Numerical Pricing of Path-Dependent Options

by

Yidong Liu

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Abstract

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Option contracts have become increasingly important in the field of finance since they possess characteristics that are attractive to both speculators and hedgers. One important problem is determining the "fair value" of an option efficiently and accurately.

In this thesis we first review basic option pricing theory and several popular numerical methods for option pricing. After discussing the analytic Black-Scholes formula for option prices, we focus on numerical methods for option pricing, including Monte Carlo methods, Tree methods and Finite Difference methods.

The main result of this thesis is an extension of the Hull-White tree method for pricing Asian Options. The original Hull-White method is applicable to options in which the averaging period is taken from the beginning to the end of the life of the option. We relax this restriction to any averaging period that ends with the life of the option. The Hull-White tree method is a special case of our method.

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Introduction

1.1 Derivative Securities

An option is a contract which confers the right, but not the obligation, to buy or sell an asset by a given date at a predetermined price.

A call option gives its holder the right to buy the underlying asset by a certain date for a certain price. A put option gives its holder the right to sell the underlying asset by a certain date for a certain price. The price in the contract is known as the *exercise price* or *strike price*; the date in the contract is known as the *expiration date, exercise date, or maturity.* While *American options* can be exercised at any time up to the expiration date, *European options* can only be exercised on the expiration date. An option gives the holder the right to do something, but the holder does not have to exercise this right. This fact distinguishes options from forward and futures contracts where the holder is obligated to buy or sell the underlying asset. An option may be abandoned and the buyer just loses what was paid for the right.

There are two sides to every option contract. On one side is the investor who has taken the *long position* (i.e., has bought the option). On the other side is the investor who has taken a *short position* (i.e., has sold or written the option).

If K is the strike price and S_T is the final price of the underlying asset at the moment of expiration, the payoff from a long position in a European call option is

$$C_l = \max\{S_T - K,\ 0\}. \tag{1.1}$$

Equation (1.1) reflects the fact that the option will be exercised if $S_T > K$ and will not be exercised if $S_T \leq K$. The payoff to the holder of a short position in the European call option is

$$C_s = -\max\{S_T - K,\ 0\} = \min\{K - S_T,\ 0\}. \tag{1.2}$$

Similarly, the payoff to the holder of a long position in a European put option is

$$P_l = \max\{K - S_T,\ 0\}, \tag{1.3}$$

and the payoff from a short position in a European put option is

$$P_s = -\max\{K - S_T,\ 0\} = \min\{S_T - K,\ 0\}. \tag{1.4}$$

From equations (1.1, 1.2, 1.3, 1.4), we know the loss of an option buyer is limited (the premium) while his profit may be unlimited. At the same time, the profit of an option writer is limited (the premium), but his loss may be unlimited.

Any securities whose value depends on the value of other more basic underlying assets are called *derivative securities.* Forward contracts, swaps, and many different types of options are all derivative securities. Derivative securities are also known as *contingent claims.*

Options have two primary uses: *speculation* and *hedging.* An investor who believes that a particular stock is going to rise can purchase some shares in that company. If he is correct, he makes money; if he is wrong, he loses money. This investor is speculating. Alternatively, if he thinks that the share price is going to rise within the next couple of months, he can speculate by buying a call with a specific exercise price and expiry date. On the other hand, if the investor thinks a particular stock is going to fall, he can speculate by selling shares or buying puts.

If an investor speculates in an option market, his profit or loss can be magnified to a very high percent of his investment. Options allow speculators to obtain very high returns, but also expose them to high risk.

Market makers usually can not afford to bear the risk of taking exposed positions. Their profit comes from selling at slightly above the 'true value' and buying at slightly below; the less risk associated with this policy, the better. Reducing risk is the subject of hedging.

A portfolio that only contains assets falls when the asset price falls, while one that is all put options rises. Somewhere in between these two extremes is a portfolio in which a small movement in the asset does not result in any movement in the value of the portfolio. This portfolio is instantaneously risk-free. The reduction of risk by taking advantage of such correlations between the asset and option price movements is called *hedging*. Options provide an effective way to hedge a portfolio.

By combining call options and put options with different exercise prices and expiry dates, one can implement various investment strategies.

1.2 An Outline of the Thesis

In this thesis we review basic option pricing theory and study numerical methods in option pricing.

In Chapter 2, we discuss some bounds on option prices, review theoretical models and deduce the Black-Scholes equation which is the cornerstone of option pricing theory. Analytical solution to the Black-Scholes equation are obtained for some final-boundary conditions.

In Chapter 3, we introduce various numerical methods for pricing options and illustrate them with practical examples.

In Chapter 4, a class of path-dependent options, including Asian options, is introduced. The application of Monte Carlo methods and the Hull-White tree method to these problems is reviewed. It is difficult to use Monte Carlo methods to price

American options. The Hull-White tree method is designed for Asian options whose averages are taken from the beginning of the option period only. We extend the Hull-White approach to regular Asian options whose averages can be taken over a period which is shorter than the life of the option. If we take our averaging period to be the life of the option, we get the same results as Hull and White.

In Chapter 5, a summary of this thesis is presented and future investigations are discussed.

Chapter 2

Pricing Derivative Securities

2.1 Basic Pricing Principles

2.1.1 Bounds on Option Prices

An option gives its holder a right with no obligation. At the same time the writer of the option has a potential obligation to comply with the contract. According to arbitrage theory the option holder has to pay for this right to the option writer: the amount he has to pay is the option price. Arbitrage theory assumes that there is no opportunity to make a sure profit without any risk— or the opportunity would disappear instantly if it did exist.

Arbitrage bounds are formalized common sense. Any reasonable person would know that if these bounds are exceeded then a risk-free profit can be made.

Assume c is the price of a European call option and C is the price of an American call option. The first bound is that a call option cannot be worth less than zero: having the right but not the obligation to do something is either worth zero or some positive amount. That is,

$$c \geq 0 \text{ and } C \geq 0. \tag{2.1}$$

The second bound is that the current call price cannot exceed the current asset price (S):

$$c \leq S \text{ and } C \leq S. \tag{2.2}$$

The reason is that the right to buy an asset cannot be worth more than that asset itself at any time. After all, if the right is exercised, it just gives the asset and no more.

The third bound applies only to American call options, i.e. those that can be exercised immediately. If the exercise price of the call options is K, then the option price, C, satisfies

$$C \geq S - K. \tag{2.3}$$

The fourth bound on call option prices is derived by considering two alternative investment strategies:

1. buy a share today at a price of S, or
2. buy a call option on the share at a price of c today and, at the same time, deposit enough money at the risk-free interest rate to yield the exercise cost, K, at time T. The sum to deposit now is the present value of K, or $PV(K)$.

At time T, the first portfolio will be worth S_T and the second will be worth

$$\max(S_T - K,\ 0) + K \geq S_T. \tag{2.4}$$

Therefore, the second portfolio must cost at least as much as the first. That is,

$$c + PV(K) \geq S, \tag{2.5}$$

or, rearranging,

$$c \geq S - PV(K). \tag{2.6}$$

Two more limits on a call price may be appreciated. The first is that, if the asset price, S, is zero, then the call price must be zero. An option on a worthless asset is itself worthless. The second is that, if the asset price, S, is very high relative to the exercise price, K, then the call price approaches its lower bound of $S - PV(K)$. The reason is that the probability of eventual exercise of the call approaches 100%, so it has no extra uncertainty value remaining.

Similarly, put options have the following price bounds.

$$p \geq 0 \text{ and } P \geq 0, \tag{2.7}$$

$$p \leq K \text{ and } P \leq K, \tag{2.8}$$

$$P \geq K - S, \tag{2.9}$$

where p is the value of a European put option and P is the value of an American put option, respectively. The bounds imply that: a put cannot have a negative value (2.7); the price of a put cannot exceed the exercise price, because that is what it would pay at maturity if S were zero (2.8); and an American put has an intrinsic value of $(K - S)$ (2.9).

Similar to (2.6) a put value satisfies

$$p \geq PV(K) - S. \tag{2.10}$$

The arbitrage argument to prove (2.10) uses the following two portfolios:

1. buy a share today and hold it; or

2. sell a put today and invest the present value of its exercise price.

The payoff for the first approach is at least as much as for the second one, so the cost of the first approach must be at least as much as the cost of the second one.

Finally we note

$$C \geq c \tag{2.11}$$

$$P \geq p \tag{2.12}$$

since American options provide more rights (early exercise) than the corresponding European options. However, it can be shown that an American call option on a non-dividend-paying stock is worth the same as the corresponding European option on the same stock [12]. That is,

$$C = c. \tag{2.13}$$

To see that, note that, if r is the risk-free interest rate, then

$$PV(K) = e^{-r(T-t)}K, \tag{2.14}$$

and the relation (2.6) becomes

$$c \geq S - Ke^{-r(T-t)} \tag{2.15}$$

and consequently the relation (2.11) becomes

$$C \geq S - Ke^{-r(T-t)}. \tag{2.16}$$

Since $r > 0$, it follows from (2.16) that $C > S - K$ when $T > t$. If the call option were exercised at time t, C would equal $(S - K)$ which is less than $(S - Ke^{-r(T-t)})$. Therefore, it is never optimal to exercise an American call option on a non-dividend-paying stock prior to expiration. Since an American call option on a non-dividend-paying stock should be exercised at its expiry date and its payoff is the same as its corresponding European call option, their prices should be the same.

On the other hand, it may be optimal under some circumstances to exercise an American put option on a non-dividend-paying stock early [12]. That is, it is possible that

$$P > p. \tag{2.17}$$

For example, if S becomes zero, it is optimal to exercise the put option immediately since the asset S would never decrease any more.

2.1.2 Put-Call Parity

Although call and put options are superficially different, in fact they can be combined in such a way that they are perfectly correlated. Here we demonstrate the put-call parity by following the argument in [23].

Suppose that at time $t = 0$ we buy one asset and one European put option on the asset, and at the same time sell one European call option on the asset with the same expiry date, T and exercise price, K. Denote the value of this portfolio at any time $t \in [0, T]$ by Π. Then

$$\Pi = S + p - c, \tag{2.18}$$

where S, p and c are the prices of the asset, the put option and the call option, respectively, at time t. It should be noted that S, p and c are random variables at time t.

The payoff of this portfolio at the expiry date T is

$$S + \max\{K - S,\ 0\} - \max\{S - K,\ 0\} = \begin{cases} K \text{ if } S \leq K \\ K \text{ if } S \geq K \end{cases}$$

that is, this portfolio gives a guaranteed value K at time $t = T$ no matter which values S, p and c are at time $t = T$.

According to arbitrage-based theory, the present value of the portfolio at time t is $Ke^{-r(T-r)}$ if r is the risk-free interest rate. Thus we have

$$S + p - c = Ke^{-r(T-t)}, \tag{2.19}$$

which is known as *put-call parity* and is true at any time $t \in [0,\ T]$.

From equation (2.19) we note that the values of European put and call options on the same asset can be determined from each other if they have the same exercise price and expiry date.

Finally we should note that the put-call parity usually is not true for American put and call options. Actually if the put-call parity were true for American put and call options as well, then from relation (2.13) the American put would have the same value as the corresponding European put. However, the American put usually has a higher price than the corresponding European put (2.17).

2.2 Fundamental Theory

2.2.1 Theoretical Assumptions

The bounds on option prices discussed above are helpful, but knowing the bounds alone is not enough. In real markets, it is of great importance to determine option prices more accurately, so we review basic option pricing theory first.

In a competitive market, a primary hypothesis is that stock prices accurately reflect all available information. This hypothesis is known as the *efficient market hypothesis*.

The efficient market hypothesis basically makes two assumptions.

1. The past price history is fully reflected in the present price, and so the past price history does not hold any further information.

2. Markets respond immediately to any new information about an asset.

Under these assumptions, change in the asset price is a *Markov Process*.

Since the absolute change in an asset price is not a good measure of change, a relative measure of the change, the *return*, is usually used. The return of an asset is defined to be the change in the price of the asset divided by the original price.

A model of stock price behavior is usually expressed as a *Wiener process*, which is a particular type of Markov process [12]. The behavior of a random variable, X, which follows a Wiener process, can be understood by considering the changes in its value in small intervals of time. Consider a small interval of time of length Δt and define ΔX as the change in X during Δt. For X to be a Wiener process ΔX must have the following properties:

1. ΔX is a random variable, drawn from a normal distribution;

2. the mean of ΔX is zero;

3. the variance of ΔX is Δt.

A generalized Wiener process for a random variable S can be defined in terms of dX as follows:

$$dS = a\,dt + b\,dX \tag{2.20}$$

where a and b are constants.

2.2.2 Theoretical Models

Suppose at time t an asset price is S. After a small subsequent time interval, dt, S changes to $S + dS$. To model the corresponding return on the asset, dS/S, the most common model decomposes this return into two parts. One is predictable, deterministic return akin to the return on money invested in a risk-free bank. It gives a contribution

$$\mu dt$$

to the return dS/S, where μ is a measure of the average rate of growth of the asset price, also known as the *drift*. In simple models, μ is taken to be a constant. In more complicated models, μ can be a function of S and t. The second contribution to dS/S models the random change in the asset price is response to external effects, such as unexpected news. It is represented by a random sample drawn from a normal distribution with mean zero and standard deviation σ. It takes the form

$$\sigma dX$$

where σ is called the *volatility*. In a simple model, σ is taken to be constant. In more complicated models, σ can be a function of time t and/or the underlying S. The quantity dX is sampled from a normal distribution.

Putting these contributions together, we obtain the stochastic differential equation

$$\frac{dS}{S} = \mu dt + \sigma dX, \tag{2.21}$$

which is the mathematical representation of simple recipe for generating asset prices and the term dX is a Wiener process.

If $\sigma = 0$, then the asset prices do not have randomness and the equation (2.21) becomes an ordinary differential equation

$$\frac{dS}{S} = \mu dt. \tag{2.22}$$

One way of writing dX is

$$dX = \phi\sqrt{dt}, \tag{2.23}$$

where ϕ is a random variable drawn from a standardized normal distribution.

2.3 Analytic Methods

2.3.1 Ito's Lemma

Ito's Lemma is one of the most important results about the manipulation of random variables in mathematical finance. It is to functions of random variables what Taylor's theorem is to functions of deterministic variables.

