On A Network Creation Game & On Nash Equilibria For a Network Creation Game

Fabrikant *et al.* & Albers *et al.*

Social Networks Seminar

October 2nd, 2007

Motivation

Fabrikant *et al.* [1]: The Internet

- Nodes select their connections.
- Nodes pay a price for each connection, and want to minimize their expenses.
- Each node wants to be "well connected":

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Social Network

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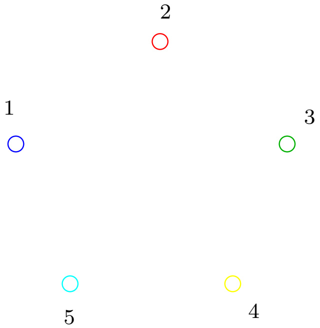
  Minimize *sum* of all costs
- How far from socially optimal?

  Price of Anarchy (Koutsoupias & Papadimitriou [3])

A Network Formation Game [1]

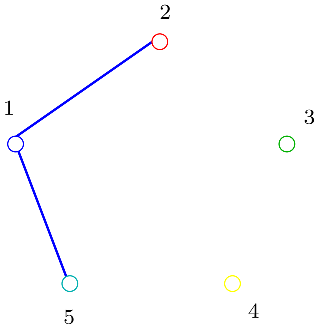
n nodes, in set $[n] = \{1, 2, \ldots, n\}$



$[5] = \{1, 2, 3, 4, 5\}$

A Network Formation Game [1]

Strategy S_i of node i: $S_i \subseteq [n] \setminus \{i\}$



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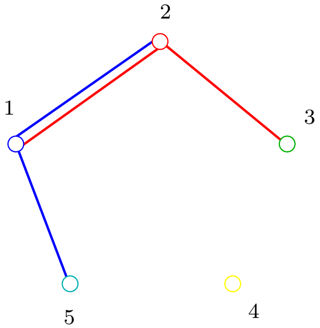

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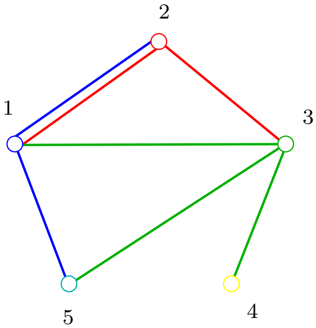

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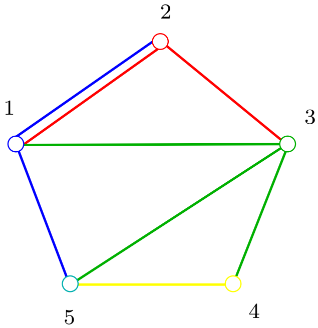

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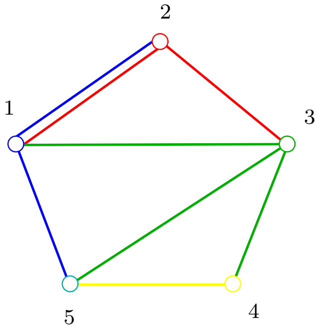
Joint Strategy: $\vec{S} = [S_1, S_2, \ldots, S_n]$



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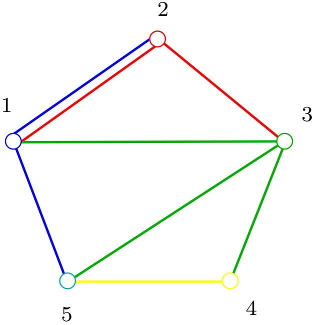
A Network Formation Game [1]

Joint Strategy: $\vec{S} = [S_1, S_2, \ldots, S_n]$
$\vec{S}$ induces an undirected multigraph $G(\vec{S})$ on $[n]$.



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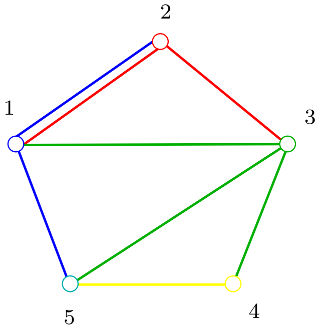
A Network Formation Game [1]

