ is an edge, or else $G_i \leftrightarrow G_k$. It is also the case that the complement of K is transitively orientable so that H is an interval and a permutation graph. To see this, assume, in order to obtain a contradiction, that H is the smallest such graph with $\bar{H}$ not transitively orientable. Note that the G_j's dominated by different sovereign G_i's are unattached. Hence, $s \geqslant 2$ sovereign G_i's means s disjoint components of H each of which is an interval as well as a permutation graph. Arbitrarily order the disjoint component into $H_1, H_2, \ldots, H_s$. We have a transitive orientation of $\bar{H}$ as follows: For $x, y \in H_i$, use the transitive orientation in H_i. For x, y in different components, orient them as H_i's are oriented. Therefore, we must have $s = 1$, i.e., there is only one sovereign subgraph G^*.

Let H' be the graph derived from H by deleting the vertex G^*. Then H' is an interval and permutation graph by the minimality of H. Noting that G^* is adjacent to all other vertices in H, we see that the transitive orientation on $\bar{H}'$ directly gives a transitive orientation on $\bar{H}$.

Let C separate G into $G_i = G(C \cup V_i)$, $1 \leqslant i \leqslant s$. A subgraph G_i is a *neighboring subgraph of* $v \in C$ if some vertex in V_i is adjacent to v.

Here is the main theorem of this section.

THEOREM 12. (Separator Theorem). *Assume clique C separates $G=(V, E)$ into subgraphs $G_i = G(C \cup V_i)$, $1 \leqslant i \leqslant s$, $s \geqslant 2$.*

(a) *G is a DE graph if and only if each G_i is DE, and the G_i's can be 2-colored such that no antipodal pairs have the same color.*

(b) *G is a DV graph if and only if each G_i is DV, and the G_i's can be 2-colored such that no antipodal pairs have the same color.*

(c) *G is a UEH graph if and only if G_i is UEH, and the G_i's can be 2-colored such that no antipodal pairs have the same color.*

(d) *G is a UV graph if and only if each G_i is UV, and the G_i's can be colored such that no antipodal pairs have the same color, and for each $v \in C$, the set of subgraphs neighboring v is 2-colored.*

(e) *G is both a UE and a CHORDAL graph if and only if each G_i is both UE and CHORDAL, and the G_i's can be 3-colored such that no antipodal pairs have the same color, and for each $v \in C$, the set of subgraphs neighboring v is 2-colored.*

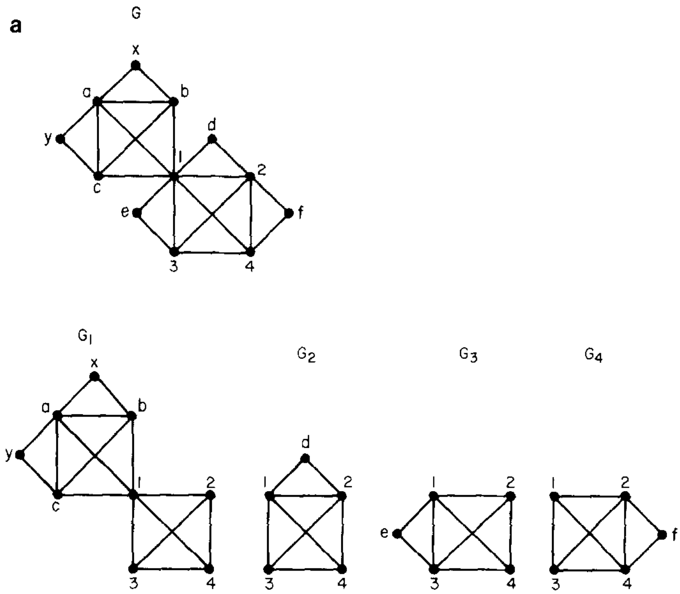
(f) *G is a chordal graph if and only each G_i is chordal.*

(g) *G is an RDV graph if and only if each G_i is RDV, and the G_i's can be two-colored such that no antipodal pairs have the same color, and that in one color every subgraph has an RDV clique tree rooted at C, and that in the other color no two subgraphs are unattached and every subgraph (with one possible exception) has an RDV clique tree rooted at a relevant clique. The exception subgraph, should it exist, is dominated by every other subgraph of the same color, and it has an RDV clique tree in which the node C has out-degree zero.*

It is understood that k-coloring means using k of fewer colors, where $k=2$ or 3.

The proof of the Separator Theorem is left for the next section. Here, we provide some insight into the ideas behind this theorem. Consider the clique tree T of a DE graph G. Separating G by a clique C corresponds to dismantling T by the edge C into several pieces. The edges corresponding to the cliques of a subgraph constitute a subtree that lies entirely within one piece. A piece may contain more than one subtree. Subtrees of antipodal subgraphs must lie on opposite sides of the edge C. This motivates the necessity of 2-coloring; the fact that this condition is also sufficient is left to the proof. (See Figs. 1 and 2.)

The intuition for UEH graphs is similar. For DV graphs, separating G by a clique C corresponds to dismantling its clique tree T by the vertex C into several pieces. The subgraphs can be grouped into two categories



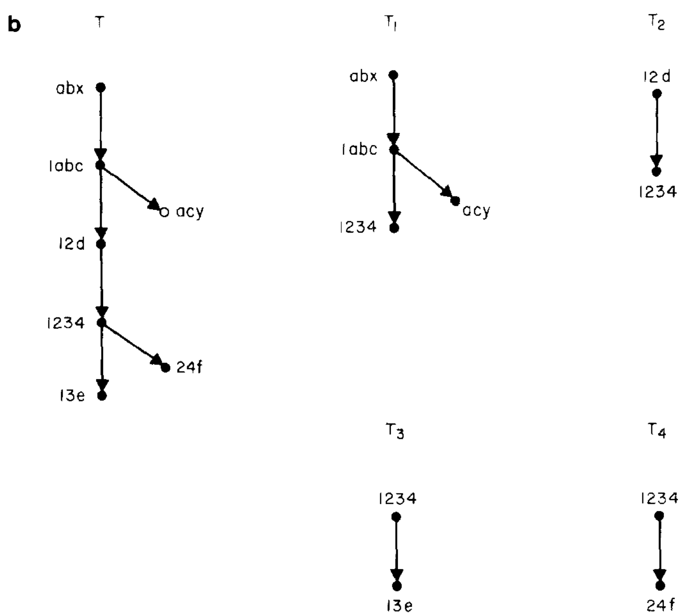


FIG. 5. (a) An RDV graph and its separation by $C = \{1, 2, 3, 4\}$. (b) The RDV clique trees for the graphs in (a).

depending on whether the piece they are in contains an edge leading into the node C or not. Two antipodal subgraphs must be in opposite categories.

The situation for UV gaphs is slightly different. Separating a UV graph G by a clique C corresponds to dismantling its clique tree T by the vertex C, but the subgraphs are grouped into as many categories as the degree of the vertex C. Consider T as consisting of branches emanating from the vertex C; the subgraphs with subtrees on the same branch are grouped together. Two antipodal subgraphs must be in different categories. For any $v \in C$, the path corresponding to v extends into at most two branches. This motivates the coloring conditions for *UV* graphs.

The situation for RDV graphs is illustrated in Figs. 5a and b. Figure 5a illustrates the separation of an RDV graph G by the clique $C = \{1, 2, 3, 4\}$. The clique trees for G and its subgraphs are shown in Fig. 5b. Each vertex is labelled by the clique to which it corresponds. The relationships between the subgraphs are: $G_1 < G_2$, $G_1 < G_3$, $G_1 \sim G_4$, $G_2 \leftrightarrow G_3$, $G_2 \leftrightarrow G_4$, and $G_3 \sim G_4$. We can use one color on G_1 and G_2, and another color on G_3 and G_4. The conditions of the Separator Theorem are satisfied by making G_1 the exception.

On the basis of the Separator Theorem, we can outline a general algorithm for recognizing these graphs:

Step 1. Obtain a clique decomposition for $G = (V, E)$.

Step 2. Verify that the atoms satisfy the Atom Theorem.

Step 3. Verify that each separation in the decomposition satisfies the Separator Theorem.

In the previous section, we have shown that Steps 1 and 2 can be done in polynomial-time for all of these graphs. Step 3 is executed at most $|V|$ times. For each separation, there are at most $|V|$ subgraphs, so it takes polynomial-time to set up the antipodal relationships between the subgraphs. This can be done by (1) determining the cliques in the subgraphs, (2) deciding antipodality between all pairs of cliques, and (3) resolving antipodality among subgraphs.

So the complexity of recognizing these graphs depends on how difficult it is to check the conditions of the Separator Theorem. For CHORDAL graphs there is nothing to check. For DV, DE, and UEH graphs this involves a 2-coloring and can be done efficiently. (We note that the statements of the Separator Theorem for DV, DE, and UEH graphs are identical except for the names of the classes; therefore these graphs differ, in a sense, only in their atoms.) The RDV graphs and UV graphs can be efficiently recognized using algorithms given by Gavril [9, 10] and by Dietz [30]. The efficiencies of recognizing RDV graphs and UV graphs under the framework of the Separator Theorem are left as open problems. Algorithms for generating clique trees (or UE representation trees in the case of UE $\cap$ CHORDAL graphs) are presented in the next section.

