Global Error Measures for Large N-body Simulations

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"Prediction can be difficult, particularly about the future."
— Mark Twain.

1. Introduction and Motivation

N-body systems are chaotic, which implies that small perturbations to a solution, such as numerical errors, are exponentially magnified with the passage of time. Although this is widely recognized, its impact on qualitative properties of numerical N-body simulations is not well understood. Animated movies of large N-body simulations, like spiral galaxies or cosmological systems, are very exciting to watch and often look quite reasonable, but it is little more than an "article of faith" that the results are qualitatively correct (Heggie, 1988).

Can these simulation results really be trusted? What conditions must a simulation meet for its accuracy to be assured? Is there a limit on the length of time a system can be followed accurately? What measures can be used to ascertain the accuracy of a simulation? More fundamentally, what do we *mean* by "accuracy" and "error" in these simulations, given that numerical errors are exponentially magnified with the passage of time? How do we know if a simulation is unreasonable, unless it is *grossly* so? In summary, *how badly are we allowed to integrate?*

There are several formal methods of measuring the accuracy of an ODE integration. *Forward* error measures the divergence of the numerical solution from the true solution with the same initial conditions; *backward* or *defect* error asks about the existence of a true solution to a slightly different problem, *i.e.*, the field is allowed small time-dependent perturbations. Finally, although we know that a numerical solution diverges exponentially away from the exact solution with the same initial conditions in the exact field, there remains the (unintuitive) possibility that there exists another exact solution, in the exact field, but with slightly different *initial conditions*, that remains close to the numerical solution. Such an exact solution, if it exists, is called a *shadow*, which we discuss below.

At the least stringent end of the global error spectrum, quantities like energy and momentum should be well conserved, but it is unclear if this is enough, because there are an infinite number of solutions that have exactly the same energy and momentum but vastly different phase space trajectories. Symplectic integrators (*e.g.*, Sanz-Serna, 1992) may provide better error control, but it is far from proven that this is enough. A less rigorous, but almost certainly satisfactory requirement, might simply be that a slightly "out-of-focus" animation of the N-body simulation be indistinguishable from an "out-of-focus" animation of the exact solution. Even less stringent, we might require that statistical "clumpiness" be reproduced correctly, *e.g.*, if we are simulating a spiral galaxy, we may require that the time evolution of the Fourier spectrum of the density distribution in cylindrical co-ordinates be reproduced reasonably accurately. This is one of the least stringent properties with which we can imagine one might be satisfied. If such a statistic is not reproduced by a simulation, then the numerical results are probably worthless.

2. Shadowing N-body Systems

A true trajectory of an ODE system lying near a numerical one is called a *shadow*. In this paper, we consider shadows of numerical solutions to the N-body problem. If you are interested only in general dynamics of solutions chosen at random, and not in the behaviour of a particular solution, then the existence of shadows implies that the numerical simulation may be dynamically valid in a very strong sense. (However, there still remains the possibility that shadows are not typical of true solutions chosen at random.)

A shadow found by our algorithm (and, most likely, shadows in general) typically stays quite close to the numerical solution for some period of time, and then it abruptly diverges at a point called a *glitch*. The relevant measures of error are the proximity of the shadow to the numerical solution, and the length of time before a glitch occurs. In gravitational N-body systems, glitches seem to occur most often at close encounters between particles (Quinlan & Tremaine, 1992). Glitches do not necessarily imply no shadow exists, only that our algorithm failed to find one.

Shadowing was initially studied in the context of gravitational N-body systems by Quinlan & Tremaine (1992), who considered a simplified system with one particle moving "pinball fashion" amongst 99 fixed ones. Not surprisingly, they found that shadows generally lasted longer, and usually remained closer to the numerical trajectory, as the local accuracy of the numerical solution increased. For a typical integration accuracy, they found that they could shadow the single particle for about 10 crossing times before a glitch occurred. They also found that the shadow length was moderately extended when the gravitational potential was "softened".

However, real N-body systems have more than one moving particle! Our work extends the results of Quinlan & Tremaine (1992) by studying shadow lengths for systems with more moving particles. To this end, we first spent several months speeding up Quinlan and Tremaine's shadowing algorithm by about a factor of 100 (Hayes, 1995). We then considered M particles moving amongst $N - M$ fixed ones.

Figure 1 plots various shadow lengths as a function of M, for systems with $N = 100$ particles. We use the standardized units of Heggie & Mathieu (1986). As can be seen in the first plot, the shadow length for an unsoftened potential seems to scale as $1/M^2$, which is not encouraging. However, perhaps we are not scaling the problem in a realistic way. The second plot shows that shadow length for a softened potential stays roughly constant, or very slowly decreases, with M. A still better way to scale the problem is to leave the potential unsoftened but to increase N, the total number of particles. This makes the problem extremely computationally intensive, and so we have only attempted this up to $M = 10$

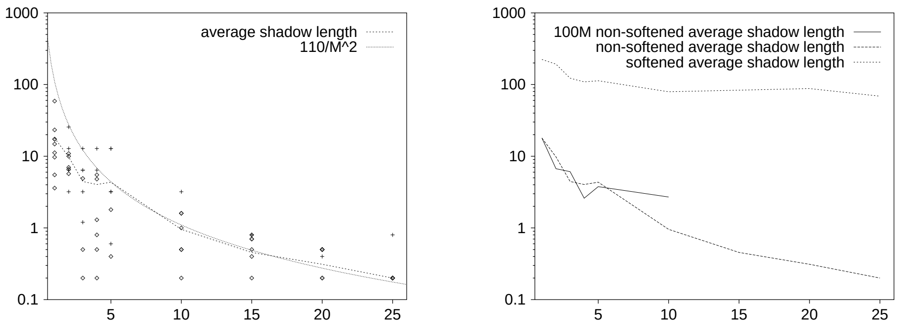

Figure 1. On the left, shadow lengths in crossing times as a function of M, the number of moving particles, while $100 - M$ particles remain fixed. The potential is unsoftened. Each point represents a shadow length for a particular orbit, while a line joins their averages. $110/M^2$ is plotted for comparison. On the right, we re-plot the average from the left, and add the average for a softened potential and the average for an unsoftened system that is scaled so that the total number of particles is $100M$ while the total mass of each system remains constant.

and $N = 1{,}000$, but it appears that the shadow length is longer in this case than for the smaller $N = 100$ particle system.

3. Further Work on Shadowing

There are many directions to pursue. First, the algorithm used here does not allow time step perturbations. During a close encounter between two stars, two solutions that differ in a direction along the orbit will temporarily separate greatly as the stars speed past each other. The difference leaves the "linear regime", and shadowing fails. Van Vleck (1995) and others have shown that, for various problems, allowing time step perturbations can increase shadow lengths *tremendously*. Other possibilities include extending our computations to larger N and M, shadowing of simpler equations that don't require ODE integrations, and testing the dependence of shadowing on the choice of integration method. Still more possibilities are listed in Hayes (1995).

References

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