Suppose a random variable, G, is described by a stochastic differential equation

$$dG = A(G,t)dX + B(G,t)dt. \tag{2.24}$$

where dX is a Wiener process. At time t the asset price is G. After a small subsequent time interval, dt, G changes to $G + dG$. Let $f(t, G)$: $[0,\ T] \times R \to R$ have continuous partial derivatives, then from (2.24) we have

$$\begin{aligned}
& f(t+dt, G+dG) - f(t,G) \\
= & \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial G}dG + \frac{1}{2}\frac{\partial^2 f}{\partial G^2}(dG)^2 + o(dt) \\
= & \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial G}[A(G,t)dX + B(G,t)dt] + \frac{1}{2}\frac{\partial^2 f}{\partial G^2}[A(G,t)dX + B(G,t)dt]^2 + o(dt) \\
= & [\frac{\partial f}{\partial t} + \frac{\partial f}{\partial G}B(G,t)]dt + \frac{1}{2}\frac{\partial^2 f}{\partial G^2}A(G,t)^2(dX)^2 + \frac{\partial f}{\partial G}A(G,t)dX + o(dt). \qquad (2.25)
\end{aligned}$$

It can be shown that [23]

$$dX^2 \to dt \tag{2.26}$$

with probability one. Substituting this into (2.25) and retaining only those terms which are at least as large as $O(dt)$, we get

$$df = [\frac{\partial f}{\partial t} + \frac{\partial f}{\partial G}B(G,t) + \frac{1}{2}\frac{\partial f^2}{\partial G^2}A(G,t)^2]dt + \frac{\partial f}{\partial G}A(G,t)dX, \tag{2.27}$$

which is well known as *Ito's Lemma.*

As an example, we consider the following special case. Suppose an asset price, S, follows the stochastic differential equation

$$dS = \mu S dt + \sigma S dX, \tag{2.28}$$

where dX is a Wiener process. According to Ito's lemma, the function

$$f(S) = \ln S \tag{2.29}$$

satisfies

$$df = \sigma dX + (\mu - \frac{1}{2}\sigma^2)dt, \tag{2.30}$$

which is a constant coefficient stochastic differential equation.

A sequence of sums of independent identically distributed (i.i.d.) random variables is called a *random walk.* The Stochastic differential equation (2.28) is sometimes called a *log-normal random walk* since the function $f(S) = \ln S$ follows a random walk.

From equation (2.30) we have

$$\ln S_T - \ln S \sim \phi[(\mu - \frac{\sigma^2}{2})(T-t), \sigma\sqrt{T-t}], \tag{2.31}$$

where S_T is the stock price at a future time T; S is the stock price at the current time t; and $\phi(m,s)$ denotes a normal distribution with mean m and standard deviation s.

From the properties of the normal distribution, it follows from (2.31) that

$$\ln S_T \sim \phi[\ln S + (\mu - \frac{\sigma^2}{2})(T-t), \sigma\sqrt{T-t}]. \tag{2.32}$$

In a world where investors are risk-neutral, the expected return on all securities is the risk-free interest rate, r, and equation (2.32) becomes

$$\ln S_T \sim \phi[\ln S + (r - \frac{\sigma^2}{2})(T-t), \sigma\sqrt{T-t}]. \tag{2.33}$$

2.3.2 Black-Scholes Equation

The Black-Scholes differential equation must be satisfied by the price, f, of any derivative security dependent on a non-dividend-paying stock. It is one of the most important equations in option pricing theory.

To derive the Black-Scholes differential equation, the following assumptions are made:

1. The stock price, S, follows the log-normal random walk (2.28).
2. The risk-free interest rate r and the asset volatility σ are known functions of time over the life of the option.
3. There are no transaction costs associated with hedging a portfolio.
4. There are no dividends during the life of the derivative security.
5. There are no risk-free arbitrage opportunities.
6. Trading of the underlying asset can take place continuously.
7. Short selling is permitted and the assets are divisible.

Here we follow the deviation in [12]. Suppose that under the above assumptions we have a derivative security whose value $f(S,t)$ depends only on S and t. Using

Ito's lemma (2.27), we can write

$$df = \sigma S \frac{\partial f}{\partial S} dX + \left(\mu S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t} \right) dt. \tag{2.34}$$

This gives the random walk followed by $f(S,t)$.

Now construct a portfolio consisting of one option and $(-\Delta)$ shares of the underlying asset, where Δ is as yet unspecified. The value of this portfolio is

$$\Pi = f(S,t) - \Delta S. \tag{2.35}$$

The change in the value of this portfolio in one time-step is

$$d\Pi = df - \Delta dS. \tag{2.36}$$

Here Δ is held fixed during the time-step; if it were not then $d\Pi$ would contain terms in $d\Delta$. Putting (2.28), (2.34) and (2.36) together, we find that Π satisfies

$$d\Pi = \sigma S \left(\frac{\partial f}{\partial S} - \Delta \right) dX + \left(\mu S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t} - \mu S \Delta \right) dt. \tag{2.37}$$

Now we can eliminate the random component in (2.37) by choosing

$$\Delta = \frac{\partial f}{\partial S}. \tag{2.38}$$

Note that Δ is the value of $(\partial f / \partial S)$ at the start of the time-step dt. This choice yields a portfolio whose increment is wholly deterministic:

$$d\Pi = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt. \tag{2.39}$$

We now appeal to the concepts of arbitrage and supply and demand, with the assumption of no transaction costs. The return on an amount Π invested at the risk-free interest rate r is $r\Pi dt$ in a time dt. If the right side of (2.39) were greater than this amount, an arbitrager could make a guaranteed risk-less profit by borrowing an amount Π to invest in the portfolio. The return for this strategy would be greater than the cost of borrowing. Conversely, if the right side of (2.39) were less than $r\Pi dt$, then the arbitrager would short the portfolio and invest Π in the bank. Either way the arbitrager would make a risk-less profit. The existence of such arbitragers with the ability to trade at low cost ensures that the return on the portfolio and on the risk-less account are equal. Thus, we have

$$r\Pi dt = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right) dt. \tag{2.40}$$

Substituting (2.35) and (2.38) into (2.40) we arrive at

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} - rf = 0. \tag{2.41}$$

This is the *Black-Scholes partial differential equation.*

The Black-Scholes equation (2.41) is a backward parabolic equation. To make the equation have a unique solution, proper final and boundary conditions must be posed. One specific example of boundary and final conditions is

$$f(S,t) = f_a(t) \ \ on \ \ S = a, \tag{2.42}$$

$$f(S,t) = f_b(t) \ \ on \ \ S = b, \tag{2.43}$$

$$f(S,t) = f_T(S) \ \ on \ \ t = T, \tag{2.44}$$

where $f_a(t)$, $f_b(t)$ and $f_T(S)$ are known functions.

2.3.3 Explicit Solutions to the Black-Scholes Equation

Usually it is impossible to get the closed form solution to the Black-Scholes equation (2.41), but for European options, we can solve the equation analytically. Here we seek the closed form solution of a European call option [23].

The value $c(S,t)$ of a European call option satisfies the differential equation

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} + rS\frac{\partial c}{\partial S} - rc = 0. \tag{2.45}$$

with

$$c(0,t) = 0, \quad c(S,t) \approx S \text{ as } S \to \infty, \tag{2.46}$$

and

$$c(S,T) = \max(S - K,\ 0), \tag{2.47}$$

where K is the exercise price, T the expiry date, r the risk-free interest rate and σ the volatility.

To make the equation (2.45) dimensionless, we set

$$S = Ke^x,\ t = T - \tau/(\frac{1}{2}\sigma^2),\ c = Kv(x,\tau). \tag{2.48}$$

Under the transformation (2.48), we get

$$\frac{\partial c}{\partial t} = -\frac{1}{2}\sigma^2 K \frac{\partial v}{\partial \tau}, \tag{2.49}$$

$$\frac{\partial c}{\partial S} = \frac{K}{S}\frac{\partial v}{\partial x}, \tag{2.50}$$

$$\frac{\partial^2 c}{\partial S^2} = \frac{K}{S^2}(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}), \tag{2.51}$$

and the original problem becomes

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k_1 - 1)\frac{\partial v}{\partial x} - k_1 v, \tag{2.52}$$

with initial condition

$$v(x,0) = \max(e^x - 1,\ 0). \tag{2.53}$$

Here we note $k_1 = r/(\frac{1}{2}\sigma^2)$ is the only parameter in the new differential equation (2.52). We will show that the boundary conditions (2.46) are satisfied automatically later.

Under the further transformation

$$v = e^{-\frac{1}{2}(k_1-1)x-\frac{1}{4}(k_1+1)^2\tau}u(x,\tau), \tag{2.54}$$

equation (2.52) and its initial condition (2.53) become

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2},\ -\infty < x < \infty,\ \tau > 0, \tag{2.55}$$

and

$$u(x,0) = u_0(x) = \max(e^{\frac{1}{2}(k_1+1)x} - e^{\frac{1}{2}(k_1-1)x},\ 0) = \begin{cases} e^{\frac{1}{2}(k_1+1)x} - e^{\frac{1}{2}(k_1-1)x}, & x > 0, \\ 0, & x \le 0 \end{cases} \tag{2.56}$$

The solution to the diffusion equation (2.55) with initial condition (2.56) can be shown to be [23]

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} u_0(s)e^{-(x-s)^2/4\tau}ds. \tag{2.57}$$

Directly integrating the right side of (2.57) we get

$$\begin{aligned}
u(x,\tau) &= \frac{1}{2\sqrt{\pi\tau}}\int_{0}^{\infty} u_0(s)e^{-(x-s)^2/4\tau}ds \\
&= -\frac{1}{\sqrt{2\pi}}\int_{x/\sqrt{2\tau}}^{-\infty} u_0(x-y\sqrt{2\tau})e^{-\frac{1}{2}y^2}dy \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x/\sqrt{2\tau}} u_0(x-y\sqrt{2\tau})e^{-\frac{1}{2}y^2}dy \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x/\sqrt{2\tau}} e^{\frac{1}{2}(k_1+1)(x-y\sqrt{2\tau})}e^{-\frac{1}{2}y^2}dy \\
&\quad -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x/\sqrt{2\tau}} e^{\frac{1}{2}(k_1-1)(x-y\sqrt{2\tau})}e^{-\frac{1}{2}y^2}dy \\
&= e^{\frac{1}{2}(k_1+1)x+\frac{1}{4}(k_1+1)^2\tau}N(d_1) - e^{\frac{1}{2}(k_1-1)x+\frac{1}{4}(k_1-1)^2\tau}N(d_2).
\end{aligned} \tag{2.58}$$

where

$$d_1 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k_1+1)\sqrt{2\tau}, \tag{2.59}$$

$$d_2 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k_1 - 1)\sqrt{2\tau}, \tag{2.60}$$

and

$$N(d_i) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_i} e^{-\frac{1}{2}s^2} ds,\ i = 1, 2. \tag{2.61}$$

Note $k_1 = r/\frac{1}{2}\sigma^2$, $x = \ln(S/K)$, and $\tau = \frac{1}{2}\sigma^2(T - t)$. So from the transformations (2.48) and (2.54) we have

$$\begin{aligned} c &= Kv(x, \tau) \\ &= Ke^{-\frac{1}{2}(k_1-1)x-\frac{1}{4}(k_1+1)^2\tau}u(x, \tau) \\ &= Ke^x N(d_1) - Ke^{-k_1\tau}N(d_2) \\ &= SN(d_1) - Ke^{-r(T-t)}N(d_2) \end{aligned} \tag{2.62}$$

where

$$d_1 = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)(T - 1)}{\sigma\sqrt{(T - t)}}, \tag{2.63}$$

$$d_2 = \frac{\ln(S/K) + (r - \frac{1}{2}\sigma^2)(T - 1)}{\sigma\sqrt{(T - t)}}. \tag{2.64}$$

From the formulas (2.62), (2.63) and (2.64), we see that, as S goes to ∞, c goes to ∞, since both $N(d_1)$ and $N(d_2)$ go to one. Similarly, as S goes to zero, c goes to zero, since both $N(d_1)$ and $N(d_2)$ go to zero. Therefore, the boundary conditions (2.46) are satisfied by solution (2.62).

We can derive the price formula for a European put in a similar way, or use the put-call parity to determine the price formula.

2.3.4 Price Formulas from Integration

We can also obtain price formulas for European options by direct integration in a simple and straightforward way. For example, consider a European call option and assume the asset price S follows the stochastic differential equation (2.28).

The value of a European call option with maturity T in a risk-neutral world should be

$$c = e^{-r(T-t)} E[\max(S_T - K,\ 0)], \tag{2.65}$$

where r is the risk-free interest rate, t is the current time, K is the exercise price and $E[\cdot]$ denotes the expectation over the probability space at time T. From equation (2.65), we have

$$c = e^{-r(T-t)} \int_{-\infty}^{\infty} [\max(S_T - K,\ 0)] dP_{S_T} \tag{2.66}$$

Suppose the density function of S_T is $g(S_T)$, then equation (2.66) becomes

$$\begin{aligned} c &= e^{-r(T-t)} \int_{-\infty}^{\infty} [\max(S_T - K,\ 0)] g(S_T) dS_T \\ &= e^{-r(T-t)} \int_{K}^{\infty} [S_T - K] g(S_T) dS_T. \end{aligned} \tag{2.67}$$

Here we determine $g(S_T)$ first and then evaluate the integral in relation (2.67).

According to equation (2.33) we know S_T is log-normally distributed, so

$$S_T = e^y, \tag{2.68}$$

where

$$y \sim \phi[m_0, \sigma_0] = \phi\left[\ln S + (r - \frac{\sigma^2}{2})(T-t), \sigma\sqrt{T-t}\right]. \tag{2.69}$$

Here S is the underlying asset at time t.

From probability theory, if a random variable x has a density function $f(x)$, then the density function of the random variable $y = g(x)$ is

$$f(g^{-1}(y)) \frac{1}{|g'(g^{-1}(y))|}, \tag{2.70}$$

provided $g(x)$ is differentiable and monotonic.

Here the density function of y is

$$\frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{(y-m_0)^2}{2\sigma_0^2}}, \tag{2.71}$$

so the density function, $g(S_T)$ of S_T is

$$g(S_T) = \frac{1}{\sqrt{2\pi}\sigma_0 S_T} e^{-\frac{(\ln S_T - m_0)^2}{2\sigma_0^2}}. \tag{2.72}$$

From equations (2.67) and (2.72) we have

$$\begin{aligned}
c &= e^{-r(T-t)} \int_K^\infty [S_T - K] g(S_T) dS_T \\
&= e^{-r(T-t)} \left[\int_K^\infty S_T g(S_T) dS_T - \int_K^\infty K g(S_T) dS_T \right] \\
&= e^{-r(T-t)} \left[\int_K^\infty S_T g(S_T) dS_T - K \int_K^\infty g(S_T) dS_T \right]. \qquad (2.73)
\end{aligned}$$

From direct integration, we have

$$\begin{aligned}
\int_K^\infty g(S_T) dS_T &= \int_K^\infty \frac{1}{\sqrt{2\pi}\sigma_0 S_T} e^{-\frac{(\ln S_T - m_0)^2}{2\sigma_0^2}} dS_T \\
&= \frac{1}{\sqrt{2\pi}\sigma_0} \int_{\ln K}^\infty e^{-\frac{(x-m_0)^2}{2\sigma_0^2}} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma_0} \int_{\ln K - m_0}^\infty e^{-\frac{x^2}{2\sigma_0^2}} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{(\ln K - m_0)/\sigma_0}^\infty e^{-\frac{x^2}{2}} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-(\ln K - m_0)/\sigma_0} e^{-\frac{x^2}{2}} dx \\
&= N[-\frac{\ln K - m_0}{\sigma_0}], \qquad (2.74)
\end{aligned}$$

where $N[x]$ is the cumulative probability function for a standard normal distribution at x.