Not surprisingly, we can prove the NP-completeness of recognizing UE $\cap$ CHORDAL graphs by transforming the 3-coloring problem into the graph recognition problem in a manner similar to [15]. Let $H=(U, D)$ be any connected graph. Without loss of generality, let every vertex of H have degree at least two. (If not, the vertex could be deleted and colored consistently later.) Let $G=(V, E)$ be defined by $V=U \cup D$, $E=\{[u, d] \in U \times D$: d is incident to u in $H\} \cup D \times D$. Then G has $|U|+1$ cliques, namely D and $C_u=\{u\} \cup \{d$: d is incident to u in $H\}$, for each $u \in U$. Clearly D separates G into subgraphs $G_u=G(D \cup C_u)$, $u \in U$. Each G_u is an atom and a UE $\cap$ CHORDAL graph since it has only two cliques. Furthermore, G_u is antipodal to G_w if and only if $[u, w] \in D$. For each $d \in D$, $d=[u, w]$ and there are exactly two neighboring subgraphs, G_u and G_w, to d. The second condition of the Separator Theorem for UE $\cap$ CHORDAL graphs is satisfied by any coloring of the subgraphs. Therefore, by the Separator Theorem, H is 3-colorable if and only if G is a UE $\cap$ CHORDAL graph. Hence it is NP-complete to recognize UE $\cap$ CHORDAL graphs. Since CHORDAL graphs are polynomial-time recognizable [10], it is NP-complete to recognize UE graphs.

In summary, we have the following theorem.

THEOREM 13. *The RDV, DV, RDE, DE, UEH, and CHORDAL graphs are recognizable in polynomial-time. The problem of recognizing UE graphs is NP-complete.*

On the basis of the Separator Theorem, we have the following intersecting relationships.

COROLLARY 14. (a) *DV = UEH $\cap$ CHORDAL.*

(b) *DE = UEH $\cap$ PERFECT.*

(c) *UE $\cap$ CHORDAL = UE $\cap$ UV.*

Proof of Part (a). By Theorem 2, DV $\subset$ UEH $\cap$ CHORDAL. To see the reverse containment, we note that by the Atom Theorem if G is a chordal atom, then G is DV. Otherwise, let G be a non-atom UEH and chordal graph. Separate G into subgraphs $G_i=G(V_i \cup C)$, $1 \leqslant i \leqslant s$. By the Separator Theorem for chordal graphs, each G_i is chordal. By the Separator Theorem for UEH graphs, each G_i is UEH and they are 2-colorable. By induction, each G_i is DV. Hence by the Separator Theorem for DV graphs, G is DV. ∎

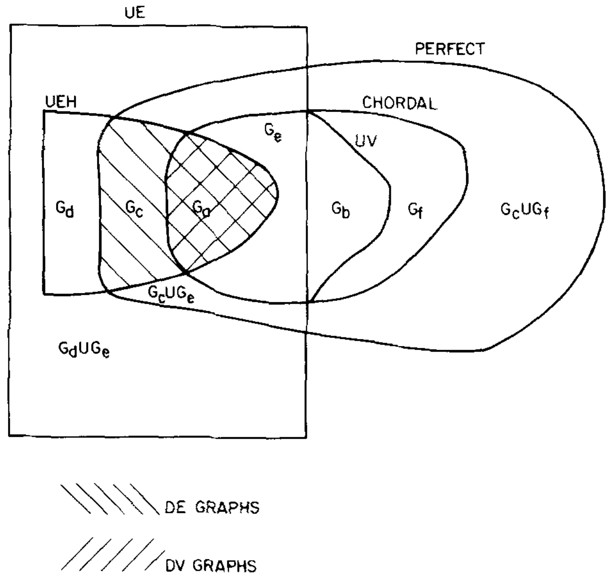


FIG. 6. Intersecting relationships among various classes of graphs. (Examples of graphs from Fig. 1 are shown in appropriate regions.)

Proof of Part (b). By Theorem 2, $\mathrm{DE} \subset \mathrm{UEH} \subset \mathrm{UE}$. By Corollary 7, a UE graph is perfect if and only if it does not contain an odd chordless cycle of length five of more, and all DE graphs are perfect. Hence $\mathrm{DE} \subset \mathrm{UEH} \cap$ PERFECT. Let $G' = (V, E)$ be a graph in UEH\DE. It remains to show that G' contains an odd chordless cycle of length at least five.

By the Separator Theorem, since G' is in UEH\DE, some atom G of G' is the line graph of a multigraph H, where H is triangle free but not bipartite. Since H is not bipartite it contains an odd cycle. The smallest odd chordless cycle must be of length at least five since H is triangle free. Hence G has an odd chordless cycle of the same length, and so does G' since separators cannot break such a chordless cycle. ∎

Proof of Part (c). This is straightforward by combining Theorem 2, the Atom Theorem, and the Separator Theorem for UV graphs and $\mathrm{UE} \cap$ CHORDAL graphs. ∎

The inclusion and intersection relationships among various classes of path graphs are summarized in Fig. 6. Examples of graphs from Fig. 1 are shown in appropriate regions. Regions not shown are empty. Note that the set of RDV (=RDE) graphs is strictly contained in the set of DV graphs since G_a is DV but not RDV.

5. Proof of the Separator Theorem

In this section, we prove the Separator Theorem. We shall proceed in the sequence UEH, UV, DE, DV, RDV, and UE$\cap$CHORDAL graphs. The proof for CHORDAL graphs is trivial and is omitted. The algorithm for constructing a clique tree or representing tree is given also.

Proof of the Separator Theorem for UEH Graphs. If $G=(V, E)$ is a UEH graph and $x \in V$, let $\pi(x)$ denote the corresponding path in the clique tree. The proof will be given as a series of propositions. Throughout, let C separate G into $G_i = G(V_i \cup C)$, $1 \leqslant i \leqslant s$.

Proposition 1. *If G is UEH, then each G_i is UEH with a clique tree having C as a leaf-edge.*

Proof. Let T be a clique tree for G, and we will construct a clique tree T_i for each G_i. Let $\pi(V_i)$ be the subtree consisting of edges traversed by paths (vertices) in V_i. Since $G(V_i)$ is connected, so is $\pi(V_i)$. There is a unique path of vertices in T given by $\pi^* = \{v_0,..., v_n\}$ with v_0 in $\pi(V_i)$, $[v_{n-1}, v_n] = C$, and $v_1,..., v_{n-2}$ not in $\pi(V_i)$. Note that when C is connected to $\pi(V_i)$ we have $n=1$. Construct T_i by augmenting $\pi(V_i)$ with one edge $[v_0, u]$, where u is a new vertex. Let $[v_0, u]$ correspond to clique C. There is a one-to-one correspondence between cliques of G_i and edges of T_i. Furthermore, $\pi'(x) = \pi(x)$ for $x \in V_i$, $\pi'(x) = \{[v_0, u]\} \cup (\pi(x) \cap \pi(V_i))$ for $x \in C$ are the representing paths. Hence, T_i is a UEH clique tree for G_i with leaf-edge C. ∎

Proposition 2. *Each G_i contains no antipodal pair of cliques (with respect to C).*

Proof. From the previous proof, we see that each G_i is a UEH graph which has a clique tree T_i with C as a leaf. Suppose that a subgraph G_i contains cliques C_1 and C_2, where $C_1 \leftrightarrow C_2$; then consider $x \in (C_1 \backslash C_2) \cap C$, $y \in (C_2 \backslash C_1) \cap C$, and $z \in C_1 \cap C_2 \cap C$. The path $\pi(z)$ in T_i contains the cliques C, C_1, and C_2 in either sequence (C, C_1, C_2) or (C, C_2, C_1). In the former case, the path $\pi(y)$ contains both C and C_2, hence must contain C_1. But $y \notin C_1$, a contradiction. In the latter case $\pi(x)$ produces a similar contradiction. ∎

In a UEH clique tree T, let $\pi(C', C'')$ denote the path (consisting of edges) between the edges in the tree corresponding to cliques C' and C''. Observe that C' and C'' are *on the same side* of C in the tree if and only if $C \notin \pi(C', C'')$. Let $\pi \oplus \pi' = \{x\colon x \text{ in either } \pi \text{ or } \pi' \text{ but not both}\}$. We state a few useful facts before proceeding with the proof.

