A Small World Threshold for Economic Network Formation
&
A Network Formation Game for Bipartite Exchange Economies
Even-Dar et al.

Social Networks Seminar Fall 2007

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Introduction

- Structural properties in social networks
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- Decentralized network formation
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• Decentralized network formation
  • stochastic models
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  - stochastic models
  - game-theoretic models
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- Properties of the resulting networks
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  • expected diameter
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  • expected diameter
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A Small World Threshold for Economic Network Formation

Model
Nash Equilibrium and Link Stable
Diameter at Equilibrium

A Network Formation Game for Bipartite Exchange Economies

Bipartite Exchange Economies
Network Formation Game
Characterization of Nash Equilibrium Graphs

Discussion
Model

- $n$ players located on a grid, each is uniquely identified with a grid point $(x, y)$, where $1 \leq x, y \leq \sqrt{n}$. 
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- The joint action of all players is $a = a_1 \times \cdots \times a_n$.
- The joint action $a$ defines an undirected graph $G(a) = (V, E(a))$, where $V = \{v_1, \ldots, v_n\}$ and $E(a) = \bigcup_{i \in \{1, \ldots, n\}} E_i(a_i)$, $E_i(a_i) = \{(v_i, v_j) \mid a_i(j) = 1\}$.
Model

Grid distance \( \delta(v_i, v'_i) = |x - x'| + |y - y'| \), where \( v_i \) at \((x, y)\) and \( v'_i \) at \((x', y')\).
Model

Grid distance $\delta(v_i, v_i') = |x - x'| + |y - y'|$, where $v_i$ at $(x, y)$ and $v_i'$ at $(x', y')$.

Edge cost function

$$c(v_i, v_j) = \begin{cases} 0 & \delta(v_i, v_j) = 1 \\ C\delta(v_i, v_j)^\alpha & \text{otherwise} \end{cases}$$
Model

Grid distance $\delta(v_i, v_i') = |x - x'| + |y - y'|$, where $v_i$ at $(x, y)$ and $v_i'$ at $(x', y')$.

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\end{cases}$$

Overall cost function $c_i$ of player $v_i$ is defined as

$$c_i(a) = c_i(a_i, a_{-i}) = \sum_{e \in E_i(a_i)} c(e) + \sum_{j=1}^{n} \Delta_G(a)(v_i, v_j)$$

where $\Delta_G(a)(u, v)$ is the shortest distance between $u$ and $v$ in $G(a)$. 
Nash Equilibrium

Definition
A joint action $a = a_1 \times \cdots \times a_n$ is a Nash equilibrium if for every player $v_i$ and any alternative action $\hat{a}_i \in \{0, 1\}^n$

$$c_i(a_i, a_{-i}) \leq c_i(\hat{a}_i, a_{-i})$$
Nash Equilibrium

Definition
A joint action \( a = a_1 \times \cdots \times a_n \) is a \textit{Nash equilibrium} if for every player \( v_i \) and any alternative action \( \hat{a}_i \in \{0, 1\}^n \)

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c_i(a_i, a_{-i}) \leq c_i(\hat{a}_i, a_{-i})
\]

If \( a \) is a Nash equilibrium, we say that \( G(a) \) is an equilibrium graph.
Link Stable

Definition
A joint action $a = a_1 \times \cdots \times a_n$ is *link stable* if for every player $v_i$ and any alternative action $\hat{a}_i \in \{0, 1\}^n$ that differs from $a_i$ in exactly one coordinate

\[ c_i(a_i, a_{-i}) \leq c_i(\hat{a}_i, a_{-i}) \]
Link Stable

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A joint action \( a = a_1 \times \cdots \times a_n \) is *link stable* if for every player \( v_i \) and any alternative action \( \hat{a}_i \in \{0, 1\}^n \) that differs from \( a_i \) in exactly one coordinate

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c_i(a_i, a_{-i}) \leq c_i(\hat{a}_i, a_{-i})
\]

If \( a \) is link stable, we say that \( G(a) \) is a stable graph.