Similarly,

$$\begin{aligned}
\int_K^\infty S_T g(S_T) dS_T &= \int_K^\infty S_T \frac{1}{\sqrt{2\pi}\sigma_0 S_T} e^{-\frac{(\ln S_T - m_0)^2}{2\sigma_0^2}} dS_T \\
&= \frac{1}{\sqrt{2\pi}\sigma_0} \int_{\ln K}^\infty e^x e^{-\frac{(x-m_0)^2}{2\sigma_0^2}} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma_0} \int_{\ln K - m_0}^\infty e^{x+m_0} e^{-\frac{x^2}{2\sigma_0^2}} dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} e^{m_0} \int_{(\ln K - m_0)/\sigma_0}^{\infty} e^{\sigma_0 x} e^{-\frac{x^2}{2}} dx \\
&= \frac{1}{\sqrt{2\pi}} e^{m_0} e^{\sigma_0^2/2} \int_{(\ln K - m_0)/\sigma_0}^{\infty} e^{-\frac{(x-\sigma_0)^2}{2}} dx \\
&= \frac{1}{\sqrt{2\pi}} e^{m_0} e^{\frac{\sigma_0^2}{2}} \int_{\frac{\ln K - m_0}{\sigma_0} - \sigma_0}^{\infty} e^{-\frac{x^2}{2}} dx \\
&= \frac{1}{\sqrt{2\pi}} e^{\frac{\sigma_0^2}{2} + m_0} \int_{-\infty}^{-(\frac{\ln K - m_0}{\sigma_0} - \sigma_0)} e^{-\frac{x^2}{2}} dx \\
&= e^{\frac{\sigma_0^2}{2} + m_0} N[-(\frac{\ln K - m_0}{\sigma_0} - \sigma_0)] \\
&= e^{\frac{\sigma_0^2}{2} + m_0} N[\sigma_0 - (\frac{\ln K - m_0}{\sigma_0})]. \qquad (2.75)
\end{aligned}
$$

Combining equations (2.73), (2.74) and (2.75) we have

$$
c = e^{-r(T-t)} \left\{ e^{\frac{\sigma_0^2}{2} + m_0} N[\sigma_0 - (\frac{\ln K - m_0}{\sigma_0})] - K N[-\frac{\ln K - m_0}{\sigma_0}] \right\}. \qquad (2.76)
$$

Noting

$$
\begin{aligned}
m_0 &= \ln S + (r - \frac{\sigma^2}{2})(T-t), \qquad (2.77) \\
\sigma_0 &= \sigma\sqrt{T-t}, \qquad (2.78)
\end{aligned}
$$

in equation (2.69) and letting

$$
\begin{aligned}
d_1 &= \sigma_0 - (\frac{\ln K - m_0}{\sigma_0}), \qquad (2.79) \\
d_2 &= -\frac{\ln K - m_0}{\sigma_0}, \qquad (2.80)
\end{aligned}
$$

we obtain from equation (2.76) that

$$
\begin{aligned}
d_1 &= \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \qquad (2.81) \\
d_2 &= \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \qquad (2.82)
\end{aligned}
$$

and

$$
c = S N(d_1) - K e^{-r(T-t)} N(d_2). \qquad (2.83)
$$

Here S is the current underlying asset. These are exactly the same formulas as we obtained above, ((2.62), (2.63) and (2.64)).

Chapter 3

Numerical Methods

There is no analytic solution to many derivative pricing problems, so numerical methods are used frequently to approximate the solution. We review some of the basic numerical methods for pricing derivative securities in this chapter.

3.1 Finite Difference Methods

3.1.1 Basic Principles

In many cases, the Black-Scholes equation can not be solved analytically, so we must turn to numerical methods. Finite difference methods is one approach. To illustrate them, we consider two simple finite difference schemes that might be used to value a European put option.

Suppose the value, $p(S,t)$, of a European put option satisfies

$$\frac{\partial p}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 p}{\partial S^2} + rS\frac{\partial p}{\partial S} - rp = 0. \tag{3.1}$$

with boundary conditions

$$p(t,S) \approx K \text{ as } S \to 0, \quad p(t,S) \approx 0 \text{ as } S \to \infty, \tag{3.2}$$

and initial (final) condition

$$p(T,S) = \max(K - S,\ 0), \tag{3.3}$$

where K is the exercise price, T the expiry date, r the risk-free interest rate and σ the volatility.

We divide the time interval from current time to the maturity of the option, $[0,\ T]$, into N sub-intervals with

$$\Delta t = T/N. \tag{3.4}$$

Similarly, we divide the interval $[0,\ S_{max}]$ into M sub-intervals with

$$\Delta S = S_{max}/M \tag{3.5}$$

where S_{max} is a stock price which is sufficiently high that, when it is reached, the put has virtually no value.

This general grid is shown in Figure 3.1.

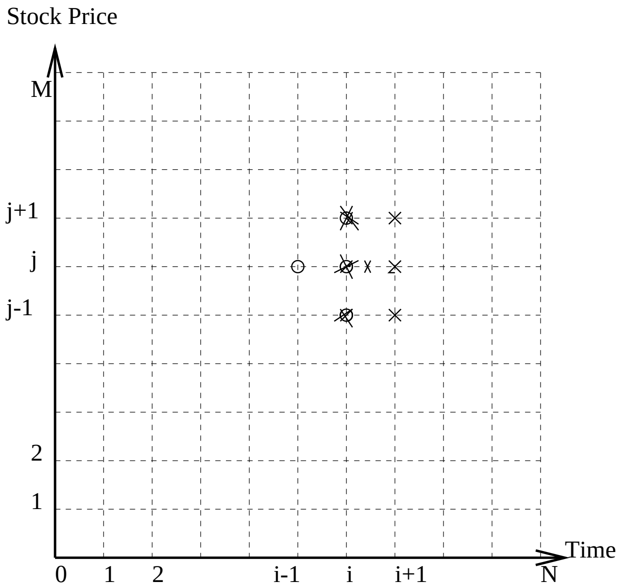

Figure 3.1: General Grid for Finite Difference Methods.

There are many ways to approximate each term of the equation (3.1) at interior grid points (i, j). One of them is to let

$$\frac{\partial p}{\partial S} \approx \frac{p(i\Delta t, (j+1)\Delta S) - p(i\Delta t, (j-1)\Delta S)}{2\Delta S}, \tag{3.6}$$

$$\frac{\partial^2 p}{\partial S^2} \approx \frac{p(i\Delta t, (j+1)\Delta S) + p(i\Delta t, (j-1)\Delta S) - 2p(i\Delta t, j\Delta S)}{\Delta S^2} \tag{3.7}$$

$$\frac{\partial p}{\partial t} \approx \frac{p(i\Delta t, j\Delta S) - p((i-1)\Delta t, j\Delta S)}{\Delta t} \tag{3.8}$$

Substituting equations (3.6), (3.7) and (3.8) into the differential equation (3.1) and noting that $S = j\Delta S$, we obtain

$$a_j p(i\Delta t, (j-1)\Delta S) + b_j p(i\Delta t, j\Delta S) + c_j p(i\Delta t, (j+1)\Delta S) \approx p((i-1)\Delta t, j\Delta S), \tag{3.9}$$

where

$$a_j = -\frac{1}{2} r j \Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t, \tag{3.10}$$

$$b_j = 1 - \sigma^2 j^2 \Delta t - r\Delta t, \tag{3.11}$$

$$c_j = \frac{1}{2} r j \Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t. \tag{3.12}$$

Now we replace (3.9) by

$$a_j p_{i,j-1} + b_j p_{ij} + c_j p_{i,j+1} = p_{i-1,j}, \tag{3.13}$$

where $p_{i,j} \approx p(i\Delta t, j\Delta S)$.

Similarly, the boundary conditions (3.2) become

$$p_{i,0} = K, \ \ p_{i,M} = 0, \ i = 0, 1, ..., N-1. \tag{3.14}$$

and the final condition (3.3) becomes

$$p_{N,j} = \max(K - j\Delta S, \ 0), \ j = 0, 1, ..., M. \tag{3.15}$$

The finite difference scheme (3.13) is *explicit* since all $p_{i,j}, \ j = 0, ..., M$, are available when $p_{i-1,j}$ is computed. Any finite difference scheme without this property is called *implicit*.

One such implicit finite difference scheme is the Crank-Nicolson method which approximates the partial derivatives at $((i+\frac{1}{2}), j)$ in (3.1) in the following way:

$$\begin{aligned}
\frac{\partial p}{\partial S} &\approx \frac{1}{2}\left[\frac{p(i\Delta t,(j+1)\Delta S)-p(i\Delta t,(j-1)\Delta S)}{2\Delta S}\right. \\
&+ \left.\frac{p((i+1)\Delta t,(j+1)\Delta S)-p((i+1)\Delta t,(j-1)\Delta S)}{2\Delta S}\right], \qquad (3.16)\\
\frac{\partial^2 p}{\partial S^2} &\approx \frac{1}{2}\left[\frac{p((i+1)\Delta t,(j+1)\Delta S)+p((i+1)\Delta t,(j-1)\Delta S)}{\Delta S^2}\right. \\
&- \frac{2p((i+1)\Delta t, j\Delta S)}{\Delta S^2} \\
&+ \left.\frac{p(i\Delta t,(j+1)\Delta S)+p(i\Delta t,(j-1)\Delta S)-2p(i\Delta t,j\Delta S)}{\Delta S^2}\right], \qquad (3.17)\\
\frac{\partial p}{\partial t} &\approx \frac{p((i+1)\Delta t, j\Delta S)-p(i\Delta t, j\Delta S)}{\Delta t}. \qquad (3.18)
\end{aligned}$$

Substituting (3.16), (3.17) and (3.18) into the differential equation (3.1) gives

$$\begin{aligned}
&a_j p(i\Delta t,(j-1)\Delta S)+(1+b_j)p(i\Delta t,j\Delta S)+c_j p(i\Delta t,(j+1)\Delta S)\\
&= -a_j p((i+1)\Delta t,(j-1)\Delta S)+(1-b_j)p((i+1)\Delta t,j\Delta S)\\
&\quad -c_j p(i+1)\Delta t,(j+1)\Delta S), \qquad (3.19)
\end{aligned}$$

where

$$\begin{aligned}
a_j &= -\frac{1}{4}\sigma^2 j^2\Delta t+\frac{1}{4}rj\Delta t, \qquad (3.20)\\
b_j &= \frac{1}{2}r\Delta t+\frac{1}{2}\sigma^2 j^2\Delta t, \qquad (3.21)\\
c_j &= -\frac{1}{4}\sigma^2 j^2\Delta t-\frac{1}{4}rj\Delta t. \qquad (3.22)
\end{aligned}$$

Now we replace (3.19) by

$$a_j p_{i,j-1}+(1+b_j)p_{i,j}+c_j p_{i,j+1} = -a_j p_{i+1,j-1}+(1-b_j)p_{i+1,j}-c_j p_{i+1,j+1}, \qquad (3.23)$$

where again $p_{i,j} \approx p(i\Delta t, j\Delta S)$.

3.1.2 Numerical Implementations

Here we implement both the explicit and the implicit finite difference schemes discussed in the previous subsection for simple examples. We solve the Black-Scholes equation with constant coefficients so that we can compare numerical results with analytic solutions.

Scheme 1: We solve the finite difference equation (3.13) with boundary conditions (3.14) and initial conditions (3.15). This is a simple explicit backward iteration and the values of $p_{0,j},\ j = 1, ..., M$, approximate the option prices $p(0, j\Delta S)$, $j = 1, ..., M$.

Scheme 2: We solve the finite difference equation (3.23) with the same initial condition and boundary conditions as in Scheme 1.

We summarize the approximate option prices for European puts and European calls for different current stock prices, $S_{current}$, in the Table 3.1 and Table 3.2, respectively. We used the following parameters in this example: $r = 10\%$, $\sigma = 30\%$, $T = 2$, $K = 50$, $S_{min} = 0$, $S_{max} = 200$, $N = 2000$ and $M = 100$. A large N is chosen since the explicit Scheme 1 does not converge for a small N.

$S_{current}$	20	30	40	50	60	70
Scheme 1	20.9767	12.7092	7.0985	3.7984	1.9980	1.0488
Scheme 2	21.1603	12.9144	7.2671	3.9155	2.0723	1.0939
Accurate	21.1611	12.9202	7.2767	3.9243	2.0785	1.0975

Table 3.1: Put Option Values Corresponding to Different Spot Prices.

$S_{current}$	20	30	40	50	60	70
Scheme 1	0.2426	2.0761	6.5660	13.3643	21.6542	30.7758
Scheme 2	0.2278	1.9820	6.3343	12.9802	21.1271	30.1203
Accurate	0.2246	1.9837	6.3402	12.9877	21.1419	30.1610

Table 3.2: Call Option Values Corresponding to Different Spot Prices.

It should be noted that the explicit finite difference method, Scheme 1, requires less computational work at each time-step than the implicit method, Scheme 2, but the implicit finite difference method is more stable. The coefficient matrix of the finite difference equation (3.23) is tridiagonal, so the Crank-Nicolson method does not result in much more computation. One can see from Tables 3.1 and 3.2 that Scheme 2 gives better approximate solutions than Scheme 1, so it is reasonable to prefer Scheme 2 even when stability is not an issue. Since Scheme 2 is also more stable than Scheme 1, it is preferred from this point-of-view too [1].

3.2 Tree Methods

3.2.1 Basic Ideas

The tree method is also called the *Lattice method*. It can be used to value quite general derivative securities, and to obtain an exact formula by taking the limit, in the case that the lattice solution converges to a continuous-time solution [8], [10].

The idea is that, if we know the price of an underlying asset now, we can build a tree or lattice of possible values of the asset up to the expiry time. This allows us to calculate the possible prices of the asset at the expiry time, and compute backward the corresponding values of the derivative security.