Let $d_G(i, j)$ be the distance of node i to j in $G(\vec{S})$

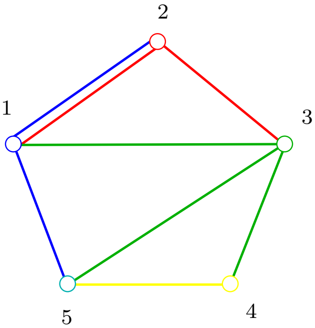


$$d_G(1, 2) = d_G(1, 3) = d_G(1, 5) = 1$$

$$d_G(1, 4) = 2$$

A Network Formation Game [1]

Cost for node i:

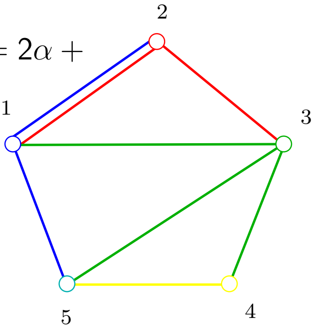


$$C_i(\vec{S}) = \alpha \cdot |S_i| + \sum_{j=1}^{n} d_G(i,j)$$

A Network Formation Game [1]

Cost for node i: $\alpha\$$ per edge

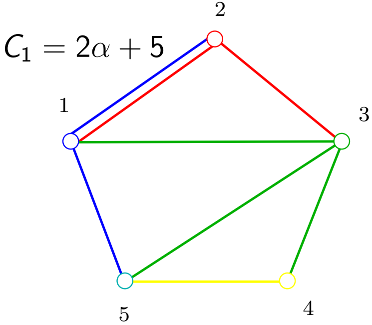
$C_1 = 2\alpha +$



$$C_i(\vec{S}) = \alpha \cdot |S_i| + \sum_{j=1}^{n} d_G(i,j)$$

A Network Formation Game [1]

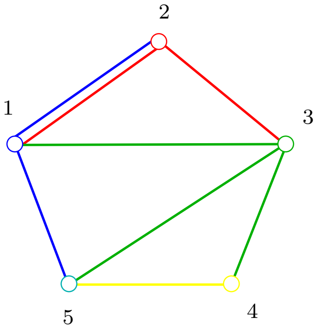
Cost for node i: $\alpha\$$ per edge + sum of all distances from i



$$C_i(\vec{S}) = \alpha \cdot |S_i| + \sum_{j=1}^{n} d_G(i,j)$$

A Network Formation Game [1]

Nodes are rational: Choose connections so that cost is minimized



$$C_i(\vec{S}) = \alpha \cdot |S_i| + \sum_{j=1}^{n} d_G(i,j)$$

Nash Equilibria

Definition

A joint strategy $\vec{S} = [S_1, \ldots, S_n]$ is a (pure) Nash equilibrium if for all $i \in [n]$

$$C_i(\vec{S}) \leq C_i([S_1, \ldots, S_{i-1}, S', S_{i+1}, \ldots, S_n]), \forall S' \subseteq [n] \setminus \{i\}$$

i.e. assuming that everyone except node i does not change their strategy, i has no incentive to deviate from S_i.

A Socially Optimal Strategy

Definition

The social cost $C(\vec{S})$ of a joint strategy is

$$C(\vec{S}) = \sum_{i=1}^{n} C_i(\vec{S}).$$

Definition

A joint strategy $\vec{S}_{opt}$ is socially optimal if its social cost is minimal, *i.e.*

$$C(\vec{S}_{opt}) \leq C(\vec{S})$$

for every joint strategy $\vec{S}$.

The Price of Anarchy

Definition (Koutsoupias and Papadimitriou [3])

The price of anarchy ρ is defined as

$$\rho = \max_{\vec{S} \in \mathcal{N}} \frac{C(\vec{S})}{C(\vec{S}_{opt})}$$

where $\mathcal{N}$ the set of Nash equilibria strategies and $\vec{S}_{opt}$ a socially optimal strategy.

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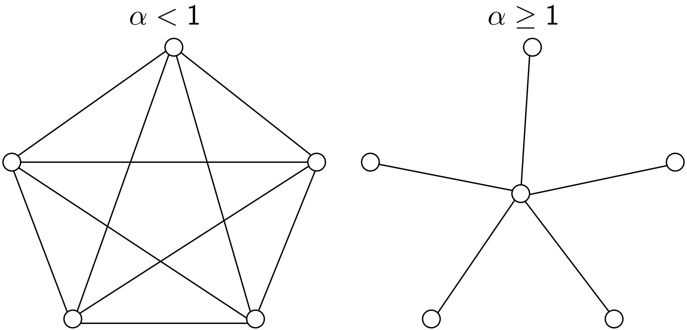
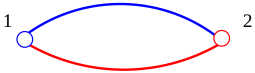
Fabrikant *et al.* [1], Albers *et al.* [2]: What is the price of anarchy in the network creation game?