PROPOSITION 3. *Let C', C'' be two cliques on the same side of C. If C' and C'' are attached then either $C' \in \pi(C, C'')$ or $C'' \in \pi(C, C')$. Furthermore, if $C' > C''$ then $C' \in \pi(C, C'')$.*

Proof. Consider $x \in C' \cap C'' \cap C$. Since C is a leaf and $\pi(x)$ is a path containing C', C'', and C, we must have either $C' \in \pi(C, C'')$ or $C'' \in \pi(C, C')$. Now let $C' > C''$. The latter case cannot happen for it would imply that $C' \in \pi(y)$ for every $y \in (C' \backslash C'') \cap C \neq \emptyset$, contradicting $C' > C''$. ∎

PROPOSITION 4. *If $C' \in \pi(C, C'')$, then $C' \geqslant C''$.*

Proof. For every $x \in C \cap C''$, we have $C' \in \pi(C, C'') \subseteq \pi(x)$, hence $x \in C'$. ∎

PROPOSITION 5. *Let C_i and C_i' be two (relevant) cliques of a subgraph G_i; then every clique in $\pi(C_i, C_i')$ is in G_i.*

Proof. Consider $x \in C_i \backslash C$ and $y \in C_i' \backslash C$; then there is a path $\mathbf{p}$ (consisting of vertices) in $G(V_i)$ connecting x and y. Let $\pi(\mathbf{p})$ be the union of all paths (consisting of edges) in T corresponding to members of $\mathbf{p}$; then $\pi(C_i, C_i') \subseteq \pi(\mathbf{p})$. Every clique in $\pi(C_i, C_i')$ contains some vertex of V_i, hence is in G_i because vertices from different subgraphs are nonadjacent. ∎

PROPOSITION 6. *The G_i's can be 2-colored such that no antipodal pairs have the same color.*

Proof. Consider a clique tree T of G. From the previous proof we know that, for each i, the cliques of G_i are on the same side of the edge C. Therefore, we can partition the G_i's into two parts according to which side of C they are on. We show that neither part contains antipodal subgraphs.

For the remainder of this proposition, we consider only cliques on one side of C. Let G_1 and G_2 be antipodal and on the same side of C; we will prove this leads to a contradiction. There are four cases:

Case 1. *$C_1 \leftrightarrow C_2$ for some C_1 in G_1, and some C_2 in G_2.* Since C_1 and C_2 are attached either $C_1 \in \pi(C, C_2)$ or $C_2 \in \pi(C, C_1)$ by Proposition 3. Then by Proposition 4, $C_1 \geqslant C_2$ or $C_2 \geqslant C_1$, a contradiction.

Case 2. *$C_1 > C_2$, $C_1' < C_2'$, for some C_1, C_1' in G_1, and some C_2, C_2' in G_2.* We have $C_1 \in \pi(C, C_2)$ and $C_2' \in \pi(C, C_1')$ by Proposition 3. Note that $\pi(C_1, C_1') = \pi(C, C_1) \oplus \pi(C, C_1')$ and $\pi(C_2, C_2') = \pi(C, C_2) \oplus \pi(C, C_2')$ because C is a leaf. By Proposition 5, $C_2' \notin \pi(C_1, C_1')$, hence $C_2' \in \pi(C, C_1)$. Similarly, we obtain $C_1 \in \pi(C, C_2')$. But the last two inclusion conditions are contradictory.

Case 3. $C_1 > C_2$, $C'_1 \geqslant C'_2$, $C'_1 | C''_2$, *for some* C_1, C'_1 *in* G_1, *and some* C_2, C'_2, C''_2 *in* G_2. Since $C'_1 \notin \pi(C'_2, C''_2) = \pi(C, C'_2) \oplus \pi(C, C''_2)$ By Proposition 5, and $C'_1 \notin \pi(C, C''_2)$ by Proposition 4, we have $C'_1 \notin \pi(C, C'_2)$; hence $C'_1 \sim C'_2$ and $C'_2 \in \pi(C, C'_1)$ by Proposition 3. But by Proposition 5, $C'_2 \notin \pi(C_1, C'_1) = \pi(C, C_1) \oplus \pi(C, C'_1)$; therefore $C'_2 \in \pi(C, C_1)$. But $C_1 \in \pi(C, C_2)$ by Proposition 3, and $C_1 \notin \pi(C_2, C'_2) = \pi(C, C_2) \oplus \pi(C, C'_2)$ by Proposition 5. So $C_1 \in \pi(C, C'_2)$, a contradiction.

Case 4. $C_1 \sim C_2$, $C_1 | C'_2$, $C''_2 \sim C'_1$, $C''_2 | C''_1$ *for some* C_1, C'_1, C''_1 *in* G_1 *and some* C_2, C'_2, C''_2 *in* G_2. From $C_1 \notin \pi(C, C'_2)$ by Proposition 4, and $C_1 \notin \pi(C_2, C'_2) = \pi(C, C_2) \oplus \pi(C, C'_2)$ by Proposition 5, we have $C_1 \notin \pi(C, C_2)$; hence $C_2 \in \pi(C, C_1)$ by Proposition 3. Then $C_2 \notin \pi(C_1, C'_1) = \pi(C, C_1) \oplus \pi(C, C'_1)$. Hence $C_2 \in \pi(C, C'_1)$. Symmetrically, we obtain $C'_1 \in \pi(C, C_2)$, a contradiction.

Combining these four cases, we complete the proof. ∎

This completes the proof of the forward part of the Separator Theorem for UEH graphs.

PROPOSITION 7. *Let C be a clique in the UEH graph G. If C is not a separator, then C is a leaf in any clique tree of G.*

Proof. If C is not a leaf, then there are (not necessarily relevant) cliques C_1 and C_2 incident to C from opposite ends of the edge C. Let $x \in C_1 \backslash C$, $y \in C_2 \backslash C$. Then x and y are not connected in $G(V \backslash C)$ because if they are, then $\pi(C_1, C_2)$ and C form part of a cycle. Hence, C is a separator. ∎

PROPOSITION 8. *Let C separate G into $G_i = G(V_i \cup C)$, $1 \leqslant i \leqslant s$. Assume each G_i is UEH, and that $G_1, G_2, \ldots, G_t$ are pairwise non-antipodal. Then $G(V_1 \cup \cdots \cup V_t \cup C)$ is a UEH graph with a clique tree having C as a leaf.*

Proof. By Lemma 11 we can arrange the first $t < s$ subgraphs so that $G_i > G_j$ implies $i < j$. We will recursively construct a UEH clique tree for $G(C \cup V_1 \cup \cdots \cup V_n)$ for $n = 1, 2, \ldots, t$ with C as a leaf.

By Lemma 4, C is not a separator of $G_i = G(C \cup V_i)$, for any i, $1 \leqslant i \leqslant s$. By Proposition 7, C is a leaf in any clique tree of G_i, $1 \leqslant i \leqslant s$. In particular, $G(C \cap V_1)$ has a clique tree with C as a leaf.

Assume $G(C \cup V_1 \cup \cdots \cup V_{n-1})$ has a clique tree $T^{(n-1)}$ with C as a leaf. The subgraph $G(C \cup V_n)$ has a clique tree T_n with C as a leaf. Let C correspond to leaves $[v_0, v_1]$ and $[v'_0, v'_1]$ in $T^{(n-1)}$ and T_n, where v_0 and v'_0 have degree 1.

First consider the case where G_n is not dominated by any G_i, $1 \leqslant i < n$. Then G_n must be unattached to any G_i, $1 \leqslant i < n$. Merge the two C-edges in $T^{(n-1)}$ and T_n to make a bigger tree $T^{(n)}$ by identifying v_0 with v'_0 and v_1

with v_1'. By Lemma 3, the edges in $T^{(n)}$ correspond to cliques in $G(C \cup V_1 \cup \cdots \cup V_n)$. Let $v \in C \cup V_1 \cup \cdots \cup V_n$. If $v \in V_1 \cup \cdots \cup V_{n-1}$, then the cliques containing v form a path in $T^{(n-1)}$, hence in $T^{(n)}$ also. If $v \in V_n$, then the cliques containing v form a path in T_n, hence in $T^{(n)}$ also. If $v \in C$ there are two cases: If v is not adjacent to any vertex in $V_1 \cup \cdots \cup V_n$, then there is only one clique, C, containing v in $T^{(n)}$. Otherwise, $v \in C$ is either adjacent to some vertex in $V_1 \cup \cdots \cup V_{n-1}$ or to some vertex in V_n. (If v adjacent to both sets, then $G_n \leftrightarrow G_i$ for some $1 \leqslant i < n$.) In the first case, the cliques containing v form a path in $T^{(n-1)}$, and there is only one clique, C, containing v in T_n. Hence the clique containing v is a path in $T^{(n)}$; the second case is similar.

Next consider the other case, where G_n is dominated by some G_i, $1 \leqslant i < n$. Let $\mathbf{A} = \{C_1, \ldots, C_k\}$, be the set of cliques with each C_j, $1 \leqslant j \leqslant k$, in some G_i, $1 \leqslant i < n$, and each C_j dominating every clique of G_n. Also, let $C \in \mathbf{A}$. We claim that, for any $v \in C$, v is adjacent to some vertex in V_n, we have $\mathbf{A} = \pi(v)$ in $T^{(n-1)}$. On one hand, for any $C' \in \mathbf{A}$, we have $C' \geqslant C''$, where C'' is any (relevant) clique in G_n containing v. Hence $C' \in \pi(v)$. On the other hand, consider any clique $C' \in \pi(v)$, C' in some G_a, $1 \leqslant a < n$. Then G_a and G_n are attached via v, hence $G_a \geqslant G_n$. Since C' is attached to the cliques in G_n containing v, C' dominates all cliques of G_n and $C' \in \mathbf{A}$.