Tree methods arise from discrete random walk models of the underlying asset. It is supposed that the lifetime T of the derivative security can be divided into M discrete time-steps, $\Delta t = T/M$. It is assumed that at each time-step the price of the asset can jump from its old value at the start of the time-step to a finite number N of new values at the end of the time-step. When $N=2$, this process produces a *binomial tree* and, when $N = 3$, it produces a *trinomial tree*. These are the two most common cases.

All tree methods assume a risk-neutral world where an investor's risk preferences are irrelevant to derivative security valuation.

3.2.2 Building a Tree

To explain the basic ideas behind lattice methods, we build a binomial tree first.

Suppose the price of the underlying asset is S at time 0. Denote this by $S_{1,1}$.

After a time-step Δt, the price may jump to either Sd or Su $(u > d)$, which are called states $S_{2,1}$ and state $S_{2,2}$ respectively (see Figure 3.2). Here we assume the corresponding transition probabilities from state $S_{1,1}$ to state $S_{2,1}$ and state $S_{2,2}$ are $(1-p)$ and p, respectively.

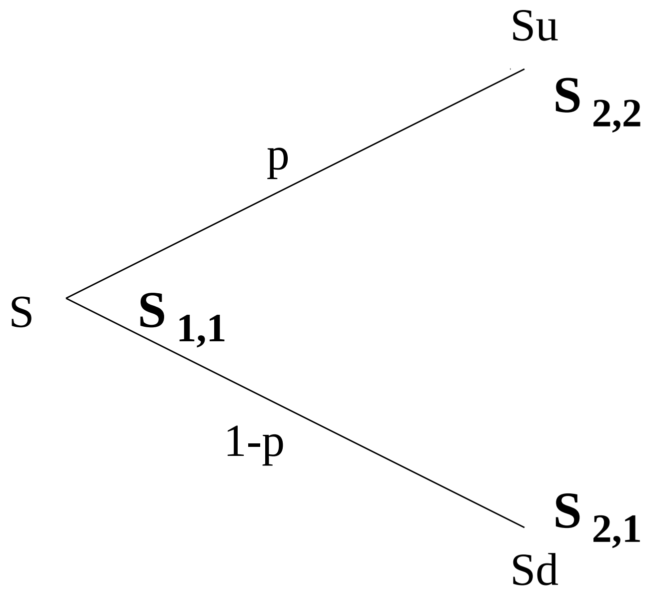

Figure 3.2: One Period of a Binomial Tree.

To avoid the number of states in a tree growing exponentially with the number of time-steps, we assume the movement factors, u and d, are constants. Under these assumptions, we can build a binomial tree (see Figure 3.3).

Now we review some basic relations for tree methods. First, since we are working in a risk-neutral world, the expected return from the underlying asset is the risk-free interest rate, r. Hence the expected value of the underlying asset at the end of a time-step Δt is $Se^{r\Delta t}$, where S is the price of the underlying asset at the beginning of the time interval Δt (see Figure 3.2). It follows that

$$Se^{r\Delta t} = pSu + (1-p)Sd \tag{3.24}$$

or

$$e^{r\Delta t} = pu + (1-p)d. \tag{3.25}$$

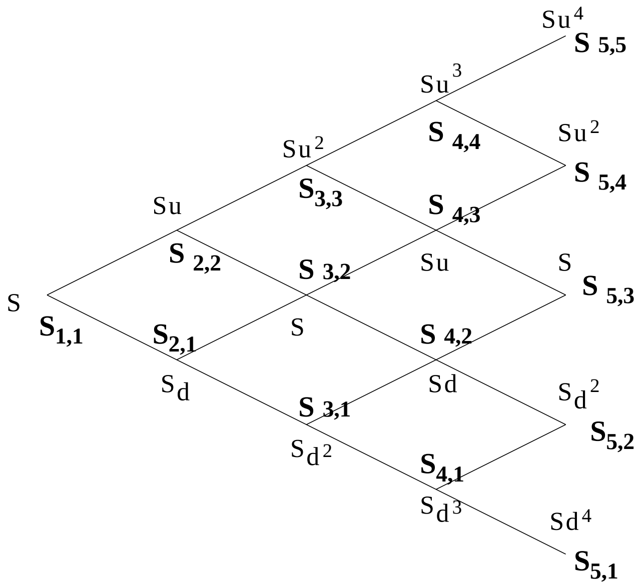

Figure 3.3: Multiple Periods of a Binomial Tree.

Second, if we assume the underlying random walk for S is log-normally distributed,

$$dS = Srdt + S\sigma dX \tag{3.26}$$

then the continuous random walk (3.26) and the discrete random walk (see Figure 3.2) should have the same variance. Thus,

$$S^2 e^{2r\Delta t}(e^{\sigma^2\Delta t} - 1) = pS^2u^2 + (1-p)S^2d^2 - S^2[pu + (1-p)d]^2 \tag{3.27}$$

or

$$e^{2r\Delta t}(e^{\sigma^2\Delta t} - 1) = pu^2 + (1-p)d^2 - [pu + (1-p)d]^2, \tag{3.28}$$

where the left side of (3.27) is the variance of the continuous random walk [12] and the right side of (3.27) is the variance of the discrete random walk.

It follows from (3.28) that

$$pu^2 + (1-p)d^2 - [pu + (1-p)d]^2 = \sigma^2\Delta t + O(\Delta t^2). \tag{3.29}$$

We have two equations ((3.25) and (3.29)) for three unknown variables, u, d and

p. To solve for u, d and p, another frequently used condition is

$$u = \frac{1}{d} \text{ or } ud = 1. \tag{3.30}$$

Condition (3.30) is frequently chosen since it leads to a balanced lattice, in which after two time-steps the asset price can return to its starting value. Other choices of u and d do not necessarily have this property.

Now let us solve the system of equations (3.25), (3.29) and (3.30) approximately (up to $O(\Delta t^{3/2})$).

From equations (3.25) and (3.29) we get

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{\sigma^2 \Delta t + e^{2r\Delta t} - d^2}{u^2 - d^2}. \tag{3.31}$$

From equation (3.31) we have

$$u + d = \frac{\sigma^2 \Delta t + e^{2r\Delta t} - d^2}{e^{r\Delta t} - d}. \tag{3.32}$$

From equation (3.30) and equation (3.32) we have

$$d^2 - [e^{r\Delta t} + (1 + \sigma^2 \Delta t)e^{-r\Delta t}]d + 1 = 0. \tag{3.33}$$

From equation (3.33) and condition (3.30) ($u > d$) we get

$$\begin{aligned} d &= \frac{1}{2}\left[e^{r\Delta t} + (1+\sigma^2\Delta t)e^{-r\Delta t} - \sqrt{[e^{r\Delta t} + (1+\sigma^2\Delta t)e^{-r\Delta t}]^2 - 4}\right] \\ &= \frac{1}{2}\left[e^{r\Delta t} + (1+\sigma^2\Delta t)e^{-r\Delta t} - \sigma\sqrt{\Delta t} + O(\Delta t^{3/2})\right] \\ &= 1 - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t + O(\Delta t^{3/2}) \\ &= e^{-\sigma\sqrt{\Delta t}} + O(\Delta t^{3/2}). \end{aligned} \tag{3.34}$$

Dropping the $O(\Delta t^{3/2})$ from (3.34) and combining it with (3.30) and (3.31) produces

$$u = e^{\sigma\sqrt{\Delta t}}, \tag{3.35}$$

$$d = e^{-\sigma\sqrt{\Delta t}}, \tag{3.36}$$

$$p = \frac{a-d}{u-d}, \tag{3.37}$$

where

$$a = e^{r\Delta t}. \tag{3.38}$$

Here we should note that $(1-p)$ may become negative under some conditions, for example if the step size is too big.

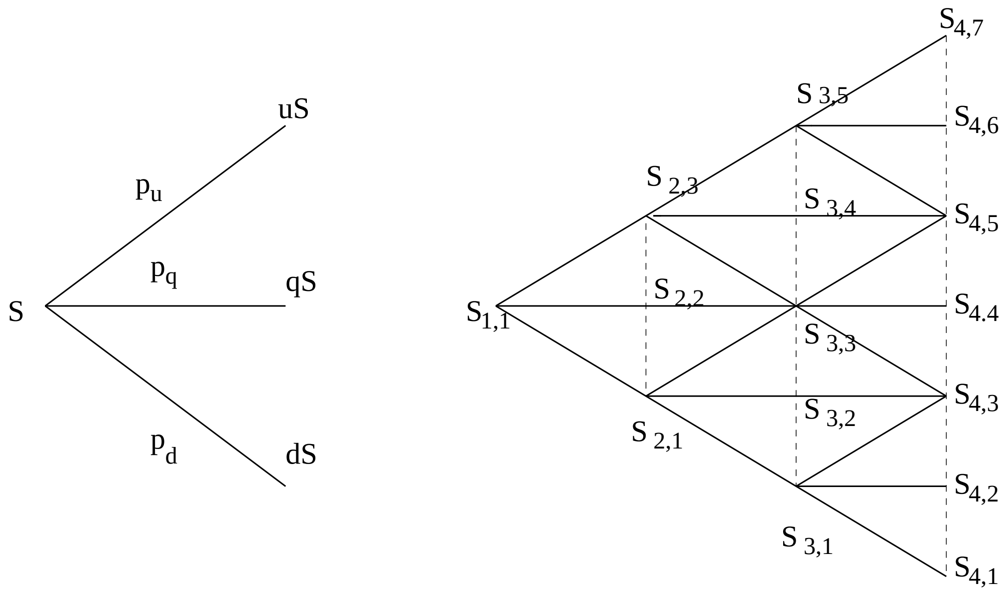

Figure 3.4: Trinomial Trees of One Period and Multiple Period.

Similarly, for a trinomial tree (see Figure 3.4) we should have

$$p_u + p_q + p_d = 1, \text{ (probability)} \tag{3.39}$$

$$p_u u + p_q q + p_d d = e^{r\Delta}, \text{ (expectation)} \tag{3.40}$$

$$p_u u^2 + p_q q^2 + p_d d^2 = \sigma^2\Delta t - e^{2r\Delta t}, \text{ (variance)}. \tag{3.41}$$

Thus we have three equations for six unknowns. Even if we make the assumption that $q = 1$ and $ud = 1$, there still remain more unknowns than equations. One possibility is to choose

$$p_u = p^2,\ p_q = 2p(1-p),\ q_d = (1-p)^2, \tag{3.42}$$

$$u = e^{2\sigma\sqrt{\Delta t/2}},\ q = 1,\ d = e^{-2\sigma\sqrt{\Delta t/2}}, \tag{3.43}$$

with

$$p = \frac{e^{r\Delta t/2} - e^{-r\sqrt{\Delta t/2}}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}}. \tag{3.44}$$

One advantage of choosing the parameters in this way is that each step of the trinomial tree is equivalent to two steps of the binomial tree with step size $\Delta t/2$.

3.2.3 Implementation of Tree Methods

After we build a tree, options can be evaluated by starting at the end of the tree (time T) and working backward.

Let $f_{m,n}$ denote the value of a European option (put or call) at state $S_{m,n}$.

Here $f_{M,n}$ $(1 \le n \le M+1)$ is determined from

$$f_{M+1,n} = \max\{S_{M+1,n} - K,\ 0\},\ \ 1 \le n \le M+1, \tag{3.45}$$

for a call option (American or European) and

$$f_{M+1,n} = \max\{K - S_{M+1,n},\ 0\},\ \ 1 \le n \le M+1, \tag{3.46}$$

for a put option (American or European). Here K is the exercise price of the option.

According to the assumption of a risk-neutral world, for a binomial tree we have

$$e^{r\Delta t} f_{m,n} = p f_{m+1,n+1} + (1-p) f_{m+1,n},\ 1 \le n \le m,\ 1 \le m \le M. \tag{3.47}$$

This gives

$$f_{m,n} = e^{-r\Delta t}(p f_{m+1,n+1} + (1-p) f_{m+1,n}),\ \ 1 \le n \le m,\ \ 1 \le m \le M. \tag{3.48}$$

To price an American option, equation (3.48) should be modified as follows:

$$C_{m,n} = \max[S_{m,n} - K,\ c_{m,n}],\ \ 1 \le n \le m,\ \ 1 \le m \le M, \tag{3.49}$$

$$P_{m,n} = \max[K - S_{m,n},\ p_{m,n}],\ \ 1 \le n \le m,\ \ 1 \le m \le M, \tag{3.50}$$

where $c_{m,n}$ or $p_{m,n}$ is obtained from equation (3.48).

For a zero-coupon European stock option (call or put), if its spot price S, its strike price K, the risk-free interest rate r and the stock price volatility σ are given, we can use tree methods to value the option. The basic steps for a binomial tree are as follows.

1. Divide the expiry time T into M equal intervals with time-step $\Delta t = T/M$.
2. Build a binomial tree as in Figure 3.3 with M periods.
3. Use equations (3.35), (3.36) and (3.37) to compute u, d and p respectively.
4. Compute the stock price $S_{M+1,n}$ $(1 \le n \le M+1)$ from $S_{M+1,n} = d^{M-n+1}u^{n-1}S = Su^{2n-M-2}$ (from the bottom to the top) since $ud = 1$.
5. Compute the option price $f_{M+1,n}$ at each state $S_{M+1,n}$ $(1 \le n \le M+1)$ from the strike price K and the stock price at the state from equation (3.45) or (3.46).
6. Compute the option prices backward from equation (3.48).

We can obtain similar formulas for trinomial trees. For a European option, we have

$$f_{m,n} = e^{-r\Delta t}(p_u f_{m+1,n+2} + p_q f_{m+1,n+1} + p_d f_{m+1,n}),\ \ 1 \le n \le 2m-1,\ \ 1 \le m \le M, \tag{3.51}$$

where $f_{m,n}$ denotes the prices of a European option (call or put) and p_u, p_q and p_d are defined in equation (3.42). For an American option, we have the same formulas as (3.49) and (3.50) but with $c_{m,n}$, $p_{m,n}$ determined by formula (3.51).

Following the above basic steps, we can implement trinomial trees in a similar way.

Next we show the numerical results for a practical example. Assume the interest rate $r = 12\%$, volatility $\sigma = 50\%$, maturity $T = 1.0$, current stock price $S = 50$ and exercise price $K = 55$, then, according to the analytic formula, we know the value of a European put option is 9.152223. The approximate option prices for different values of M are shown in Table 3.3.

M	Binomial Tree	Trinomial Tree
16	9.26786	9.21698
32	9.21698	9.16432
64	9.16432	9.14213
128	9.14213	9.16037
256	9.16036	9.15193
512	9.15193	9.15050

Table 3.3: European Put Prices Corresponding to Different Step Sizes T/M.