Two Nash Equilibria

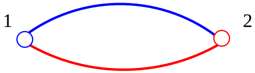


Not unique!



Lemma (Simple Edges Lemma)

If $\vec{S}$ is a Nash equilibrium, $G(\vec{S})$ is a simple graph, i.e. no connection is paid for twice.



Lemma (Simple Edges Lemma)

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Proof.

Each node has an incentive to remove its edge. □

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If $\vec{S}$ is a Nash equilibrium, the diameter of $G(\vec{S})$ is at most $\alpha + 1$

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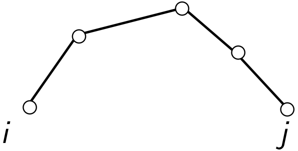
$$d(i,j) > \alpha + 1$$



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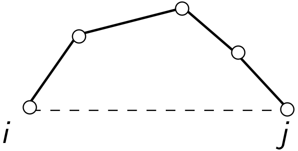

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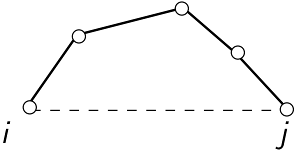
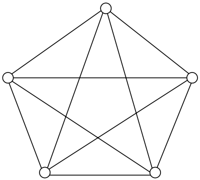


$$C_i' = C_i - d(i,j) + 1 + \alpha < C_i$$

$\square$

Corollary

For $\alpha < 1$, $\vec{S}$ is a Nash equilibrium if and only if $G(\vec{S})$ is the complete graph.



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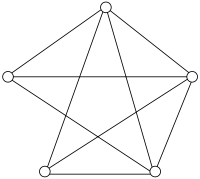

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$\Leftarrow$: Removing an edge increase the cost of the node that paid for it by $1 - \alpha > 0$.

□

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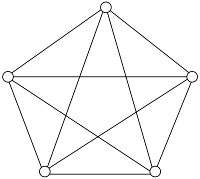
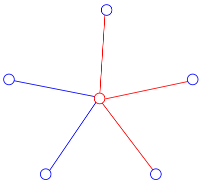


Proof.

$\Leftarrow$: Removing an edge increase the cost of the node that paid for it by $1 - \alpha > 0$.
$\Rightarrow$: The diameter of $G(\vec{S})$ is at most $\alpha + 1$, hence it is 1. □

Lemma

For $\alpha \geq 1$, if $G(\vec{S})$ is a star then $\vec{S}$ is a Nash equilibrium.



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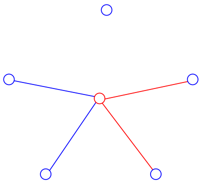

Proof.

Removing a red edge makes the cost of the red (center) node ∞.

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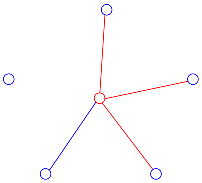

Proof.

Removing a red edge makes the cost of the red (center) node ∞.
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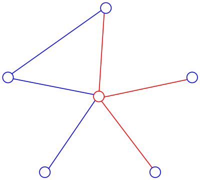

Proof.

Removing a red edge makes the cost of the red (center) node ∞.
Removing a blue edge makes the cost of a blue (leaf) node ∞.
There is no incentive to add an edge between two blue nodes, as this changes the cost by $\alpha - 1 \geq 0$. $\square$

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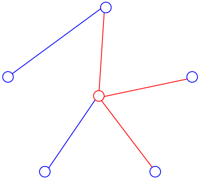
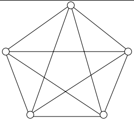
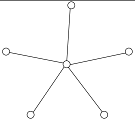
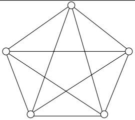
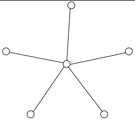
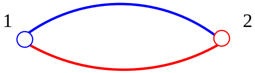


Proof.

Removing a red edge makes the cost of the red (center) node ∞. Removing a blue edge makes the cost of a blue (leaf) node ∞. There is no incentive to add an edge between two blue nodes, as this changes the cost by $\alpha - 1 \geq 0$. Finally, moving a blue edge to a leaf also increases the cost by $n - 3$. □

The Price of Anarchy

	$\alpha < 1$	$1 \leq \alpha < 2$	$2 \leq \alpha$
$\vec{S}_{opt}$			
Wost-Case Nash			?
Price of Anarchy	1	4/3	?