Let C and $C^* = [u_0, u_1]$ be the two ends of the path $\mathbf{A}$ in $T^{(n-1)}$, where u_0 is closer to C than u_1 is, Let $C = [w_0, w_1]$ in T_n, where w_0 has degree 1. Then we can construct $T^{(n)}$ by identifying u_0 with w_0 and u_1 with w_1 to merge $T^{(n-1)}$ and T_n. The resulting $T^{(n)}$ is a UEH clique tree for $G(C \cup V_1 \cup \cdots \cup V_n)$. ∎

For example, the subgraphs $G_1 > G_2$ in Fig. 4a are both UEH; their clique trees are shown in Fig. 4b. In $T^{(1)} = T_1$, the path $\mathbf{A}$ consists of the edges $\{1, 2, a\}$ and $C = \{1, 2, 3, 4\}$. The trees can be glued together by coalescing the $\{1, 2, a\}$ edge of T_1 and the C edge of T_2.

PROPOSITION 9. *Let C separate G into $G_i = G(V_i \cup C)$, $1 \leqslant i \leqslant s$. If the G_i's are UEH and they are 2-colorable such that no antipodal pairs have the same color, then G is UEH.*

Proof. By Proposition 8, for each color, we can construct a clique tree with C as a leaf. Let C be the edge $[v_0, v_1]$ in one tree and $[v_0', v_1']$ the edge in the other, where v_0 and v_1' have degree 1. Then make a bigger tree T by identifying v_0 with v_1' and v_1 with v_0'. Then T is a clique tree for G; hence G is a UEH graph by the Clique Tree Theorem. ∎

This completes the proof of the Separator Theorem for UEH graphs. Propositions 8 and 9 outline an algorithm to construct a clique tree for UEH graphs.

Proof of the Separator Theorem for UV Graphs. The proof proceeds as a series of propositions in a manner similar to the UEH case.

PROPOSITION 1′. *If G is UV, then each G_i is UV with a clique tree T_i having C as a leaf node.*

Proof. Let T be a UV clique tree for G; we will construct a UV clique tree T_i for each G_i. Let $\pi(V_i)$ be the subtree consisting of vertices traversed by paths corresponding to the vertices in V_i. Since $G(V_i)$ is connected, so is $\pi(V_i)$. There is a unique path $\pi^* = \{v_0, ..., v_n\}$ in T with $v_0 = C$, $v_1, ..., v_{n-1} \notin \pi(V_i)$, and $v_n \in \pi(V_i)$. Construct T_i by augmenting $\pi(V_i)$ by a new vertex v^* and a new edge $[v^*, v_n]$. Then T_i is a UV clique tree for G_i. ∎

In the tree T, let $\pi(C', C'')$ denote the path (consisting of vertices) from C' to C''. Observe that C' and C'' are *on the same branch* (with respect to root C) if and only if $C \notin \pi(C', C'')$. Recall that the symmetric sum is defined as $\pi \oplus \pi' = \{x$: x is in either π or π' but not both$\}$. Using this notation, we can easily derive analogies of Propositions 2–5.

PROPOSITION 2′. *Each G_i contains no antipodal cliques (with respect to C).*

PROPOSITION 3′. *Let C' and C'' be two cliques on the same branch of C. If C' and C'' are unattached, either $C' \in \pi(C, C'')$ or $C'' \in \pi(C, C')$. Furthermore, if $C' > C''$ then $C' \in \pi(C, C'')$.*

PROPOSITION 4′. *If $C' \in \pi(C, C'')$, then $C' \geqslant C''$.*

PROPOSITION 5′. *Let C_i and C_i' be two (relevant) cliques of a subgraph G_i; then every clique in $\pi(C_i, C_i')$ is in G_i.*

PROPOSITION 6′. *The G_i's can be colored such that no antipodal pairs have the same color, and for each $v \in C$, the neighboring subgraphs of v are 2-colored.*

Proof. Consider a clique tree T of G. Consider T as a rooted tree with C as root. For each i, $1 \leqslant i \leqslant s$, the cliques of G_i are contained in one branch of the tree T. Let there be k branches emanating from C. We will color the subgraphs using k colors. A subgraph is colored according to which branch of the tree it is on.

Assuming subgraphs $G_1 \leftrightarrow G_2$, and that G_1 and G_2 have the same color, we will exhibit a contradiction. There are four cases to consider.

Case 1. *$C_1 \leftrightarrow C_2$ for some C_1 in G_2, and some C_2 in G_2.* Since C_1 and

C_2 are attached, Proposition 3′ implies that either $C_1 \in \pi(C, C_2)$ or $C_2 \in \pi(C, C_1)$. By Proposition 4′ either $C_1 \geqslant C_2$ or $C_2 \geqslant C_1$, a contradiction.

Case 2. $C_1 > C_2$, $C_1' < C_2'$, *for some* C_1, C_1' *in* G_1, *and some* C_2, C_2' *in* G_2. We have $C_1 \in \pi(C, C_2)$ and $C_2' \in \pi(C, C_1')$ by Proposition 3′. Note that $\pi(C_1, C_1') = \pi(C, C_1) \oplus \pi(C, C_1')$ and $\pi(C_2, C_2') = \pi(C, C_2) \oplus \pi(C, C_2')$. By Proposition 5′, $C_2' \notin \pi(C_1, C_1')$, hence $C_2' \in \pi(C, C_1)$. Similarly, we obtain $C_1 \in \pi(C, C_2')$. The last two conclusions are contradictory.

Case 3. $C_1 > C_2$, $C_1' \geqslant C_2'$, $C_1' \mid C_2''$, *for some* C_1, C_1' *in* G_1, *and some* C_2, C_2', C_2'' *in* G_2. Since $C_1' \notin \pi(C_2', C_2'') = \pi(C, C_2') \oplus \pi(C, C_2'')$ by Proposition 5′, and $C_1' \notin \pi(C, C_2'')$ by Proposition 4′, we have $C_1' \notin \pi(C, C_2')$; hence $C_1' \sim C_2'$ and $C_2' \in \pi(C, C_1')$ by Proposition 3′. But $C_2' \notin \pi(C_1, C_1') = \pi(C, C_1) \oplus \pi(C, C_1')$; therefore $C_2' \in \pi(C, C_1)$. But $C_1 \in \pi(C, C_2)$ by Proposition 3′, and $C_1 \notin \pi(C_2, C_2') = \pi(C, C_2) \oplus \pi(C, C_2')$ by Proposition 5, so $C_1 \in \pi(C, C_2')$, a contradiction.

Case 4. $C_1 \sim C_2$, $C_1 \mid C_2'$, $C_2'' \sim C_1'$, $C_2'' \mid C_1''$, *for some* C_1, C_1', C_1'' *in* G_1, *and some* C_2, C_2', C_2'' *in* G_2. $C_1 \notin \pi(C, C_2')$ by Proposition 4′, and $C_1 \notin \pi(C_2, C_2') = \pi(C, C_2) \oplus \pi(C, C_2')$ by Proposition 5′, so we have $C_1 \notin \pi(C, C_2)$; hence $C_2 \in \pi(C, C_1)$ by Proposition 3′. Similarly we have $C_1' \in \pi(C, C_2'')$. Then $C_1' \notin \pi(C_2', C_2'') = \pi(C, C_2') \oplus \pi(C, C_2'')$ by Proposition 5′, and $C_1' \in \pi(C, C_2')$. Similarly $C_2' \in \pi(C, C_1')$, a contradiction.

Combining these four cases we have that no antipodal subgraphs have the same color. For each $v \in C$, the path in T corresponding to v is contained in at most two branches from the root C. Therefore, the neighboring subgraphs of v have at most two colors among them. ∎

This completes the forward part of the Separator Theorem for UV graphs. We can easily derive these analogies to Propositions 7 and 8.

PROPOSITION 7′. *Let C be a clique in the UV graph G. If C is not a separator, then C is a leaf node* (*i.e., a node with degree* 1) *in any clique tree of G.*

PROPOSITION 8′. *Let C separate G into $G_i = G(V_i \cup C)$, $1 \leqslant i \leqslant s$. Assume each G_i is UV, and that $G_1, ..., G_t$ are pairwise non-antipodal. Then $G(V_1 \cup \cdots \cup V_t \cup C)$ is a UE graph with a clique tree T having C as a leaf node.*

Proof. By Lemma 11, we can arrange the subgraphs $G_1, ..., G_t$ so that $G_i > G_j$ implies $i < j$. We will recursively construct a clique tree for $G(C \cup V_1 \cup \cdots \cup V_n)$, $n = 1, 2, ..., t$, with C as a leaf node.

By Lemma 4, C is not a separator of G_i, $1 \leqslant i \leqslant s$. By Proposition 7′, C is a leaf node in the clique tree T_i of G_i, $1 \leqslant i \leqslant s$. In particular, G_1 has a clique tree with C as a leaf node.

Assume $G(C \cup V_1 \cup \cdots \cup V_{n-1})$ has a clique tree $T^{(n-1)}$ with leaf node C. Let C correspond to node v in $T^{(n-1)}$ and to node v' in T_n.