From Table 3.3 we note the option prices from the trinomial tree for each M equal the corresponding price from the binomial tree with the number of subintervals being $2M$. This is consistent with our specific choice of the parameters for the trinomial tree.

With the same parameters as in the above example, we can compute the approximate prices of the corresponding American put option. The results are shown in the Table 3.4.

M	Binomial Tree	Trinomial Tree
16	10.3568	10.2802
32	10.3026	10.2556
64	10.2684	10.2495
128	10.2609	10.2638
256	10.2682	10.2612
512	10.2635	10.2610
2000	10.2632	10.2622
4000	10.2625	10.2627

Table 3.4: American Put Prices Corresponding to Different Step Sizes T/M.

From Tables 3.3 and 3.4 we note the American put is more expensive than the corresponding European put. At the same time, our numerical results (not exhibited here) show that the American call and the corresponding European call have the same price. This is consistent with our analysis of the relation between the prices of American options and European options.

Unlike European options, for American options, we do not note an obvious relation between the numerical results for the binomial tree and the trinomial tree in Table 3.4. The reason is that, when one step of the trinomial tree is divided into two steps in the corresponding binomial tree, the American feature allows the option to be exercised within the two steps.

From the viewpoint of software development, it would be of interest to estimate option prices to within a user-specified accuracy. From our numerical examples presented here, it seems that the relation between the approximate errors and step sizes is not trivial.

3.3 Monte Carlo Methods

3.3.1 Monte Carlo Methods and Efficiency

The Monte Carlo approach has proved to be a valuable computational tool in numerical pricing of modern financial securities. Monte Carlo simulation is simple and flexible in the sense that it can easily be modified to accommodate different stochastic processes governing the underlying stock returns. It is especially useful for derivative securities where the payoff is dependent on the history of the underlying variable or where there are several underlying variables. Generally, the Monte Carlo approach consists of the following steps.

1. Simulate sample paths of the underlying state variable e.g., underlying asset prices (and/or interest rates) over the relevant time horizon.

2. Evaluate the discounted cash flows of a security on each sample path, as determined by the structure of the security in question.

3. Average the discounted cash flows over sample paths.

Before using Monte Carlo methods for a practical example, we analyze the efficiency of Monte Carlo methods. Suppose we want to compute a parameter θ — for example, the price of a derivative security. Assume we can generate by Monte Carlo methods an independent identically distributed (i.i.d.) sequence $\{\hat{\theta}_i,\ i = 1, 2, ...\}$, where each $\hat{\theta}_i$ has expectation θ and variance σ^2. A natural estimator of θ based on n replications is then the sample mean

$$\frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_i. \tag{3.52}$$

By the central limit theorem, for large n this sample mean is approximately normally distributed with mean θ and variance σ^2/n. Probabilistic error bounds in the form of confidence intervals follow readily from the normal approximation, and indicate that the error in the estimator is proportional to $\sigma/\sqrt{n}$.

Suppose, now, that we have a choice between two types of Monte Carlo estimates which we denote by $\{\hat{\theta}_i^{(1)}$, i=1, 2, ...$\}$ and $\{\hat{\theta}_i^{(2)}$, i=1, 2, ...$\}$. Suppose that both are unbiased, so that

$$E[\hat{\theta}^{(1)}] = E[\hat{\theta}^{(2)}] = \theta, \text{ but } \sigma_1 < \sigma_2, \tag{3.53}$$

where $\sigma_j^2 = Var[\hat{\theta}^{(j)}]$, j=1, 2. From previous observations it follows that a sample mean of n replications of $\hat{\theta}^{(1)}$ gives a more precise estimate of θ than does a sample mean of n replications of $\hat{\theta}^{(2)}$. But this analysis oversimplifies the comparison because it fails to capture possible differences in the computational effort required by the two estimators. Generating n replications of $\hat{\theta}^{(1)}$ may be more expensive than generating n replications of $\hat{\theta}^{(2)}$. So smaller variance is not sufficient ground for preferring one estimator over another [5].

Boyle et al. [5] give a good discussion of this issue. Suppose the work required to generate one replication of $\hat{\theta}^{(j)}$ is a constant b_j, j=1, 2. Within computing time t, the number of replications of $\hat{\theta}^{(j)}$ that can be generate is about t/b_j. The two estimators available within computing time t are therefore

$$\frac{b_1}{t}\sum_{i=1}^{t/b_1}\hat{\theta}^{(1)}, \text{ and } \frac{b_2}{t}\sum_{i=1}^{t/b_2}\hat{\theta}^{(2)}. \tag{3.54}$$

For large t, these are approximately normally distributed with mean θ and with standard deviations

$$\sigma_1\sqrt{\frac{b_1}{t}} \text{ and } \sigma_2\sqrt{\frac{b_2}{t}}. \tag{3.55}$$

Thus, for large t, the first estimator should be preferred over the second if

$$\sigma_1^2 b_1 < \sigma_2^2 b_2. \tag{3.56}$$

Equation (3.56) provides a sound basis for trading-off estimator variance and computational expense. Thus, it is reasonable to take the product of variance and work per run as a measure of *efficiency*.

3.3.2 Variance Reduction Techniques

Equipped with a basis for evaluating potential efficiency improvements, we can consider specific variance reduction techniques, including antithetic variates, control variates, moment matching and so on. Here we mainly study the techniques of antithetic variates and moment matching, since the method of control variates is problem-dependent and must estimate unknown parameters first.

The method of antithetic variates is a widely used techniques in financial pricing problems. Consider the problem of computing the risk-neutral price of a European call option on a no-dividend stock. Although we can compute the price exactly, we use this option as an example here. Independent replications of the terminal stock price under the risk-neutral measure can be generated from the formula

$$S_T^{(i)} = S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z_i},\ i = 1, ..., n, \tag{3.57}$$

where S_0 is the initial stock price, r is the risk-less interest rate, σ is the stock's volatility, T is the option's maturity, and the $\{Z_i\}$, $i = 1, .., n$ are independent samples from the standard normal distribution, $N(0, 1)$. Based on n replications, an unbiased estimator of the price of a call option with strike K is given by

$$c = \frac{1}{n}\sum_{i=1}^{n} c_i \equiv \frac{1}{n}\sum_{i=1}^{n} e^{-rT}\max\{0, S_T^{(i)} - K\}. \tag{3.58}$$

In this context, the method of antithetic variates [4] is based on the observation that if Z_i has a standardized normal distribution, then so does $(-Z_i)$. The price $\hat{S}_T^{(i)}$ obtained from (3.57) with Z_i replaced by $(-Z_i)$ is thus a valid sample from the terminal stock price distribution. Similarly, each

$$\hat{c}_i = e^{-rT}\max\{0, \hat{S_T}^{(i)} - K\} \tag{3.59}$$

is an unbiased estimator of the option price, as is therefore

$$c_{av} = \frac{1}{n}\sum_{i=1}^{n} \frac{c_i + \hat{c}_i}{2}. \tag{3.60}$$

A heuristic argument for preferring c_{av} notes that the random inputs obtained from the collection of antithetic pairs $\{(Z_i, -Z_i)\}$ are more regularly distributed than a collection of $2n$ independent samples. In particular, the sample mean over the antithetic pairs always equals the population mean, 0, whereas the mean over finitely many independent samples is almost surely different from 0. If the inputs are made more regular, it may be hoped that the outputs are more regular as well. Indeed, a large value of $S_T^{(i)}$ resulting from a large Z_i will be paired with a small value of $\hat{S}_T^{(i)}$ obtained from $(-Z_i)$. A more detailed argument can be found in [5].

The technique of moment matching was proposed by Barraquand [2]. We again use the pricing of a European call option introduced in (3.57) as an example.

Let $Z_i, i = 1, ..., n$, denote the independent standard normals used to drive a simulation. The sample moments of the n Z's will not usually exactly match those of the standard normal distribution. The idea of moment matching is to transform the Z's to match a finite number of the moments of the underlying population. For example, the first moment of the standard normal can be matched by defining

$$\tilde{Z}_i = Z_i - \bar{Z}, \ i = 1, ..., n, \tag{3.61}$$

where $\bar{Z}_i = \sum_{i=1}^{n} Z_i/n$ is the sample mean of the Z's. Note that the $\tilde{Z}_i$'s are normally distributed if the Z_i's are normal. However, the $\tilde{Z}_i$'s are not independent. As before, terminal stock prices are generated from the formula

$$\tilde{S_T}^{(i)} = S_0 e^{(r-\frac{1}{2}\sigma^2)+\sigma\sqrt{T}\tilde{Z}_i}, \ i = 1, ..., n. \tag{3.62}$$

An unbiased estimator of the call option price is the average of the n values

$$\tilde{c_i} = e^{-rT}\max\{\tilde{S_T}^{(i)} - K, 0\}. \tag{3.63}$$

In the standard Monte Carlo method, confidence intervals for the true value c can be estimated from the sample mean and variance of the estimator. This cannot be done here since the n values of $\tilde{Z}$ are not independent, and hence the values $\tilde{c_i}$ are not independent. This is one drawback of the moment matching method: confidence intervals are not as easy to obtain.

Equation (3.61) showed one way to match the first moment of a distribution with mean zero. If the underlying population does not have a zero mean, transformed Z's could be generated using $\tilde{Z}_i = Z_i - \bar{Z} + \mu_Z$, where μ_Z is the population mean. The idea can easily be extended to match two moments of a distribution. In this case, an approximate transformation is

$$\tilde{Z}_i = (Z_i - \bar{Z})\frac{\sigma_Z}{s_Z} + \mu_Z, \ i = 1, ..., n, \tag{3.64}$$

where s_Z is the sample standard deviation of the Z_i's and σ_Z is the population standard deviation. Of course, for a standard normal, $\mu_Z = 0$ and $\sigma_Z = 1$. An estimator of the call option price is the average of the n values $\tilde{c_i}$ which are produced from the Z_i's given by (3.64).

Using the transformation (3.64), the $\tilde{Z}_i$'s are not normally distributed even if Z_i's are normal. Hence, the corresponding $\tilde{c}_i$ are biased estimators for the true option value. For most financial problems of practical interest, this bias is likely to be small. However, the bias can be arbitrarily large in extreme circumstances [5].

The moment matching method is an example of the idea "use what you know". In this simple European option example, the mean and variance of the terminal stock price S_T are also known. So the moment matching idea could also be applied to the simulated terminal stock values $S_T^{(i)}$. In this case, to match the first moment, define

$$\tilde{S_T}^{(i)} = S_T^{(i)} - \bar{S_T} + \mu_{S_T}, \tag{3.65}$$

where $\mu_{S_T} = S_0 e^{rT}$ and $\bar{S_T}$ is the sample mean of the $S_T^{(i)}$'s. To match the first two moments, define

$$\tilde{S_T}^{(i)} = (S_T^{(i)} - \bar{S_T})\frac{\sigma_{S_T}}{S_{S_T}} + \mu_{S_T}, \tag{3.66}$$

where $\sigma_{S_T} = S_0\sqrt{e^{2rT}(e^{\sigma^2 T} - 1)}$ and S_{S_T} is the sample standard deviation of the $S_T^{(i)}$'s.

3.3.3 Numerical Examples of Monte Carlo Methods

Here we illustrate the techniques introduced above.

Suppose the stock price at the expiry date satisfies the equation (3.57), then the exact price of a European call option can be obtained from equation (2.83).

To test our crude Monte Carlo method, we use the following steps.

1. Produce a random series with standard normal distribution. The code used here comes from [20].

2. Compute the stock price at the expire date, T, using equation (3.57).

3. Compute the price of a European call option, $\max\{S_T - K, 0\}$.

4. Take the arithmetic average of the sample prices of the European call option. This is the approximate value of the price we need.

To use the technique of antithetic variates, we use both $\{Z_i\}$ and $\{-Z_i\}$, $i = 1, ..., n$, as random numbers of the random series in step 1 above.

To use the technique of moment matching about $\{Z_i\}$, we produce a series of random numbers, $\{Z_i\}$, $i = 1, ..., n$, first and compute its mean $\hat{Z}$ and/or standard deviation S_Z, then we can produce a new random series $\tilde{Z}_i$ using formula (3.61) (matching the first moment) or formula (3.64) (matching both the first moment and the second moment).

Similarly, to use the technique of moment matching about $\{S_T^{(i)}\}$, we compute its mean $\bar{S}_T$ and/or standard deviation S_{S_T} first and use transformations (3.65) or (3.66) to produce the series of different $\tilde{S}_T^{(i)}$.

For $T = 0.2$, $r = 0.10$, $K = 100$, $n = 100$, we implemented different techniques described above for different volatilities and we show their standard errors in Table 3.5. Standard errors are estimated based on $m = 1000$ simulations for each case. The results are consistent with the results in the Table 1 in [5].

σ	S_0/K	No Variance Reduction	Using $\{-Z_i\}$	MM1 eqn(3.61)	MM2 eqn (3.64)	MM1 eqn(3.65)	MM2 eqn(3.66)
0.2	1.0	0.62	0.29	0.30	0.09	0.27	0.10
0.4	1.0	1.23	0.65	0.67	0.19	0.57	0.23
0.6	0.9	1.42	0.91	0.95	0.37	0.86	0.28
0.6	1.0	1.87	1.07	1.08	0.32	0.89	0.39
0.6	1.1	2.38	1.13	1.11	0.26	0.82	0.49

Table 3.5: Standard Errors of Monte Carlo Methods for European Call Options.

Note that the standard errors depend on the choices of the seeds and are not exactly the same as the results reported in [5]. From Table 3.5 we can see that the variance reduction techniques work well. Of course, the techniques with smaller standard errors require more computation. This is a complex problem and we defer a deeper analysis of this topic to a later date.

Chapter 4

Pricing Path-Dependent Options

4.1 Path-Dependent Options

A *path-dependent option* is an option whose payoff at exercise or expire date depends, in some non-trivial way, on the past history of the underlying asset price as well as its price at exercise or expiry. American options, Asian options and lookback options are all path-dependent options.

An American option is a path-dependent option since there is a positive probability of the option being exercised before expiry and thus ceasing to exist. An American early exercise feature turns any option into a potentially path-dependent option [23].

A *lookback option* has a payoff which depends not only on the asset price at expiry but also on the maximum or the minimum of the underlying asset price over some time prior to expiry.

Asian options are path-dependent exotic options. They have payoffs which depend on the history of the random walk of the underlying asset price via some sort of average. One such option is the *average strike call*, where the payoff is the difference between the asset price at expiry and its average over some period prior to expiry if this difference is positive, and zero otherwise.