Note: an equilibrium graph is a link stable graph.
Comparison to Other Models

• Fabrikant et al.:
Comparison to Other Models

- **Fabrikant *et al.:***
  - constant edge cost $\alpha$
Comparison to Other Models

- Fabrikant *et al.*:
  - constant edge cost $\alpha$
  - no notion of locality
Comparison to Other Models

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  - longer-distance edges have lower probability vs. higher cost
Comparison to Other Models

• Fabrikant et al.:
  • constant edge cost $\alpha$
  • no notion of locality
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• Kleinberg:
  • longer-distance edges have lower probability vs. higher cost
  • probability & cost both in a power law form
Diameter of equilibrium or link stable graphs

Summary:
- $0 < \alpha < 2$: constant
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- $\alpha > 2$: $\Omega(\sqrt{n}^{-\frac{\alpha-2}{\alpha+1}})$
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- $\alpha = 2$: $O\left(\sqrt{n^2/\log n}\right)$
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• $0 < \alpha < 2$: constant
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Theorem
For any constant $\epsilon > 0$, if $\alpha = 2 - \epsilon$, then there exists a constant $c(\alpha)$ such that for any $n$, all Nash Equilibria or link stable graphs over $n$ players have diameter at most $c(\alpha)$. 
Constant Diameter Bound for $\alpha < 2$

- $c(\alpha) = 6c'(\alpha)$, $c'(\alpha) = 3^{\frac{1+2\epsilon_1}{\epsilon_1}}$, $\epsilon_1 = \frac{\epsilon}{2(2-\epsilon)}$
Constant Diameter Bound for $\alpha < 2$

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- as $\alpha \rightarrow 2$, the constant diameter bound $c(\alpha)$ blows up.

- better bound for $\alpha \leq 1$: 
  - if $\alpha < 1$, then $\Delta_G(a)(u,v) \leq 5$;
  - if $\alpha = 1$, then $\Delta_G(a)(u,v) \leq 2\lceil C_2 + 4 \rceil$. 

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Discussion
Bipartite Exchange Economies

- A bipartite exchange economy consists of a bipartite graph $G = (B, S, E)$, where nodes in $B$ represent buyers, nodes in $S$ represent sellers, and all edges in $E$ are between $B$ and $S$. Each buyer has an infinitely divisible endowment of 1 unit of cash; each seller has an infinitely divisible endowment of 1 unit of wheat. Buyers have utility $x$ for $x$ units of wheat and 0 utility of cash; sellers have utility $x$ for $x$ units of cash and 0 utility of wheat. Buyer $b_i$ can trade with seller $s_j$ if and only if there is an edge between $b_i$ and $s_j$. 
Bipartite Exchange Economies

- A bipartite exchange economy consists of a bipartite graph $G = (B, S, E)$, where nodes in $B$ represent buyers, nodes in $S$ represent sellers, and all edges in $E$ are between $B$ and $S$.
- Each buyer has an infinitely divisible endowment of 1 unit of cash.
Bipartite Exchange Economies

- A *bipartite exchange economy* consists of a bipartite graph $G = (B, S, E)$, where nodes in $B$ represent buyers, nodes in $S$ represent sellers, and all edges in $E$ are between $B$ and $S$.
- Each buyer has an infinitely divisible endowment of 1 unit of *cash*;
- each seller has an infinitely divisible endowment of 1 unit of *wheat*. 
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Buyer $b_i$ can trade with seller $s_j$ if and only if there is an edge between $b_i$ and $s_j$. 
Exchange Equilibrium

Let $\omega^s_j$ denote the exchange rate (or price), in terms of cash per unit wheat, that seller $s_j$ is offering; Let $\omega^b_i$ denote the exchange rate, in terms of wheat per unit cash, that buyer $b_i$ is offering. Let $x_{ij}$ denote the amount of seller $s_j$’s wheat that buyer $b_i$ consumes.
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**Definition**

A set of exchange rates, $\{\omega^b_i\}$ and $\{\omega^s_j\}$, and consumption plans, $\{x_{ij}\}$, constitutes an *exchange equilibrium* for $G$ if the following two conditions hold:

1. The market clears, i.e., supply equals demand. More formally, for each seller $s_j$, $\sum_{b_i \in N(s_j)} x_{ij} = 1$, where $N(s_j) = \{b_i | (i, j) \in E\}$.
2. For each buyer $b_i$, the consumption plan $\{x_{ij}\}$ is optimal. That is, $x_{ij} > 0$ if and only if $\omega^s_j = \min_{s_k \in N(b_i)} \omega^s_k$. 