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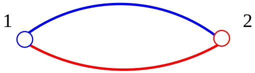

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Proof.

Dropping the extra edge can only reduce the social cost. □

A Lower Bound on the Social Cost

Let $\vec{S}$ be a strategy in which no edge is paid for twice. Let E be the edge set of $G(\vec{S})$. Then

$$C(\vec{S}) = \alpha|E| + \sum_{i,j} d_G(i,j)$$

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The bound becomes tight if the diameter of $G(\vec{S})$ is at most 2

Socially Optimal Strategies and the Price of Anarchy

$$C(\vec{S}) \geq 2n(n-1) + (\alpha - 2)|E|$$

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  by the Diameter Lemma.

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  Therefore, $\rho = C(\text{star})/C(\text{clique}) \leq 4/3$.

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  Therefore, $\rho = C(\text{star})/C(\text{clique}) \leq 4/3$.
- For $\alpha \geq 2$, the star is a socially optimal strategy.

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Theorem (Fabrikant *et al.* [1])

For $\alpha > n^2$, the price of anarchy $\rho = O(1)$. For $\alpha < n^2$, the price of anarchy is $\rho = O(\sqrt{\alpha})$.

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Theorem (Fabrikant *et al.* [1])

If, for $\vec{S}$ a Nash equilibrium, $G(\vec{S})$ is a tree, $C(\vec{S})/C(\vec{S}_{opt}) < 5$.

The Tree Conjecture

Conjecture (Fabrikant *et al.* [1])

There is a constant A such that, for all $\alpha > A$ every non-transient Nash equilibrium is a tree.

The Tree Conjecture

Conjecture (Fabrikant *et al.* [1])

There is a constant A such that, for all $\alpha > A$ every non-transient Nash equilibrium is a tree.

Albers *et al.[2]*: The conjecture is not true. For every α, n_0, there exists for some $n \geq n_0$ a non-transient equilibrium that contains a cycle.

Theorem (Albers *et al.* [2])

If $\alpha < \sqrt{n}$ *or* $\alpha > 12n\lceil \log n \rceil$, $\rho = O(1)$. *For values in between, it is* $O\left(1 + \min\left(\frac{\alpha^2}{n}, \frac{n^2}{\alpha}\right)^{1/3}\right)$.

In particular, for constant α, the price of anarchy is $O(1)$.

Conclusions

- The price of anarchy for the network formation game is bounded for non-trivial values of the price α.

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- Selfish nodes create a network not too far from the socially optimal.

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- Non-infinity distance for disconnected nodes.
- More general cost functions.
- Dynamic: Nodes arrive in stages.

References

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Transient Equilibria

Definition

A Nash equilibrium $\vec{S} = [S_1, \ldots, S_n]$ is called *weak* if for some $i \in [n]$ there exists an $S' \subseteq [n] \setminus i$ such that

$$C_i(\vec{S}) = C_i([S_1, \ldots, S_{i-1}, S', S_{i+1}, \ldots, S_n]).$$

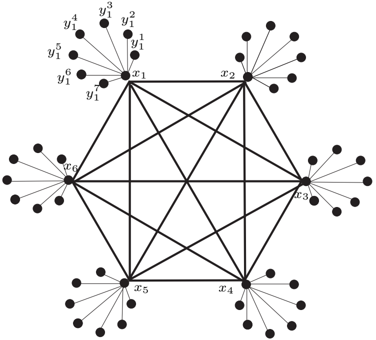
i.e. assuming that everyone except node i does not change their strategy, i has no incentive to deviate from S_i but there is a strategy towards which it is indifferent.

Definition

An equilibrium $\vec{S}$ is called *transient* if (a) it is weak and (b) there exists a sequence of single player strategy changes which do not alter the changer's payoff leading eventually to a non-equilibrium strategy.

A Transient Nash Equilibrium that is not a Tree

Albers *et al.* A (k, ℓ) clique of stars, with $\alpha = \ell$: All edges are bought by the clique nodes.



An NP-Hardness Result

Theorem (Fabrikant *et al.*)

It is NP-hard, given $\vec{S}$ a joint strategy in $[n-1]$, to compute the best response of an additional node n.

Proof.

Reduction from DOMINATING SET for $1 < \alpha < 2$. $\square$