In the over-the-counter market, many options are written on average prices. For example, the payoff at maturity for a one-year average sterling call option is the maximum of zero and the difference between the arithmetic average of the daily exchange rate, averaged over some specified period, and the exercise price. Thus, the value of the option depends upon the average price rather than the spot price.

There are many possible uses for such options. For corporations that receive or pay foreign currency claims on a regular basis, a foreign currency option based on the average of the exchange rate represents one way to reduce its foreign currency exposure. Similar comments also apply to interest rate instruments.

Average options, by their design, reduce the significance of possible abnormal price movements at the maturity of the options. Thus, average options provide a way to meliorate any possible distortions that might arise because of a lack of depth in the market of the underlying asset.

The definition of the average is affected by the following factors:

1. The period of averaging. Over what time range before expiry is the average taken?

2. Arithmetic or geometric averaging. The average can be defined as the mean of the underlying asset (the arithmetic average, $\sum_{i=1}^{N} S_i/N$) or the exponent of the mean of the logarithm of the underlying asset price (the geometric average, $(\Pi_{i=1}^{N} S_i)^{1/N}$).

3. Weighted or unweighted averaging. Is the average simply the mean of asset prices over the averaging period or are some prices given a greater weighting in the average? We might for example choose to give a greater weighting to recent prices.

4. Discrete or continuous sampling of the underlying asset price. It is easier to take the mean of a small number of asset prices than to take the average over all the realized asset prices.

The pricing and hedging of average options raises some interesting issues. First, the value of the option depends upon the history of the underlying asset price movements over the averaging period. Thus, if the binomial tree approach is used to price the option, it is necessary to keep track of 2^n possible paths, where n is the number of the steps in the tree. This becomes infeasible for large values of n unless some approximation method is used. Second, even if the asset price follows a log-normal distribution, the arithmetic average will not be log-normally distributed. Standard option models, such as the Black-Scholes approach, rely upon the assumption of log-normality. Third, once the maturity of the option is less than the averaging period, the hedging properties of the option can change quite dramatically [22], [23].

In this Chapter we mainly consider Asian options with simple arithmetic averages over different averaging periods. For Asian options with geometric averages, a formula is given in [22].

4.2 Implementation of Monte Carlo Methods

Monte Carlo methods are widely used in pricing derivative securities. In Chapter 3 we reviewed the basic steps and variance reduction techniques for Monte Carlo methods for pricing European options. Here we use Monte Carlo methods to price European-style path-dependent options.

To price an option with Monte Carlo methods, one simulates the stochastic process of the corresponding underlying asset and prices the option for each realization. From the central limit theorem, the statistical estimate from sufficiently many simulations gives a good approximation of the option price. Usually each Monte Carlo simulation is a stochastic process over the whole life of the option, so it can be used to price path-dependent options. However, it is difficult to use Monte Carlo methods to price American-style options since it is hard to produce independent identically distributed (i.i.d.) simulations when the options can be exercised early.

To demonstrate the steps of pricing path-dependent options with Monte Carlo methods, we use a European-style Asian option as an example. Other European-style path-dependent options can be priced in a similar way.

The payoff of a European-style Asian call option with arithmetic average is

$$\max(\hat{S} - K, 0), \tag{4.1}$$

where

$$\hat{S} = \sum_{i=1}^{k} \frac{S_i}{k}, \tag{4.2}$$

K is the exercise price of the option, S_i is the asset price at time $t_i = iT/k$, and T is the option maturity.

The value of the option in a risk-neutral world is

$$E[e^{-rT} \max(\hat{S} - K, 0)], \tag{4.3}$$

where r is the risk-free interest rate in the risk-neutral world and $E[\cdot]$ represents expectation.

There is no easily evaluated closed-form expression for this option value since the distribution of the arithmetic average is usually very complicated. In fact, even though stock prices are log-normally distributed, their arithmetic average is not log-normally distributed. Furthermore, the arithmetic average of stock prices is not a Markov process since it depends on the history of the underlying stock prices. Various formulas to approximate an Asian option price have been developed, but simulation is usually used to test the accuracy of the approximation.

Before we implement Monte Carlo methods for practical examples, we first assume the underlying asset of our Asian options are log-normally distributed. That is,

$$dS = Srdt + S\sigma dX, \tag{4.4}$$

where r is the risk-free interest rate, σ is the underlying asset volatility, and dX is a Winener process.

According to Ito's lemma, the underlying asset price at t_i satisfies

$$S_i = S_{i-1} e^{(r - \frac{1}{2}\sigma^2)T_i + \sigma\sqrt{T_i}Z}, \ i = 1, ..., k, \tag{4.5}$$

where S_{i-1} is the known price of the underlying asset at t_{i-1}, Z is a random variable with the standard normal distribution, and T_i is the time interval between t_{i-1} and

t_i, which is constant (T/k) here.

Following the outline of Monte Carlo simulations given in Chapter 3, we implement a Monte Carlo method for pricing a European-style Asian call option as follows:

1. Produce a random series Z_i, $i = 1, ..., nk$, with standard normal distribution.

2. Compute S_i, $i = 1, ..., k$ according to (4.5). S_0 is the current price of the underlying asset.

3. Compute the arithmetic average, $\hat{S}$ of S_i, $i = 1, ..., k$ according to (4.2) for each realization.

4. Compute the option price according to (4.3).

5. Take the arithmetic average of the n sample option prices. This is the approximate value of the Asian call option.

The techniques of both antithetic variates and moment matching that we reviewed in Chapter 3 can be used here to reduce the variance of the simulation. To use the technique of antithetic variates, we use both Z_i and $(-Z_i)$, $i = 1, ..., nk$, as the random numbers of the random series in step 1 and replace n by $2n$ in step 5.

To evaluate this Asian option, k random numbers are needed to simulate one option payoff, so nk random numbers are needed in total for n simulations. The moment matching technique can be applied to all nk random numbers as a whole (the whole group approach), or it can be applied k times to the n numbers used to generate each S_i at time t_i (the subgroup approach). Here we implement the subgroup approach only since numerical results show the subgroup approach gives better results than the whole group approach [5].

The numerical results of our simulation are shown in Table 4.1. The parameters used here are: the number of simulations used to estimate the option price, $n = 100$; the exercise price, $K = 100$; the interest rate, $r = 10\%$; the time to maturity, $T = 0.2$ year; the number of steps over which the average is taken, $k = 5$; and the number of simulations used to estimate the standard error, $m = 200$. Our numerical results are consistent with those in [5].

σ	S_0/K	No Variance Reduction	Using $\{-Z_i\}$	Matching the First Moment	Matching the First Two Moments
0.4	1.0	0.76	0.41	0.44	0.37
0.6	1.0	1.15	0.69	0.63	0.53

Table 4.1: Standard Errors for the Monte Carlo method for an Asian Call Option.

We do not test more cases here since it is very expensive to compute accurate approximations of the true option prices which are used to estimate the standard errors. It should be noted that the techniques of variance reduction work well here. For example, when we used the technique of antithetic variate, the number of the random series is doubled. According to the central limit theorem, the corresponding standard error should decrease by a factor $\sqrt{2}$. From Table 4.1, we note the standard errors decrease by more than that factor ($0.76/0.41 \approx 1.85 > \sqrt{2} \approx 1.414$, $1.15/0.69 \approx 1.67 > 1.414$).

4.3 Implementation of a Tree Method

Monte Carlo methods provide a simple approach to price path-dependent options, but they have difficulty in pricing American-style options and they converge slowly. Hull and White [13] propose a tree method, which we describe below, for pricing path-dependent options.

Assume the underlying asset price for an Asian option follows a log-normal random walk. Then we can build a binomial tree, as shown in Figure 4.1, similar to the one exhibited in Chapter 3. Partition the life of the option $[0,\ T]$ into N parts with step size $\Delta t = T/N$ and denote the underlying asset at level i of the tree by $S_{i,j},\ j = 1, ..., i$ (note the nodes at each level are indexed from the bottom to the top of the level). At level 1, the underlying asset is the current price of the underlying asset. That is, $S_{1,1} = S$.

According to [13], the payoff of an Asian call option at maturity is

$$\max\left[\frac{\sum_{i=1}^{N+1} S_{i,k_i}}{N+1} - K,\ 0\right], \tag{4.6}$$

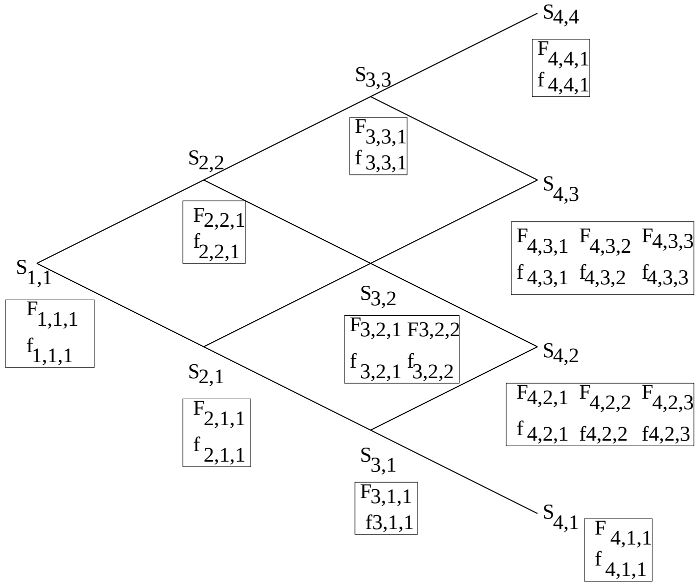

Figure 4.1: A Binomial Tree for a Path-Dependent Option.

where K is the exercise price and $(k_1, k_2, ..., k_{N+1})$ is the path of the underlying asset price during the option's life. Unlike (4.1), the payoff here involves price of the underlying asset from present to the end of the option. It should be noted that the number of possible paths doubles at each step. So at the expiry date there are 2^N paths.

It is impractical to keep all the paths and their corresponding averages of the underlying asset prices at each level of the tree as shown in Figure 4.1, so Hull and White propose an approximate method [13]. At each level i of the tree the maximum and minimum averages of the underlying asset price are computed first. That is,

$$A_{i,max} = \frac{\sum_{j=1}^{i} S_{j,j}}{i} = \frac{\sum_{j=1}^{i} Su^{j-1}}{i}, \tag{4.7}$$

$$A_{i,min} = \frac{\sum_{j=1}^{i} S_{j,1}}{i} = \frac{\sum_{j=1}^{i} Sd^{j-1}}{i}. \tag{4.8}$$

Instead of computing all the possible averages inside the interval $[A_{i,min},\ A_{i,max}]$, Hull and White approximate these averages by the values

$$Se^{kh},\ k = M_{min}, ..., M_{max}, \tag{4.9}$$

where S is the current price of the underlying asset, h is a partition parameter which determines the accuracy of the approximation, and the interval $[Se^{M_{min}h},\ Se^{M_{max}h}]$ contains the interval $[A_{i,min},\ A_{i,max}]$. In practice, the maximum and minimum averages of each level i ($i > 1$) are computed from the maximum and minimum averages of the previous level, so formulas (4.7) and (4.8) are replaced by

$$A_{i,max} = \frac{(i-1)A_{i-1,max} + S_{i,i}}{i} = \frac{(i-1)A_{i-1,max} + Su^{i-1}}{i}, \tag{4.10}$$

$$A_{i,min} = \frac{(i-1)A_{i-1,min} + S_{i,1}}{i} = \frac{(i-1)A_{i-1,min} + Sd^{i-1}}{i}, \tag{4.11}$$

After the possible averages at each level are approximated in this way, one can evaluate the option price starting from level $(N+1)$ and working backward through the tree to level 1. At level $(N+1)$, the call option price of each approximate average Se^{kh} is determined by (4.6), so an array of option prices are computed at each node of the last level. (Actually the arrays of option prices at the nodes of the last level are all the same.)

To illustrate the computation of the option prices, we consider one step from level i to level $(i-1)$ as shown in Figure 4.2. Suppose the option prices at each node of level i are known and we need to determine the option prices corresponding to different averages at each node of level $(i-1)$. In particular, consider node $S_{i-1,j}$ as shown in Figure 4.2.

The underlying asset price at node A of level $(i-1)$ may either move up to node B or down to node C at level i with probabilities p and $(1-p)$ respectively. If the underlying asset price moves up to B, then the averages $A_{i-1,k},\ k = 1, ..., N_1$ will change to new averages

$$A^*_{i,k} = \frac{(i-1)A_{i-1,k} + S_{i,j+1}}{i},\ k = 1, ..., N_1 \tag{4.12}$$

If $A^*_{i,k}$ equals to one of the averages at level i, $A_{i,l}$ say, then the payoff, $f^u_{i,k}$, is the corresponding option price, $U_{i,l}$, which is known. If $A^*_{i,k}$ is not equal to any of the averages at level i, then $A^*_{i,k}$ is between $A_{i,l}$ and $A_{i,l+1}$ for some l because of the way

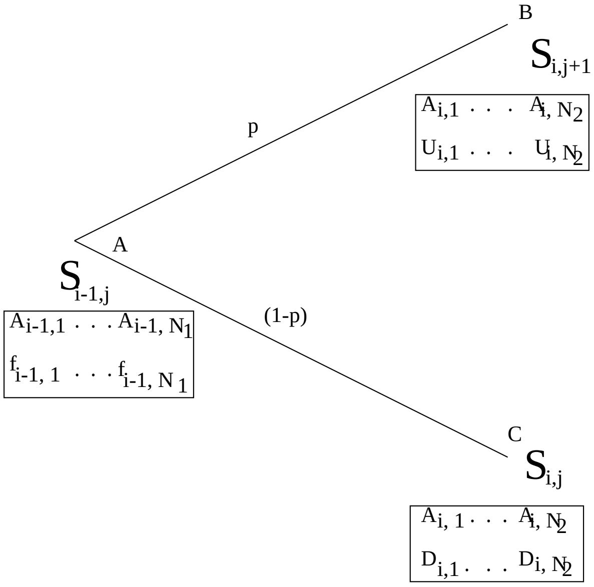

Figure 4.2: One Step of the Backward Computation in a Binomial Tree.

we built the tree. Using linear interpolation, we can approximate the corresponding payoff $f^u_{i,k}$ from $U_{i,l}$ and $U_{i,l+1}$.

Similarly, if the underlying asset price moves down to node C, for any average $A_{i,k}$ we can compute the corresponding payoff, $f^d_{i,k}$, in the same way.

Now, for a European-style Asian option we can approximate the option prices of the corresponding averages at node A by

$$f_{i-1,k} = e^{-r\Delta t}(pf^u_{i,k} + (1-p)f^d_{i,k}), \ k = 1, ..., N_1, \tag{4.13}$$

where r is the risk-free interest rate.