First consider the case where G_n is not dominated by any G_i, $1 \leqslant i < n$. Then form $T^{(n)}$ by merging v in $T^{(n-1)}$ and v' in T_n. That is, identify the two vertices into one. By Lemma 3, there is a one-to-one correspondence between the vertices of $T^{(n)}$ and the cliques of $G(C \cup V_1 \cup \cdots \cup V_n)$. If $v \in V_1 \cup \cdots \cup V_{n-1}$, then the cliques containing v are all in $G(C \cup V_1 \cup \cdots \cup V_{n-1})$ and they form a path in $T^{(n-1)}$; hence they form a path in $T^{(n)}$. If $v \in V_n$, then the cliques containing v are all in $G(C \cup V_n)$ and they form a path in T_n; hence they form a path in $T^{(n)}$. Finally, if $v \in C$, there are two cases: If v is not adjacent to any vertex in $V_1 \cup \cdots \cup V_n$, then C is the only clique in $G(C \cup V_1 \cup \cdots \cup V_n)$ which contains v. Otherwise, v is either adjacent to some vertex in $V_1 \cup \cdots \cup V_{n-1}$ or adjacent to some vertex in V_n. (If v is adjacent to both, then $G_n \leqslant G_i$ for some $1 \leqslant i < n$.) In the former, the cliques containing v form a path in $T^{(n-1)}$ and C is the only clique in T_n containing v. In the latter, C is the only clique in $T^{(n-1)}$ containing v and the cliques containing v is T_n form a path. Either way, the cliques containing v in $T^{(n)}$ form a path. Therefore, $T^{(n)}$ is a clique tree for $G(C \cup V_1 \cup \cdots \cup V_n)$. Also, every path $\pi(x)$ with $x \in C$, has C at one end in $T^{(n)}$.

Next we consider the case where G_n is dominated by some G_i, $1 \leqslant i < n$. Let $\mathbf{A} = \{C_1, ..., C_k\}$, be the set of cliques with each C_j, $1 \leqslant j \leqslant k$, in some G_i, $1 \leqslant i < n$, and each C_j dominating every clique in G_n. We can show that, for any $v \in C$, v adjacent to some vertex V_n, we have $\mathbf{A} = \pi(v)$ in $T^{(n-1)}$. On one hand, for any $C' \in \mathbf{A}$, we have $C' \geqslant C''$, where C'' contains v in G_n. Hence $C' \in \pi(v)$. On the other hand, for any $C' \in \pi(v)$ in $T^{(n-1)}$, we have C' attached to C'' in G_n, C'' containing v. Hence C' dominates every clique of G_n, $C' \in \mathbf{A}$. Let C and C^* be the two ends of $\mathbf{A}$. Construct $T^{(n)}$ by merging C^* in $T^{(n-1)}$ with C in T_n. ∎

PROPOSITION 9′. *Let C separate G into $G_i = G(C \cup V_i)$, $1 \leqslant i \leqslant s$. If each G_i is UV, and they can be colored in such a way that no antipodal pairs have the same color and that, for every $v \in C$, the neighboring subgraphs of v are two-colored, then G is UV.*

Proof. By Proposition 8′, for each color, we can construct a clique tree. Next we construct a clique tree T for G by "gluing" up all the clique trees of different colors. We merge all the vertices corresponding to C in the clique trees of various colors into one vertex.

If a vertex $v \in C$, then the cliques containing v form a path in one of the subtrees. If $v \in C$, then in at most two subtrees, the cliques containing v form a path with C at one end. After gluing, the cliques form a path in T.

By Lemma 3, there is a one-to-one correspondence between cliques of G and vertices of T. By the Clique Tree Theorem, G is a UV graph. ∎

This completes the proof of the Separator Theorem for UV graphs. Propositions 8′ and 9′ outline an algorithm to construct a clique tree for UV graphs.

Proof of the Separator Theorem for DE Graphs. This is similar to the proof for UEH graphs, with some extra attention paid to Propositions 8 and 9. By Proposition 8, the clique trees $T_1,..., T_t$ for the pairwise non-antipodal DE subgraphs $G_1,..., G_t$ are "glued" together to make a clique tree for $G(C \cup V_1 \cup \cdots \cup V_t)$. We accomplish this by iteratively merging the directed leaf edge $C = (w_0, w_1)$ in T_n with a specific directed edge $C^* = (u_0, u_1)$ in $T^{(n-1)}$ (which is a UE clique tree for $G(C \cup V_1 \cup \cdots \cup V_{n-1})$), where u_0 is closer to $C = (v_0, v_1)$ in $T^{(n)}$ than u_1 is. If w_0 has degree 0, then we identify w_0 with u_0 and w_1 with u_1. Otherwise, w_1 has degree 1, and we reverse the direction of all edges in T_n, then identify w_0 with u_1 and w_1 with u_0. The resulting $T^{(n)}$ is the desired DE clique tree. Other portions of the proof can be modified by this technique of reversing the directions of all arcs when necessary. ∎

Proof of the Separator Theorem for DV Graphs. This is similar to the proof for UV graphs, with the following special attention to the orientation of the edges in the clique tree.

In Proposition 6′, the UV clique tree is considered a rooted tree with C as the root, and each subgraph G_i is colored according to which branch of the tree it is on. If there are k branches, then k colors are used. For a DV clique tree, we will use only two colors. Use one color on the branches with an edge leading into the node C, and use another color for all other branches. For each $v \in C$, the path $\pi(v)$ occupies at most one branch in each color. Two like-colored subgraphs on different branches are necessarily unattached. Therefore two colors suffice.

In Propositions 8′ and 9′ when two DV clique trees are glued together to make a larger DV clique tree, sometimes we have to reverse the directions of all arcs in one tree. The proof remains valid after such modifications. ∎

Proof of the Separator Theorem for RDV Graphs. If G is RDV, consider an RDV clique tree T for G. Without loss of generality, assume C is not the root. Color each subgraph according to the branch it is on in the same way as the coloring of DV subgraphs. Two antipodal subgraphs are colored differently.

Let T^* be the subtree of T rooted at C. Then T^* is an RDV representation tree for every subgraph having the color opposite that of the subgraph containing the root clique. A clique tree with the same root can be easily derived from T^*. Next, we consider the subgraphs having the same color as the subgraph containing the root clique.

The vertices corresponding to (relevant) cliques form a contiguous part of the path from the root to C. Hence no two (relevant) cliques are unattached, and no two subgraphs are unattached. In each subgraph G_i, consider the (relevant) clique that is closest to the root. The subtree of T rooted at the vertex corresponding to this clique is an RDV representation tree for G_i, and a clique tree with the same root can be easily derived. The only possible exception is the subgraph containing the root clique, G_x, and the exception occurs exactly when the root clique is not a relevant clique. In this case, G_x is dominated by every other like-colored subgraph, and the tree obtained from T by removing T^* is an RDV representation tree for G_x. A clique tree with C as a leaf can be easily derived.

Conversely, assume each separated subgraph is RDV and that they can be 2-colored satisfying the conditions of the theorem. The RDV clique trees rooted at C for the subgraphs of the first color can be glued together by the method described in the proof of Proposition 8′ to make an RDV clique tree T' rooted at C. In the RDV clique tree for each subgraph of the second color cliques form the path from the root to the leaf C. Hence these trees can be glued together by the same method to make a clique tree T'' rooted at a relavant clique and having C as a leaf. The clique tree of the exception subgraph can be glued to T'' to form an RDV clique tree T''' in which C is a leaf. Then T' and T''' can be glued at the C nodes to obtain an RDV clique tree T for G. ▌

Proof of the Separator Theorem for $UE \cap CHORDAL$ Graphs. Again, the proof proceeds in a series of propositions. But first, we need some more definitions.

A path π *traverses* a (relevant) clique C' if (1) π contains C' when C' is an edge clique; or (2) π contains an arm A of C' with $A \in \pi(x)$ for some $x \in C' \backslash C$, when C' is a claw clique. Let $\pi = \pi_1 \oplus \pi_2$. If neither π_1 nor π_2 traverses C', then π does not traverse C'; if either π_1 or π_2 (but not both) traverses C', then π traverses C'. The case when both π_1 and π_2 tranverse C' will not be encountered in this paper.

If C is a claw clique, we can *associate* each G_i with an arm of C as follows. If $\pi(V_i)$ does not contain any arm of the separating claw clique C, then consider the unique path π connecting $\pi(V_i)$ and C. π cannot emanate from the center of C while keeping V_i and C connected in G, so it emanates from one arm of C; associate G_i with this arm. If $\pi(V_i)$ contains exactly one arm of C, associate G_i with this arm. We claim $\pi(V_i)$ cannot contain two or more arms of C. Otherwise, there exist some $x, y \in V_i$ where $\pi(x)$ contains one arm A_1 and $\pi(y)$ contains another arm A_2. We have $x \neq y$ since $x, y \notin C$. Let A_1, A_2, A_3 also denote the completely connected set in V represented by these edges. There exist some $u \in A_1 \cap A_2 \cap C$, some $v \in (A_1 \backslash A_2) \cap C$, and some $w \in (A_2 \backslash A_1) \cap C$. Let $\mathbf{p} \subseteq V_i$ be the shortest path between x and y in $G(V_i)$. Since G_i is chordal, by focusing on the cycle formed by v, w, and $\mathbf{p}$, we can find a vertex $z \in \mathbf{p}$, z adjacent to both v and w. By focusing on the cycle formed by u and $\mathbf{p}$, we find that z is adjacent to u. Since $\pi(u)$, $\pi(v)$, and $\pi(w)$ each contain a different pair of arms of C, $\pi(z)$ must contain two arms of C in order to intersect all three paths. But $z \notin C$, a contradiction. Therefore $\pi(V_i)$ contains at most one arm of C. In the case when C is an edge clique, we define the "arms" of C to be the two sides of the edge and associate each G_i with the side it is on.