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- The equilibrium exchange rates are unique, and at equilibrium, if $x_{ij} > 0$ then $\omega_j^s = 1/\omega_i^b$. 
Exchange Equilibrium

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- The equilibrium exchange rates are unique, and at equilibrium, if $x_{ij} > 0$ then $\omega_j^s = 1/\omega_i^b$.
- We call each sellers’ buyers) exchange rate $\omega_j^s(\omega_i^b)$ her wealth. There is no wealth variation at exchange equilibrium of a bipartite exchange economy when the wealth of all sellers are equal and the wealth of all buyers are equal.
Exchange Subgraph

Definition
Let $G = (B, S, E)$ be a bipartite exchange economy. Let $\{\omega^b_i\}$, $\{\omega^s_j\}$, and $\{x_{ij}\}$ be an exchange equilibrium. Then, the exchange subgraph of $G$ is $G' = (B, S, E')$, where $E' = \{(i, j) \mid x_{ij} > 0\}$.
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- Exchange subgraph $G'$ is minimal if the removal of any edge in $G'$ from $G$ changes the exchange equilibrium wealths.
Exchange Subgraph

Definition
Let \( G = (B, S, E) \) be a bipartite exchange economy. Let \( \{\omega_i^b\} \), \( \{\omega_j^s\} \), and \( \{x_{ij}\} \) be an exchange equilibrium. Then, the exchange subgraph of \( G \) is \( G' = (B, S, E') \), where \( E' = \{(i, j) \mid x_{ij} > 0\} \).

- The exchange subgraph may not be unique.
- Exchange subgraph \( G' \) is minimal if the removal of any edge in \( G' \) from \( G \) changes the exchange equilibrium wealths.
- Trading components: the connected components of the exchange subgraph. We say that a trading component is \((m, k)\) if there are \( m \) buyers and \( k \) sellers.
Fact

In a trading component \((m, k)\), the wealth of each buyer is \(k/m\), and the wealth of each seller is \(m/k\).
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- The graph \(G\) that defines the bipartite exchange economy is fixed \textit{a priori}
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- Other works have studied how the topology of \(G\) affects the variation in prices.
Fact

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- The graph \(G\) that defines the bipartite exchange economy is fixed \textit{a priori}.
- Other works have studied how the topology of \(G\) affects the variation in prices.
- The interest of this work is to allow players to construct the graph and to characterise the generated networks and the wealth variation.
Network Formation Game

- Two sets of players: buyer set $B$ and seller set $S$, where $|B| = |S| = n$.
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- An edge $(b_i, s_j)$ is bought by player $b_i$ only if $a^b_i(j) = 1$; and it is bought by $s_j$ only if $a^s_j(i) = 1$. 
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- Joint action of all players: $a = a^b_1 \times \cdots \times a^b_n \times a^s_1 \times \cdots \times a^s_n$, which defines a bipartite graph, $G(a) = (B, S, E)$, where $E$ is the set of edges that the players bought.
Network Formation Game

- Two sets of players: buyer set $B$ and seller set $S$, where $|B| = |S| = n$.
- The action of a buyer $b_i$ is denoted $a_i^b \in \{0, 1\}^n$; The action of a seller $s_j$ is denoted $a_j^s \in \{0, 1\}^n$.
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- Joint action of all players: $a = a_1^b \times \cdots \times a_n^b \times a_1^s \times \cdots \times a_n^s$, which defines a bipartite graph, $G(a) = (B, S, E)$, where $E$ is the set of edges that the players bought.
- $E_i^b(a) = \{(b_i, s_j) \mid a_i^b(j) = 1\}$ the set of edges $b_i$ buys; $E_j^s(a) = \{(b_i, s_j) \mid a_j^s(i) = 1\}$ the set of edges $s_j$ buys. Then, $E = \bigcup_{i \in [n], t \in \{b, s\}} E_i^t(a)$. 
Outline
A Small World Threshold for Economic Network Formation

A Network Formation Game for Bipartite Exchange Economies

Discussion

Nash Equilibrium

- Graph $G = G(a)$ defines a bipartite exchange economy, and an exchange equilibrium of the bipartite exchange economy determines the wealth each player earns, denoted by $\omega^t_i = \omega^t_i(G)$. The utility of each player is her wealth minus the cost for the edges she bought.
Nash Equilibrium

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- For a player of type $t \in \{b, s\}$, $u^t_i(a) = u^t_i(a^t_i, a^{t - i}) = \omega^t_i - \alpha|E^t_i|$. 
Nash Equilibrium

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  $$u^t_i(a) = u^t_i(a^t_i, a^t_{\_i}) = \omega^t_i - \alpha|E^t_i|.$$

- A joint action $a$ is a *Nash equilibrium* if for every player $t_i$ we have
  $$u^t_i(a^t_i, a^t_{\_i}) \geq u^t_i(\hat{a}^t_i, a^t_{\_i})$$
  for every action $\hat{a}^t_i$. 
Nash Equilibrium

- Graph $G = G(a)$ defines a bipartite exchange economy, and an exchange equilibrium of the bipartite exchange economy determines the wealth each player earns, denoted by $\omega_i^t = \omega_i^t(G)$. The utility of each player is her wealth minus the cost for the edges she bought.