For an American-style Asian call option, formula (4.13) should be changed to

$$f_{i-1,k} = \max[A_{i-1,k} - K, \ e^{-r\Delta t}(pf^u_{i,k} + (1-p)f^d_{i,k})], \ k = 1, ..., N_1. \tag{4.14}$$

Similarly, for an American-style Asian put option, formula (4.13) should be changed to

$$f_{i-1,k} = \max[K - A_{i-1,k}, \ e^{-r\Delta t}(pf^u_{i,k} + (1-p)f^d_{i,k})], \ k = 1, ..., N_1. \tag{4.15}$$

As a simple example, let the initial stock price $S = 50$, risk-free interest rate

$r = 10\%$, the stock price volatility $\sigma = 30\%$, the number of the time steps for the tree $N = 40$ and the maturity $T = 1$. The call option prices for European-style and American-style Asian options, respectively, for different strike prices are shown in Table 4.2.

h	Style	40	45	50	55	60
0.1	European	11.6409	7.7572	4.6798	2.6225	1.4145
	American	13.6938	9.1320	5.3154	2.8509	1.4976
0.05	European	11.5915	7.6896	4.6059	2.5243	1.2896
	American	13.3868	8.8128	5.0830	2.6858	1.3408
0.01	European	11.5479	7.6226	4.5305	2.4295	1.1838
	American	13.1700	8.5693	4.9082	2.5491	1.2164
0.005	European	11.5449	7.6157	4.5223	2.4200	1.1767
	American	13.1593	8.5552	4.8946	2.5373	1.2084

Table 4.2: Call Option Values Corresponding to Different Exercise Prices.

From Table 4.2 we note that our numerical results are consistent with the results in [13]. It should be noted that the values of American-style options are always greater than the values of the corresponding European-style options which means that it may be optimal to exercise Asian options early. We note that the relative differences between the prices of American-style options and the prices of European-style options decrease as their option prices decrease. One possible reason for this is that option prices decrease as the advantages to exercise the options decrease. If an option would not be exercised, the right to exercise the option early should not be worth much. We defer an accuracy analysis and the development of heuristics for choosing the step size h for future work.

4.4 Implementation of an Extended Tree Method

The tree method described above is designed for Asian options whose averages are taken over the whole life of the options. Hull and White [13] note that an important open problem is how to price Asian options whose averaging period is less than the life of the option. We address this problem here.

4.4.1 European-Style Asian Options

First we extend the Hull and White tree method to European-style Asian options with an averaging period that is less than the life of the option. Suppose the underlying asset for an Asian option follows a log-normal random walk and can be simulated as shown in Figure 4.3. If the maturity date of the option is T and the averaging period of the option is $T_{avg} \leq T$, we can simulate the underlying asset in N time steps with

$$\Delta t = \frac{T}{N}. \tag{4.16}$$

Without lose of generality we can assume $T_{avg}/\Delta t$ is an integer and denote

$$N_{avg} = \frac{T_{avg}}{\Delta t} \text{ and } I = N - N_{avg} + 1, \tag{4.17}$$

where I is the level of the tree from which the averages of the underlying asset prices are taken.

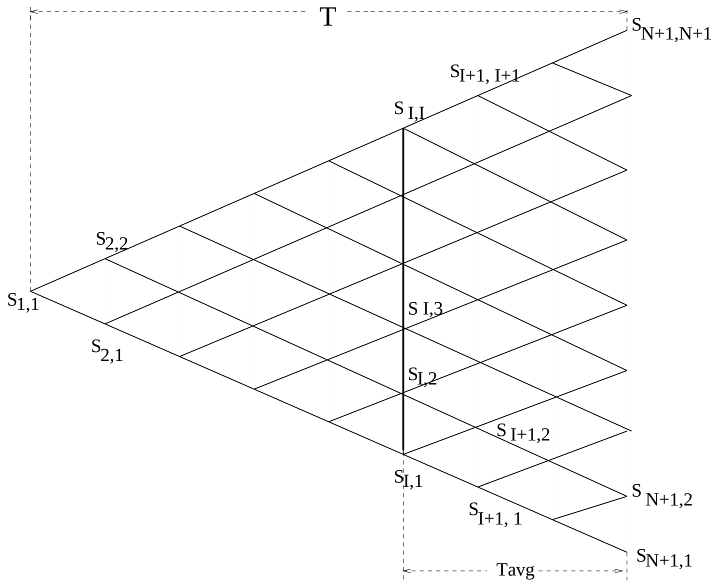

Figure 4.3: European-Style Asian Options Averaged Over Last Period.

Using the notation in Figure 4.3, the payoff of a European-style Asian call option is

$$\max\left[\frac{\sum_{l=I}^{N+1} S_{l,k_l}}{N_{avg}+1} - K,\ 0\right], \tag{4.18}$$

where K is the exercise price and $(k_I, k_{I+1}, ..., k_{N+1})$ is a path. If the risk-free interest rate is r, then the option price should be

$$e^{-rT} E\left[\max\left\{\frac{\sum_{l=I}^{N+1} S_{l,k_l}}{N_{avg}+1} - K, 0\right\}\right], \tag{4.19}$$

where $E[\cdot]$ denotes the expectation over the probability space of all possible paths.

One simple method to evaluate the option prices for this kind of Asian option is to use the Hull and White tree method directly. That is, at each node of level I in the binomial tree (see Figure 4.3), we evaluate a European-style Asian option with exercise price K, maturity T_{avg}, current underlying asset price $S_{I,j}$ and averaging period T_{avg}. After the option price at each node of level I is computed, we can evaluate the European-style options backward as we did in Chapter 3. This approach is a straightforward application of the Hull and White tree method, but it results in much duplication of computation since the common approximate averages of different nodes are not used.

Instead of using the Hull and White tree method at each node of level I in the binomial tree (Figure 4.3), we can extend their tree method. Our goal is still to find the option price at each node of level I first, but we compute it in a different way.

From Figure 4.3, we know the maximum average and minimum average at level $(I+1)$ are

$$\frac{S_{I,I} + S_{I+1,I+1}}{2} \text{ and } \frac{S_{I,1} + S_{I+1,1}}{2}, \tag{4.20}$$

respectively. Using formulas similar to (4.10) and (4.11), we can computed the maximum averages and minimum averages of level k for $I+1 < k \leq N+1$ and partition the interval $[A_{k,min},\ A_{k,max}]$ of level k, $I+1 \leq k \leq N+1$, in the same way. Following the computing process of Section 4.3, at each node of level $(N+1)$ we compute the option prices for all the approximate averages of level $(N+1)$ first. Then, at each node of level k, $(I+1) \leq k < (N+1)$, we compute the option prices for all the approximate averages at that level backward in the same way as described in Section 4.3. After the option prices at each node of level $(I+1)$ are computed, we determine a single option price at each node of level I.

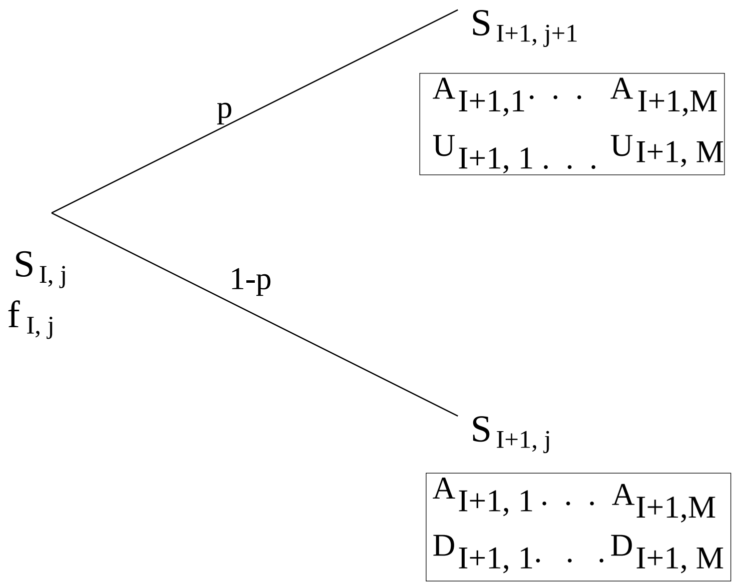

Figure 4.4: Computing Option Price at Each Node of Level I.

To compute the option price $f_{I,j}$ at level I, we know the underlying asset price $S_{I,j}$ may either move down to $S_{I+1,j}$ or up to $S_{I+1,j+1}$ after time Δt and the corresponding average will become either $(S_{I,j} + S_{I+1,j})/2$ or $(S_{I,j} + S_{I+1,j+1})/2$. Using the same method as before we can compute the option price $f_{I,j}$ at $S_{I,j}$, $j = 1, ..., I$. Note that it is inefficient to compute option prices at each node for all the averages of the level even though this approach is easily implemented and saves some computer memory. This approach is modified in our implementation of American-style Asian options.

As noted above, after we compute all the option prices at level I, we can use the procedure for evaluating European options described in Chapter 3 to compute the option prices $f_{i,j}$ backward from level $i = I - 1$ to level $i = 1$.

As a simple numerical example, we compute the prices of European-style Asian put and call options with the underlying initial asset price $S = 50$, the risk-free interest rate $r = 10\%$ and the stock volatility $\sigma = 30\%$. The number of the time steps used is $N = 40$ and the partition step size is $h = 0.005$. The option prices for different maturities and exercise prices are shown in Table 4.3.

From Table 4.3, we see that, if the averaging periods are chosen as long as the maturities of the options, we get the same results as Hull and White [13]. When the averaging periods are shorter than the corresponding maturities, our results are

Maturity	Average	Put Exercise Prices			Call Exercise Prices		
T	T_{avg}	45	50	55	45	50	55
1.0	0.1	1.880	3.503	5.732	10.913	8.012	5.717
	0.5	1.458	3.032	5.316	9.511	6.561	4.321
	1.0	0.751	2.182	4.603	7.615	4.522	2.420
2.0	0.2	2.381	3.811	5.631	15.042	12.378	10.104
	1.0	1.925	3.324	5.173	12.666	9.971	7.727
	2.0	1.104	2.407	4.342	9.582	6.792	4.633

Table 4.3: Option Values Corresponding to Different Exercise Prices.

consistent with the results from Monte Carlo simulations (not shown here). Note that for both put and call options, the option price decreases as the length of the averaging periods increases. One possible reason for this is that when the length of the averaging periods increases, the volatility of the corresponding arithmetic average decreases. This is consistent with the fact that an option price decreases as the volatility of the underlying asset decreases.

4.4.2 American-Style Asian Options

We turn now to American-style Asian options whose averages are taken over a period of length less than the life of the option.

As an example, consider an American-style Asian call option whose maturity is T, averaging period is $T_{avg} \leq T$ and exercise price is K. Assume the underlying asset follows a log-normal random walk which is simulated by a binomial tree (see Figure 4.5).

We consider two kinds of American-style Asian options here. One is an option that can not be exercised for $t < T_{avg}$ and its payoff at any time $t \geq T_{avg}$ is

$$\max\left[\frac{\sum_{j=J_1}^{J_2} S_{j,k_j}}{J_2 - J_1 + 1} - K,\ 0\right], \tag{4.21}$$

where $(k_{J_1}, ..., k_{J_2})$ is the path of the underlying asset process, and J_1 and J_2 satisfy

$$(J_2 - 1)\Delta t = t, \tag{4.22}$$

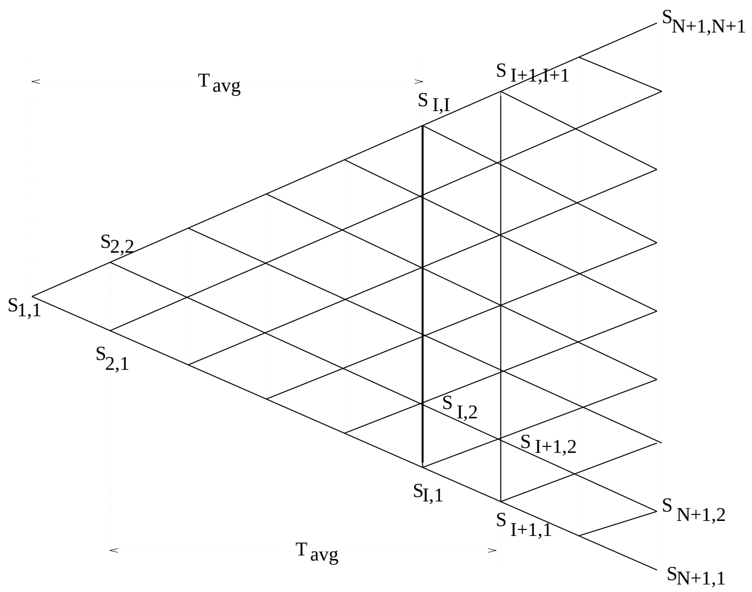

Figure 4.5: American-Style Asian Options Averaged Over a Period Shorter Than the Life of the Option.

$$(J_1 - 1)\Delta t = t - T_{avg}. \tag{4.23}$$

Here Δt is the time step for the binomial tree in Figure 4.5. If N is the number of the time steps, then we have

$$\Delta t = \frac{T}{N}. \tag{4.24}$$

The other kind is an option that can be exercised any time. If the exercise time $t \geq T_{avg}$, then its payoff is (4.21). But, if the exercise time $t < T_{avg}$, then its payoff is

$$\max\left[\frac{\sum_{j=1}^{J} S_{j,k_j}}{J} - K,\ 0\right], \tag{4.25}$$

where $(k_1, k_2, ..., k_J)$ is the path of the underlying asset process and J satisfies

$$(J-1)\Delta t = t. \tag{4.26}$$

Note that when $T_{avg} = T$, these two kinds of options become the European-style and American-style Asian options Hull and White study in [13].

To extend the Hull and White tree method for these two kinds of Asian options, we follow the steps outlined below.

1. **Build a Tree.** We first build a binomial tree with two parts as shown in Figure 4.5. The tree has N time steps, each of length Δt satisfying (4.24). The dividing line between the first and second parts of the tree is level I, where $(I-1)\Delta t = T_{avg}$.