We will 3-color the subgraphs according to their association with the arms of C, when C is a claw clique; and 2-color the subgraphs according to which side of C they are on, when C is an edge clique. If G_i is associated to an arm of C, then its (relevant) cliques are considered to be associated with this arm.

Let $\pi(C', C'')$ denote the unique path (consisting of edges), from C' to C''. Let the path start at the center vertex when C' (or C'') is a claw clique, and let the path include C' (or C'') when it is an edge clique.

PROPOSITION 1″. *If G is a UE and CHORDAL graph, then G_i is UE and CHORDAL with a UE representation tree where C is a leaf-edge.*

Proof. The proof is identical to that for UEH graphs when C is an edge clique. When C is a claw clique, we need to replace references to the "edge" for C by the "arm of C associated with G_i." The proof goes through as before. ∎

PROPOSITION 2″. *Each G_i contains no antipodal cliques (with respect to C).*

Proof. By Proposition 1″, G_i is UE and CHORDAL and has a UE representation tree T_i with C a leaf-edge. Suppose, in order to obtain a contradiction, that G_i has antipodal cliques $C' \leftrightarrow C''$, i.e., there exists $u \in C \cap C' \cap C''$, $v \in (C' \cap C) \backslash (C'' \cap C)$, and $w \in (C'' \cap C) \backslash (C' \cap C)$. Note that $\pi(u)$ contains C' (or C'') if it is an edge clique and contains two arms of C' (or C'') if it is a claw clique. If $\pi(u)$ reaches C' (resp. C'') first when starting from C, then $C' \geqslant C''$ (resp. $C'' \geqslant C'$), a contradiction. If $\pi(u)$ meets C' and C'' both for the first time on the same edge, and C' is an edge clique, C'' is a claw clique (or vice versa), then $C' \geqslant C''$ (or vice versa).

Assume $\pi(u)$ meets C' and C'' both for the first time on the same edge A in T_i and C' and C'' are both claw cliques. Then there is a four-prong star consisting of arms A, B, D, and E in T_i where C' has arms A, B, and D; C'' has arms A, B and E; $\pi(u)$ contains A and B; $\pi(v)$ contains A and D; and $\pi(w)$ contains A and E. By the definition of claw cliques, there is a path $\pi(x)$ containing B, E and a path $\pi(y)$ containing B, E. Then $\{v, w, x, y\}$ form a chordless cycle of length four, contradicting the chordality of G_i. ∎

The last of the proof shows that two attached (with respect to C) cliques of a UE$\cap$CHORDAL graph cannot share two arms.

PROPOSITION 3″. *Let C' and C'' be associated with the same arm of C. If C' and C'' are attached, then either $\pi(C, C'')$ traverses C' or $\pi(C, C')$ traverses C''. Further, if $C' > C''$ then $\pi(C, C'')$ traverses C'.*

Proof. The path $\pi(v)$, $v \in C \cap C' \cap C''$, contains C' (or C'') when it is an edge clique and contains two arms of C' (or C'') when it is a claw clique. Think of $\pi(v)$ as starting from C and going toward C' and C''. If $\pi(v)$ reaches C' first, then $\pi(C, C'')$ traverses C'. If $\pi(v)$ reaches C'' first, then $\pi(C, C')$ traverses C''. Assume $\pi(v)$ reaches both C' and C'' simultaneously. In this case, C' and C'' cannot both be edge cliques. If C' is a claw and C'' is an edge, then $\pi(C, C')$ traverses C''. C' and C'' cannot both be claws, since then they would share two arms and contradict the observation immediately following the proof of Proposition 2″.

Furthermore, if $C' > C''$, then $\pi(C, C'')$ traverses C' in all cases. ▮

PROPOSITION 4″. *If $\pi(C, C'')$ traverses C', then $C' \geqslant C''$.*

Proof. For every $x \in C \cap C''$, $\pi(x)$ contains $\pi(C, C'')$ which traverses C'. Hence $x \in C'$. ▮

PROPOSITION 5″. *Let C_i and C_i' be two (relevant) cliques of G_i. Then every clique traversed by $\pi(C_i, C_i')$ is in G_i.*

Proof. Consider $x \in C_i \backslash C$, $y \in C_i' \backslash C$. Then there is a path $\mathbf{p}$ connecting x and y in $G(V_i)$. Let $\pi(\mathbf{p}) = \bigcup_{v \in \mathbf{p}} \pi(v)$. Then $\pi(C_i, C_i') \subseteq \pi(\mathbf{p})$. Let C_0 be a clique traversed by $\pi(C_i, C_i')$. Then there is some $v \in C_0 \backslash C$ with $\pi(v) \cap \pi(C_i, C_1') \neq \emptyset$. But then v is adjacent to some vertex in $\mathbf{p}$. Hence, $v \in V_i$ and C_0 is in G_i. ▮

PROPOSITION 6″. *The G_i's can be 3-colored such that no antipodal pairs have the same color, and so that, for each $v \in C$, the neighboring subgraphs of v are 2-colored.*

Proof. Consider a UE representation tree T of G. According to our earlier discussion, each subgraph can be associated with an arm of C according to the position of $\pi(V_i)$ in the tree. We will 3-color the subgraphs according to the arm it is associated with. Any path $\pi(v)$, $v \in C$, contains at most two arms of C, so the neighboring subgraphs of v have at most two colors. It remains to show that no antipodal pairs of subgraphs have the same color.

For the remainder of the proof, we consider only one arm A of C, and the subgraphs associated with A. Note that all relevant cliques in these subgraphs are on the same side of A. Let G_1 and G_2 be antipodal and let both be associated with A; we will derive a contradiction. There are four cases:

Case 1. $C_1 \leftrightarrow C_2$ *for some* C_1 in G_1, *and some* C_2 *and* G_2. Since C_1 and C_2 are attached, $\pi(C, C_1)$ traverses C_2 or $\pi(C, C_2)$ traverses C_1 by Proposition 3″. But then $C_2 \geqslant C_1$ or $C_1 \geqslant C_2$ by Proposition 4″, a contradiction.

Case 2. $C_1 > C_2$, $C_1' < C_2'$, *for some* C_1, C_1' *in* G_1, *and some* C_2, C_2' *in* G_2. $\pi(C, C_1')$ traverses C_2' by Proposition 3″. Since all (relevant) cliques are on the same side of A, we have $\pi(C_1, C_1') = \pi(C, C_1) \oplus \pi(C, C_1')$. By Proposition 5″, $\pi(C_1, C_1')$ does not traverse C_2'. Hence $\pi(C, C_1)$ traverses C_2'. By a symmetric argument, $\pi(C, C_2')$ traverses C_1. The last two conclusions are contradictory.

Case 3. $C_1 > C_2$, $C_1' \geqslant C_2'$, $C_1' \mid C_2''$, *for some* C_1, C_1' *in* G_1, *and some* C_2, C_2', C_2'' *in* G_1. As before, $\pi(C_2', C_2'') = \pi(C, C_2') \oplus \pi(C, C_2'')$. By Proposition 5″, $\pi(C_2', C_2'')$ does not traverse C_1'. Also, $\pi(C, C_2'')$ does not traverse C_1' since $C_1' \mid C_2''$. Hence $\pi(C, C_2')$ does not traverse C_1'. Then since $C_1' \sim C_2'$, we have $\pi(C, C_1')$ traversing C_2' by Proposition 3″. Also, $\pi(C_1, C_1') = \pi(C, C_1) \oplus \pi(C, C_1')$, and by Proposition 5″, $\pi(C_1, C_1')$ does not traverse C_2'. Hence $\pi(C, C_1)$ traverses C_2'. But we know that $\pi(C_2, C_2') = \pi(C, C_2) \oplus \pi(C, C_2')$, and that $\pi(C_2, C_2')$ does not traverse C_1 by Proposition 5″, while $\pi(C, C_2)$ does by Proposition 3″. Hence $\pi(C, C_2')$ must traverse C_1, a contradiction.

Case 4. $C_1 \sim C_2$, $C_1 \mid C_2'$, $C_2'' \sim C_1'$, $C_2'' \mid C_1''$, *for some* C_1, C_1', C_1'' *in* G_1, *and some* C_2, C_2', C_2'' *in* G_2. We have $\pi(C_2, C_2') = \pi(C, C_2) \oplus \pi(C, C_2')$. By Proposition 5″, $\pi(C_2, C_2')$ does not traverse C_1. Because $C_1 \mid C_2'$, $\pi(C, C_2')$ does not traverse C_1. Then $\pi(C, C_2)$ does not traverse C_1, and $\pi(C, C_1)$ traverses C_2 because $C_1 \sim C_2$. As before, $\pi(C_1, C_1') = \pi(C, C_1) \oplus \pi(C, C_1')$ does not traverse C_2. Hence $\pi(C, C_1')$ traverses C_2. Symmetrically, we obtain that $\pi(C, C_2)$ traverses C_1', a contradiction.