- For a player of type $t \in \{b, s\}$, $u_i^t(a) = u_i^t(a_i^t, a_{-i}^t) = \omega_i^t - \alpha|E_i^t|$.

- A joint action $a$ is a Nash equilibrium if for every player $t_i$ we have $u_i^t(a_i^t, a_{-i}^t) \geq u_i^t(\hat{a}_i^t, a_{-i}^t)$ for every action $\hat{a}_i^t$.

- Equilibrium graph: the graph induced by a Nash equilibrium action $a$. 
Structure of Equilibrium Graphs

Theorem

Let $G$ be a Nash equilibrium graph of the network formation game. Then $G$ is equal to its minimal exchange subgraph.
Structure of Equilibrium Graphs

**Theorem**

Let $G$ be a Nash equilibrium graph of the network formation game. Then $G$ is equal to its minimal exchange subgraph.

The intuition: if redundant edges existed, the nodes that purchased them can remove them from the graph without affecting their wealth, and thus it is not a Nash equilibrium.
Structure of Equilibrium Graphs

Three types of graphs:

- **Perfect Matching** The class of all perfect matchings between the buyers and sellers. All exchange rates or wealths are equal to 1.
Structure of Equilibrium Graphs

- *Exploitation Graphs* Every trading component has a single player of one type "exploiting" a larger set of players of the other type. For any $k, l > 1$, $G$ consists of the union of $n_1 (1, k)$, $n_2 (1, k + 1)$, $n_3 (l, 1)$, and $n_4 (l + 1, 1)$ trading components. At most 4 different seller wealth values: $1/k$, $1/(k + 1)$, $l$, and $l + 1$. 
Structure of Equilibrium Graphs

- Balanced Graphs For any $k > 2$, $G$ consists of the union of $n_1$ trading components either $(k - 1, k)$ or $(k, k + 1)$ and $n_1$ trading components either $(k, k - 1)$ or $(k + 1, k)$. At most 4 different seller wealth values: $k/(k - 1)$, $(k + 1)/k$, $(k - 1)/k$, and $k/(k + 1)$. 

![Diagram of Balanced Network]
Main Theorem

Theorem

Let $\text{NE}(n, \alpha)$ be the set of all Nash equilibrium graphs of the network formation game for a fixed population size $n$ and edge cost $\alpha$, and let $\text{NE}$ be the union of $\text{NE}(n, \alpha)$ over all $n$ and $\alpha$. Then the set $\text{NE}$ equals the union of classes Perfect Matchings, Exploitation Graphs, and Balanced Graphs.
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Let $\text{NE}(n, \alpha)$ be the set of all Nash equilibrium graphs of the network formation game for a fixed population size $n$ and edge cost $\alpha$, and let $\text{NE}$ be the union of $\text{NE}(n, \alpha)$ over all $n$ and $\alpha$. Then the set $\text{NE}$ equals the union of classes Perfect Matchings, Exploitation Graphs, and Balanced Graphs.

Proof.

- relate the edge cost $\alpha$ to the minimum exchange equilibrium wealth in any equilibrium graph
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Proof.

- relate the edge cost $\alpha$ to the minimum exchange equilibrium wealth in any equilibrium graph ⇒ provide upper and lower bounds on the edge cost $\alpha$ in terms of the minimum exchange wealth
Main Theorem

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Let $NE(n, \alpha)$ be the set of all Nash equilibrium graphs of the network formation game for a fixed population size $n$ and edge cost $\alpha$, and let $NE$ be the union of $NE(n, \alpha)$ over all $n$ and $\alpha$. Then the set $NE$ equals the union of classes Perfect Matchings, Exploitation Graphs, and Balanced Graphs.

Proof.

• relate the edge cost $\alpha$ to the minimum exchange equilibrium wealth in any equilibrium graph $\Rightarrow$ provide upper and lower bounds on the edge cost $\alpha$ in terms of the minimum exchange wealth

• These bounds constrain the possible equilibrium graphs
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Proof.

• relate the edge cost $\alpha$ to the minimum exchange equilibrium wealth in any equilibrium graph $\Rightarrow$ provide upper and lower bounds on the edge cost $\alpha$ in terms of the minimum exchange wealth
• These bounds constrain the possible equilibrium graphs
• The remaining possibilities can all be realized
Discussion

Limitations in game-theoretical models

- one-shot, simultaneous move game
- how to reach a Nash equilibrium
- simulation results (Even-Dar and Kearns)
- convergence?
- approximation?
- bounded rationality: *satisfice* rather than *maximize* utility