   To compute the asset price averages, first let $A_{1,1}^{max} = A_{1,1}^{min} = S_{1,1}$. Then, for each node $S_{k,j}$ with $k \geq 2$ in the first part, if it is not a boundary node $(1 < j < k)$, we compute the maximum and minimum averages at the node from

$$A_{k,j}^{max} = \frac{(k-1)A_{k-1,j}^{max} + S_{k,j}}{k}, \tag{4.27}$$

$$A_{k,j}^{min} = \frac{(k-1)A_{k-1,j-1}^{min} + S_{k,j}}{k}, \tag{4.28}$$

   where $A_{k-1,j}^{max}$ and $A_{k-1,j-1}^{min}$ are the maximum average at node $S_{k-1,j}$ and the minimum average at node $S_{k-1,j-1}$, respectively. If the node $S_{k,j}$ is a boundary node ($j=1$ or $j=k$), then the maximum and minimum averages at the node are the same and are computed from

$$A_{k,1}^{max} = A_{k,1}^{min} = \frac{(k-1)A_{k-1,1}^{max} + S_{k,1}}{k} \text{ or} \tag{4.29}$$

$$A_{k,k}^{max} = A_{k,k}^{min} = \frac{(k-1)A_{k-1,k-1}^{max} + S_{k,k}}{k}, \tag{4.30}$$

   where $A_{k-1,1}^{max}$ and $A_{k-1,k-1}^{max}$ are the maximum (also minimum) averages at nodes $S_{k-1,1}$ and $S_{k-1,k-1}$ respectively. Then we approximate the possible averages between the minimum and maximum averages at each node using (4.9). A minor difference between our approach and the Hull and White tree method is that in our approach the maximum and minimum averages are computed at each node instead of each level.

   Similarly, we can compute the maximum and minimum averages at each node $S_{k,j}$ in the second part of the tree, but the averages should be computed from level $(k-I)$ to level k. To compute the maximum and minimum averages at node $S_{k,j}$ which is not a boundary node, $1 < j < k$, we know the maximum average should be computed from the maximum average at node $S_{k-1,j}$ and the minimum average should be computed from the minimum average at node $S_{k-1,j-1}$ since the path having the maximum average will go through $S_{k-1,j}$ and the path having the minimum average will go through $S_{k-1,j-1}$. For the node $S_{k,j}$, we can easily determine the beginning node for the path that gives the maximum average at the node $S_{k-1,j}$ or the minimum average at the node $S_{k-1,j-1}$ (see Figure 4.6). The beginning node for the path that gives the maximum average at the node $S_{k-1,j}$ is node $S_{k-1-I,max}$ with

$$max = \min[j,\ k-1-I], \tag{4.31}$$

and the beginning node for the path that gives the minimum average at the node $S_{k-1,j-1}$ is node $S_{k-1-I,min}$ with

$$min = \max[j-1-I, 1]. \tag{4.32}$$

From this information we have

$$A_{k,j}^{max} = \frac{IA_{k-1,j}^{max} - S_{k-1-I,max} + S_{k,j}}{I}, \tag{4.33}$$

$$A_{k,j}^{min} = \frac{IA_{k-1,j-1}^{min} - S_{k-1-I,min} + S_{k,j}}{I}, \tag{4.34}$$

where $A_{k-1,j}^{max}$ is the maximum average at $S_{k-1,j}$, $A_{k-1,j-1}^{min}$ is the minimum average at $S_{k-1,j-1}$, $S_{k-1-I,max}$ is the beginning node for the path that has the maximum average at node $S_{k-1,j}$ and $S_{k-1-I,min}$ is the beginning node for the path that has the minimum average at node $S_{k-1,j-1}$. If $S_{k,j}$ is a boundary node, $j = 1$ or $j = k$, then the maximum and minimum averages at the node are the same and are computed from

$$A_{k,k}^{max} = uA_{k-1,k-1}^{max} \text{ or} \tag{4.35}$$

$$A_{k,1}^{min} = dA_{k-1,1}^{min}, \tag{4.36}$$

where u and d are the up-factor and down-factor, respectively, used in building the tree. After we compute the maximum and minimum averages at $S_{k,j}$ we can approximate the possible averages between them in the way described above.

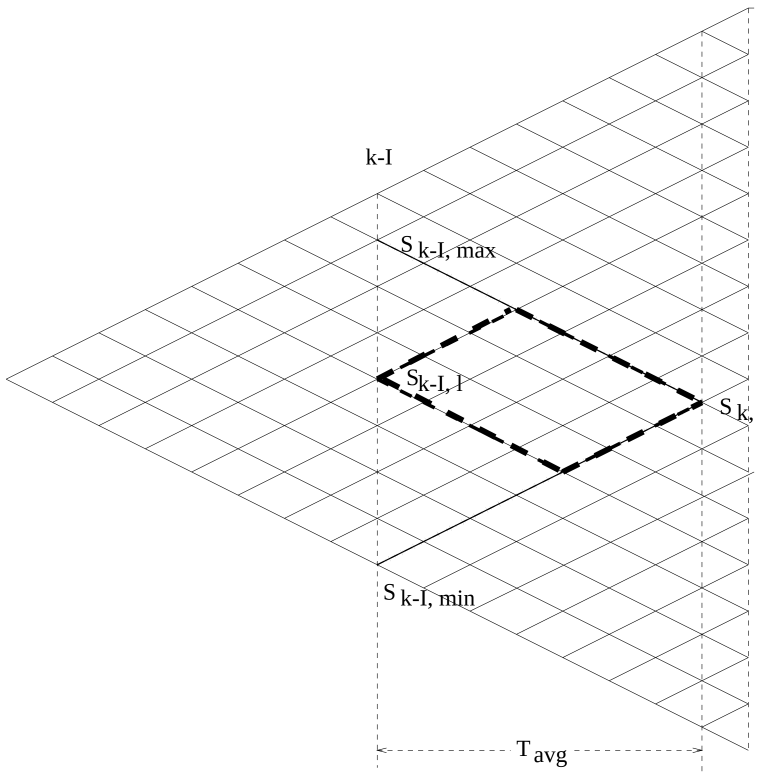

Figure 4.6: The Beginning Nodes for American-Style Asian Options Averaged Over a Period Shorter Than the Life of the Option.

2. **Determine Beginning Nodes.** When we compute option prices backward in the second part of the tree, we need the beginning node for a path having an average close to Se^{mh} for each Se^{mh} between the maximum and minimum averages at each node.

   For a node $S_{k,j}$, the beginning nodes for the paths having maximum average $A_{k,j}^{max}$ and minimum average $A_{k,j}^{min}$ are nodes $S_{k-I,max}$ and $S_{k-I,min}$, respectively, (see Figure 4.6). They can be determined from relations similar to (4.31) and (4.32).

For an approximate average Se^{mh} between the maximum average $A_{k,j}^{max}$ and the minimum average $A_{k,j}^{min}$, it is nontrivial to determine the "best" beginning node for a path having an average close to Se^{mh}. The approximate solution to this problem that we use is outlined below.

Paths with different beginning nodes may have almost the same averages over the averaging period that are all close to Se^{mh}, so we can select one of several beginning nodes here.

For each node $S_{k-I,l}$ between the nodes $S_{k-I,min}$ and $S_{k-I,max}$ (see Figure 4.6), we can easily determine the maximum average, $A_{k-I,l}^{max}$, and minimum average, $A_{k-I,l}^{min}$, over all the paths between $S_{k-I,l}$ and $S_{k,j}$ (the paths having these two averages are shown in Figure 4.6). If an approximate average Se^{mh} is between the maximum average $A_{k-I,l}^{max}$ and minimum average $A_{k-I,l}^{min}$, then node $S_{k-I,l}$ is a candidate for the beginning node of a path with average close to Se^{mh}. Each approximate average Se^{mh} may have several candidates for its beginning node.

Suppose $S_{k-I,mid}$ is a middle node between nodes $S_{k-I,min}$ and $S_{k-I,max}$, where

$$mid = \left[\frac{max + min}{2}\right], \tag{4.37}$$

and max and min are determined from equations (4.31) and (4.32), respectively. There is a high probability that a path reaching node $S_{k,j}$ begins from the middle node, so it seems reasonable to choose a node which is close to the middle node among candidate nodes for the beginning node for a path with an average close to Se^{mh}.

Let $Se^{M_1h} < \cdots < Se^{mh} < \cdots < Se^{M_2h}$ denote all the approximate averages at node $S_{k,j}$ and $A_{k-I,mid}^{max}$ and $A_{k-I,mid}^{min}$ be the maximum and minimum averages over all the paths between $S_{k-I,mid}$ and $S_{k,j}$. For each $Se^{mh} \in [A_{k-I,mid}^{min},\ A_{k-I,mid}^{max}]$, we choose $S_{k-I,mid}$ as the beginning node for the path with average close to Se^{mh}. Let $A_{k-I,mid-1}^{min}$ be the minimum average over all the paths between $S_{k-I,mid-1}$ and $S_{k,j}$, and $A_{k-I,mid+1}^{max}$ be the maximum average over all the paths between $S_{k-I,mid+1}$ and $S_{k,j}$. Then, for each $Se^{mh} \in [A^{min}_{k-I,mid-1}, A^{min}_{k-I,mid})$, we choose $S_{k-I,mid-1}$ as the beginning node for the path with average close to Se^{mh}. Similarly, for each $Se^{mh} \in (A^{max}_{k-I,mid}, A^{max}_{k-I,mid+1}]$, we choose $S_{k-I,mid+1}$ as the beginning node for the path with average close Se^{mh}. Continuing this process, we can determine the beginning node for a path with an average close to Se^{mh} for each approximate average Se^{mh}, $m = M_1, ..., M_2$.

Our method for choosing the beginning node of a path with an average close to Se^{mh} is somewhat arbitrary. We plan to investigate other schemes for selecting beginning nodes later.

3. **Computing Option Prices Backward.** First we compute the payoffs for different averages at each node of the last level $(N+1)$ from (4.21) for a call option. (Payoffs for a put option can be computed in a similar way.)

   Next, we compute the option prices for all the approximate averages at each node of the second part of the tree backward from level N to level I. If the option prices for all the averages at each node of level k are available, we can compute the option prices for all the averages at any node $S_{k-1,j}$ of level $k-1$ as we did in Section 4.3, but equation (4.12) is changed to

$$A^*_{k-1,l} = \frac{IA_{k-1,l} - S^{begin}_{k-1-I,l} + S_{k,j+1}}{I}, \; l = 1, ..., N_1, \tag{4.38}$$

   where $A^*_{k-1,l}$ denotes the new average when $S_{k-1,j}$ moves up to $S_{k,j+1}$, $S^{begin}_{k-1-I,l}$ denotes the beginning node for a path with an average close to $A_{k-1,l}$, and N_1 denotes the number of approximate averages at node $S_{k-1,j}$.

   After the option price for each node of level I is computed, we follow the procedure described at Section 4.3 to compute option prices in the first part of the tree, starting at level $I-1$ and working backward through the tree to level 1.

To compare numerical results, we repeat the example we considered before. That is, the underlying initial asset price $S = 50$, the risk-free interest rate $r = 10\%$ and the stock volatility $\sigma = 30\%$. The number of time steps in the tree is $N = 40$ and the partition step size $h = 0.005$. The option prices for different parameters are shown in Table 4.4.

Maturity	Average	Style	Put Exercise Prices			Call Exercise Prices		
T	T_{avg}	EA/AA	45	50	55	45	50	55
	0.1	EA	2.361	4.519	7.602	12.132	8.888	6.318
		AA	2.361	4.519	7.602	12.132	8.888	6.318
1.0	0.5	EA	1.994	3.799	6.639	10.360	7.046	4.745
		AA	1.994	3.840	7.174	10.619	7.072	4.745
	1.0	EA	0.751	2.182	4.603	7.615	4.522	2.420
		AA	0.826	2.633	4.603	8.550	4.892	2.536
	0.2	EA	3.231	5.355	8.212	16.637	13.655	11.116
		AA	3.231	5.355	8.212	16.637	13.665	11.116
2.0	1.0	EA	2.585	4.235	6.596	13.698	10.716	8.390
		AA	2.586	4.316	7.205	13.890	10.747	8.392
	2.0	EA	1.104	2.407	4.342	9.582	6.792	4.633
		AA	1.287	3.093	6.245	10.623	7.326	4.886

Table 4.4: Option Values Corresponding to Different Parameters.

In Table 4.4, we use EA to denote American-style Asian options that can not be exercised within the averaging period (European-feature) and AA to denote American-style Asian options that can be exercised within the averaging period (American-feature). Note that, when $T_{avg} = T$, our tree method gives the same numerical results as the Hull and White tree method. From Tables 4.3 and 4.4, we see that the American-style Asian options are more expensive than the corresponding European-style Asian options. This is consistent with our analysis of the prices of European options and American options. From Table 4.4, we observe that under certain conditions the option prices of style EA and AA are very close or even equal. This suggests that it may not be optimal to exercise American-style Asian options for $t < T_{avg}$ under certain conditions, but the reason for this conclusion is not clear to us yet.

We defer the analysis of the accuracy of our tree method to a later date, in part because we have not yet found an effective method to test the accuracy of our approach.

Chapter 5

Conclusion

5.1 Summary

After reviewing the basic option pricing theory, we study several widely used numerical methods in modern finance: Finite Difference methods, Monte Carlo methods and Tree methods and illustrate each of them with practical examples.

Monte Carlo methods are flexible schemes for pricing derivative securities, especially path-dependent options, but they are neither efficient nor suitable for pricing American-style options which can be exercised early.

Hull and White develop a tree method for pricing Asian options, but their method is designed for Asian options whose averages are taken over the whole life of the option.

The main contribution of this thesis is an extension of the Hull and White tree method for path-dependent options. We illustrate the general approach by developing a method for pricing general Asian options whose averages can be taken over the last period of the option. The Hull and White tree method is a special case of our scheme.

Numerical testing confirms the theoretical prediction that when our averages are taken over the whole life of the option, we get the same results as Hull and White. When our averages are taken over the last period of the option, our results for European-style Asian options are consistent with the results of Monte Carlo simulations, but our method is more efficient. For American-style Asian options whose averages are taken over the last period, we have not yet found an efficient method with which to compare the results from our method.

Since we use different data structures in implementing our tree method, our method is more efficient than the straight-forward Hull and White tree method described in [13].

5.2 Future Work

We studied different numerical methods for pricing options and provided some simple illustration of the schemes. But little work has been done in analyzing the accuracy and efficiency of different approaches. We intend to examining these questions later. For each promising numerical method, we plan to develop an efficient, robust, reliable program to test the scheme.

Although in this thesis we assumed throughout that the interest rate is a constant, the majority of option contracts depend on the term structure of interest rates. It is of great importance to develop efficient algorithms for pricing various path-dependent interest-rate options. We hope to extend our approach to handle this class of options.

In addition, we assumed throughout that the underlying asset for an option does not pay any dividends during the life of the option. Since this is often not true, the method should be extended to allow for dividend payments during the life of the option.

In our tree method, the beginning node of a path with an average close to Se^{mh} was chosen somewhat arbitrarily. We plan to develop and analyze other schemes for finding beginning nodes or, more generally, updating the averages.

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