Combining four cases, we complete our proof. ▌

This completes the proof of the forward part of the Separator Theorem for UE $\cap$ CHORDAL graphs.

PROPOSITION 7″. *Let C be a clique in a UE and CHORDAL graph G_i. If C is not a separator, then there exists a UE representation tree of G_i having C as a leaf.*

Proof. Let T_i be a UE representation tree for G_i with the minimum number of edges. If C is an edge, then C is a leaf edge since C is not a separator. Otherwise, assume C is a claw. Let A_1, A_2, A_3 denote the completely connected sets represented by the three arms of C. We claim that at least two A_i's are subsets of C. Suppose the opposite. Then, without loss of generality, assume there are some $x \in A_1 \backslash C$ and some $y \in A_2 \backslash C$. Let $\mathbf{p} \subseteq V_i$ be the shortest path connecting x and y in $G(V_i)$, and let $u \in A_1 \cap A_2 \cap C$, $v \in (A_1 \backslash A_2) \cap C$, $w \in (A_2 \backslash A_1) \cap C$. Focusing on the cycle consisting of v, w, and $\mathbf{p}$, we find that there is some $z \in \mathbf{p}$, z adjacent to both v and w because G_i is chordal and $\mathbf{p}$ is the shortest path between x and y. Focusing on the cycle consisting of u and $\mathbf{p}$, we find that z is adjcent to u because G_i is chordal and $\mathbf{p}$ is the shortest path between x and y. Since $\pi(u)$, $\pi(v)$, and $\pi(w)$ each contain a different pair of arms of C, $\pi(z)$ must contain two arms of C in order to intersect all three paths. But $z \notin C$, a contradiction.

Without loss of generality, assume $A_2, A_3 \subseteq C$. These two arms of C are both leaves; otherwise C would be a separator. But then we can merge these two arms and let it represent C, contradicting the minimality of T_i. Therefore, C is a leaf edge in T_i. ∎

PROPOSITION 8″. *Let C separate G into $G_i = G(C \cup V_i)$, $1 \leqslant i \leqslant s$. Assume each G_i is UE and CHORDAL and that $G_1, G_2, \ldots, G_t$ are pairwise non-antipodal. Then $G(C \cup V_1 \cup \cdots \cup V_t)$ is a UE and CHORDAL graph with a UE representation tree having C as a leaf edge.*

Proof. The proof is similar to that of Proposition 8, with minor modifications. By Lemma 11, we can rearrange the subgraphs such that $G_i > G_j$ implies $i < j$. Assume $G(C \cup V_1 \cup \cdots \cup V_{n-1})$ is a UE and CHORDAL graph with a UE representation tree $T^{(n-1)}$ having C as a leaf. By Proposition 7″, G_n has a UE representation tree T_n having C as a leaf. We wish to combine $T^{(n-1)}$ and T_n to obtain a UE representation tree $T^{(n)}$ for $G(C \cup V_1 \cup \cdots \cup V_n)$.

We are done if $n = 1$, so assume $n \geqslant 2$. If G_n is not dominated by any G_i, $1 \leqslant i < n$, then $T^{(n)}$ can be easily obtained by merging the C-leaves of $T^{(n-1)}$ and T_n because G_n is unattached to every G_i, $1 \leqslant i < n$.

Assume G_n is dominated by some G_j, $1 \leqslant j < n$. Let $\mathbf{A}$ be the set of edges of $T^{(n-1)}$ which represent completely connected sets $S \subseteq V$ satisfy $C' \cap C \subseteq S \cap C$ for every clique C' of G_n (i.e., S "dominates" every clique C' of G_n). We will show that either $\mathbf{A} = \pi(v)$ in $T^{(n-1)}$ for every $v \in C$, with v adjacent to some member of V_n in G; or we can redefine some paths in $T^{(n-1)}$ to make this assertion true.

Let $S(e)$ denote the completely connected set represented by the edge e. Consider an arbitary $v \in C$, where v is adjacent to some member of V_n. For any $e \in \mathbf{A}$, and any (relevant) clique C' of G_n, with C' containing v, we have $C' \cap C \subseteq S(e) \cap C$. Hence $v \in S(e)$, $e \in \pi(v)$, and $\mathbf{A} \subseteq \pi(v)$. On the other hand, if $\pi(v) \subseteq \mathbf{A}$ we are done. Assume there is some edge $e_1 \in \pi(v) \backslash \mathbf{A}$ for some $v \in C$, and we will show that there are exactly two such edges.

Let C^* be a clique containing $S(e_1)$. C^* must be a claw for otherwise we have $C^* = S(e_1)$ and $e_1 \in \mathbf{A}$, a contradiction. Since $v \in C^*$ and v adjacent to V_n, we have that C^* dominates every clique of G_n. Since $S(e_1)$ does not dominate every clique in G_n, there exist some $u \in C^* \cap C$, u adjacent to V_n, $u \notin S(e_1)$. Let e_1, e_2, e_3 be the three arms of C^* with e_3 closest to C; then $e_1, e_3 \in \pi(v)$ and $e_2, e_3 \in \pi(u)$. Every edge $e \in \pi(C, C^*)$ "dominates" every clique C' of G_n because $C' \cap C \subseteq C^* \cap C \subseteq S(e) \cap C$. Hence $\pi(C, C^*) \subseteq \mathbf{A}$. For any $e \in \mathbf{A}$, we have $e \in \pi(u) \cap \pi(v) = \pi(C, C^*)$. Hence $\pi(C, C^*) = \mathbf{A}$. Next we show that for any $w \in C$, w adjacent to V_n, we have $\pi(w) = \mathbf{A} \cup \{e_1\}$ or $\mathbf{A} \cup \{e_2\}$. Note that $w \in C^*$ because w is adjacent to some vertex in V_n and C^* dominates every clique of G_n.

Earlier, we have showed that $\mathbf{A} \subseteq \pi(w)$. If $\pi(w) = \mathbf{A}$, then $w \notin C^*$, also C^* cannot dominate all cliques of G_n, a contradiction. Let e' denote an arbitrary edge in $\pi(w) \backslash \mathbf{A}$, and C' be an arbitrary clique containing $S(e')$. Since C' is attached to some clique of G_n via w, C' dominates all cliques of G_n. Hence C' contains u and v. Since $\pi(u) \cap \pi(v) = \mathbf{A}$, C' is a claw clique because an edge clique outside $\mathbf{A}$ cannot contain both u and v, hence cannot dominate every clique of G_n. Furthermore, C^* is the only claw clique containing all u, v, and w in $T^{(n-1)}$ because $\pi(u) \cap \pi(v) = \mathbf{A}$, and $\pi(w)$ properly contains $\mathbf{A}$. Therefore, $C' = C^*$, and $e' = e_1$ or e_2. Since $\pi(w)$ cannot contain both e_1 and e_2, we have $\pi(w) = \mathbf{A} \cup \{e_1\}$ or $\mathbf{A} \cup \{e_2\}$. Next we show how to reroute some paths in $T^{(n-1)}$ to obtain a new UE representation $(T^{(n-1)}, \mathbf{P}')$.

Let $S^* = \{u' \in S(e_1) | \ u'$ adjacent to some vertex in $V_n\}$. Note that $S^* \subseteq C \cap C^*$. For every $u' \in S^*$, let the new path be $\pi'(u') = \pi(u') \backslash \{e_1\} + \{e_2\}$. For all other vertices, use the old paths. If there is a vertex $w \in V$, where $\pi(w)$ includes e_1 but not e_2, then we claim $e_3 \in \pi(w)$. Suppose the opposite. Then let C_0 be any clique containing v and w. Since C_0 is attached to some clique of G_n via v, C_0 dominates every clique of G_n. But $u \notin C_0$ and u is adjacent to some vertex in V_n; hence C_0 cannot dominate every clique of G_n, a contradiction. Therefore $e_3 \in \pi(w)$, and $\pi'(w)$ and $\pi'(u')$ intersect at e_3, for any $u' \in S^*$. If there is a vertex $w \in V$, $\pi(w)$ includes e_2 but not e_1, then $e_3 \in \pi(w)$ by a symmetrical argument. Hence $(T^{(n-1)}, \mathbf{P}')$ is a UE representation of $G(C \cup V_1 \cup \cdots \cup V_n)$. Furthermore, let $\mathbf{A}'$ be the set of edges e in $T^{(n-1)}$ whose represented completely connected set $S(e) \subseteq V$ satisfies $C' \cap C \subseteq S(e) \cap C$ for every clique C' of G_n. Then $\mathbf{A}' = \mathbf{A} \cup \{e_2\}$ and $A' = \pi'(v)$ for every $v \in C$, with v adjacent to some member of V_n.

By merging the last edge of the path $\mathbf{A}'$ in $T^{(n-1)}$ with the leaf edge C in T_n, we obtain $T^{(n)}$. ∎

For example, Fig. 7a shows two UEH and CHORDAL subgraphs and Fig. 7b shows their UE representation trees. Each edge is labelled by the completely connected set to which it corresponds. Both subgraphs have

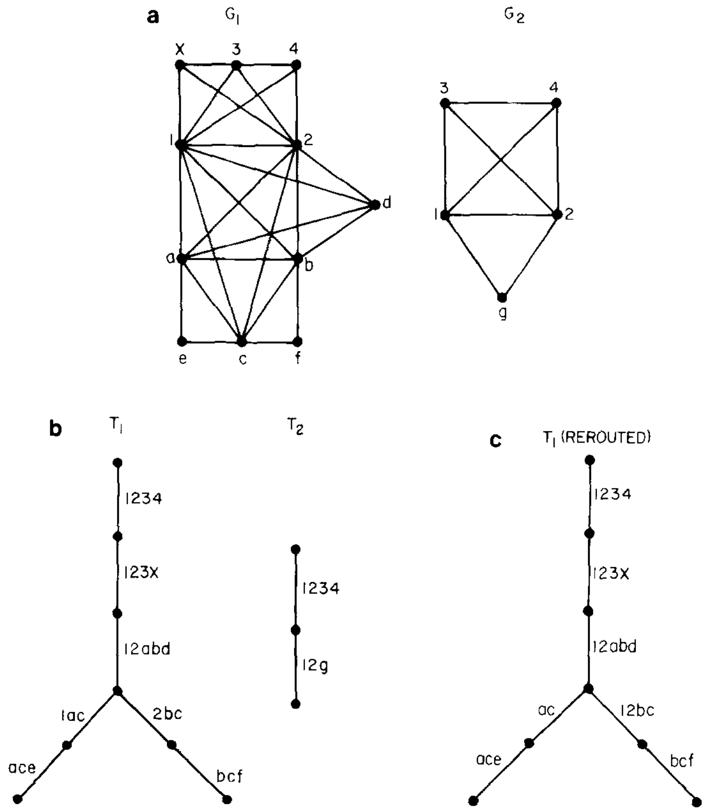


FIG. 7. (a) Two UE and CHORDAL subgraphs. (b) The UE representation trees for the graphs in (b). (c) The tree T_1 with the path $\pi(1)$ rerouted.

$C = 1234$ as a leaf. We have $\mathbf{A} = \{1234,\ 123x,\ 12abd\}$. Although $G_1 > G_2$, their representation trees cannot be readily glued together. But by rerouting the path $\pi(1)$ as shown in Fig. 7c, we have $\mathbf{A}' = \{1234,\ 123x,\ 12abd,\ 12bc\}$. Then the trees can be glued together by coalescing the edge $12bc$ in T_1 and the edge C in T_2.

PROPOSITION 9″. *Let C separate G into $G_i = G(C \cap V_i)$, $1 \leqslant i \leqslant s$. If each G_i is UE and CHORDAL, and they can be 3-colored so that no antipodal pairs have the same color, and so that, for each $v \in C$, the neighboring subgraphs of v are 2-colored, then G is UE and CHORDAL.*

Proof. G is chordal by the Separator Theorem. By Proposition 8″, for each color, we can glue all of the subgraphs of that color to obtain a UE representation with C as a leaf edge. Now, make a claw C and merge each arm with leaf edge C of the same color. The result is a UE tree for G. For each $v \in C$, $\pi(v)$ includes the two arms which have the colors of the neighboring subgraphs of v. Note that if two colors suffice, then C is an edge clique. ∎

Proposition 8″ and 9″ outline an algorithm for constructing a UE representation tree for a graph G which is both UE and CHORDAL. As stated in the previous section, the problem of recognizing UE $\cap$ CHORDAL graphs is NP-complete. So the tree-construction problem in NP-hard (due to the 3-coloring condition).

6. Concluding Remarks

We have presented a unified approach for studying various classes of graphs arising from the intersection of paths in a tree. This framework leads to an efficient algorithm for recognizing DV, DE, UEH, and CHORDAL graphs. It also indicates why recognizing UE graphs is NP-complete. The efficiencies of recognizing RDV graphs and UV graphs under this framework and the extension of the Separator Theorem to general UE graphs are open problems.

We leave as an open problem improvement on the time bound for the recognition algorithm. It may be possible to reduce the time bound to be linear in $|V|$, $|E|$, and $|\mathbf{C}|$ rather than the product of these quantities. Dietz [30] has obtained a linear-time algorithm for recognizing RDV graphs.

Other open questions are the complexity of the minimum coloring problem for UEH graphs (which can be solved in polynomial time for DE graphs and is NP-hard for UE graphs) and the minimum clique cover problem for DE and UEH graphs (which is NP-hard for UE graphs).

References

1. C. Berge, Some classes of perfect graphs, *in* "Graph Theory and Theoretical Physics" (F. Harary, Ed.), Academic Press, New York, 1967.
2. C. Berge, "Graphs and Hypergraphs," North-Holland, New York, 1973.
3. P. Buneman, A characterization of rigid circuit graphs, *Discrete Math.* **9** (1974), 205–212.
4. S. Fujishige, An efficient *PQ*-graph algorithm for solving the graph-realization problem, *J. Comput. System Sci.* **21** (1980), 63–86.
5. D. R. Fulkerson and O. A. Gross, Incidence matrices with consecutive ones property, *Bull. Amer. Math. Soc.* **70** (1964), 681–684.
6. M. R. Garey and D. J. Johnson, "Computers and Intractability: A Guide to the Theory of NP-completeness," Freeman, San Francisco, 1978.
7. F. Gavril, Algorithms for minimum coloring, maximum clique, minimum covering by cliques, and maximum independent set of a chordal graph, *SIAM J. Comput.* **1** (1972), 180–187.

8. F. GAVRIL, The intersection graphs of subtrees in trees are exactly the chordal graphs, *J. Combin. Theory Ser. B* **16** (1974), 47–56.
9. F. GAVRIL, A recognition algorithm for the intersection graphs of directed paths in directed trees, *Discrete Math.* **13** (1975), 237–249.
10. F. GAVRIL, A recognition algorithm for the intersection graphs of paths in trees, *Discrete Math.* **23** (1978), 211–227.
11. F. GAVRIL AND R. TAMARI, An algorithm for constructing edge-trees from hypergraphs, *Networks* **3** (1983), 377–388.
12. F. GAVRIL AND R. TAMARI, An algorithm for constructing trees from hypergraphs, *in* "Proceedings, Fourth Symposium uber Operations Research, Saarbrucken, 1979," published in the series Operations Research Verfahren.
13. P. C. GILMORE AND A. J. HOFFMAN, A characterization of comparability graphs and of interval graphs, *Canad. J. Math.* **16** (1964), 539–548.
14. M. C. GOLUMBIC, "Algorithmic Graph Theory and Perfect Graphs," Academic Press, New York, 1980.
15. M. C. GOLUMBIC AND R. E. JAMISON, The edge intersection graphs of paths in a tree, *J. Combin. Theory* **55** (1985), 151–159.
16. M. C. GOLUMBIC AND R. E. JAMISON, Edge and vertex intersection of paths in a tree, *Discrete Math.*, in press.
17. V. KLEE, What are the intersection graphs of arcs in a circle? *Amer. Math. Monthly* **76** (1969), 810–813.
18. D. KÖNING, "Theorie des endlichen und unendlichen Graphen," Akademische, Leipzig, 1936.
19. W. A. LOBB, Perfect graphs from paths in trees, unpublished manuscript.
20. P. L. RENZ, Intersection representations of graphs by arcs, *Pacific J. Math.* **34** (1970), 501–510.
21. F. S. ROBERTS, "Discrete Mathematical Models with Applications to Social, Biological and Environmental Problems," Prentice–Hall, Englewood Cliffs, N.J.
22. M. M. SYSLO, On characterizations of cycle graphs, *in* "Coloq. CNRS, Problems Combinatories et Theorie des Graphs, Orsay 1976," pp. 395–398, Paris, 1978.
23. M. M. SYSLO, "On Characterizations of Cycle Graphs and on Other Families of Intersection Graphs," Report Nr. N-40, Institute of Computer Science, University of Wroclaw, Poland, June 1978.
24. M. M. SYSLO, Triangulated edge intersection graphs of paths in a tree, *Discrete Math.*, in press.
25. R. TARJAN, Decomposition by clique separators, *Discrete Math.* **55** (1985), 221–231.
26. L. E. TROTTER, JR., Line perfect graphs, *Math. Programming* **12** (1977), 255–259.
27. S. TSUKIYAMA, M. IDE, M. ARIYOSHI, AND I. SHIRAWAKA, A new algorithm for generating all the maximal independent sets, *SIAM J. Comput.* **6** (1977), 505–517.
28. A. C. TUCKER, Matrix characterizations of circular arc graphs, *Pacific J. Math.* **39** (1971), 535–545.
29. S. H. WHITESIDES, An algorithm for finding clique cut-sets, *Inform. Process. Lett.* **12** (1981), 31–32.
30. P. F. DIETZ, "Intersection Graph Algorithms," Ph.D. thesis, Cornell University, August 1984.
31. P. G. H. LEHOT, An optimal algorithm to detect a line graph and output its root graph, *J. Assoc. Comput. Mach.* **21** (1974), 569–575.
32. I. HOLYER, The NP-completeness of edge-coloring, *SIAM J. Comput.* **10** (1981), 718–720.