Exact solutions and conservation laws for coupled generalized Korteweg-de Vries and quintic regularized long wave equations

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A new model equation is proposed for simulating unidirectional wave propagation in nonlinear media with dispersion. This model equation is a one-dimensional evolutionary partial differential equation that is obtained by coupling the generalized Korteweg-de Vries (gKdV) equation, which includes a nonlinear term of any order and cubic dispersion, with the quintic Regularized Long Wave (qRLW) equation, which includes fifth order dispersion. Exact solitary wave solutions are derived for this model equation. These analytical solutions are obtained for any order of the nonlinear term and for any given values of the coefficients of the cubic and quintic dispersive terms. Analytical expressions for three conservation laws and for three invariants of motion for solitary wave solutions of this new equation are also derived.

1. INTRODUCTION

The Korteweg-de Vries (KdV) equation, $u_t + uu_x + u_{xxx} = 0$, is a well-known nonlinear partial differential equation (PDE) originally formulated to model unidirectional propagation of shallow water gravity waves in one dimension; it describes the long time evolution of weakly nonlinear dispersive waves of small but finite amplitude. The original experimental observations of Scott Russell [13] in 1834 and the pioneering studies by Boussinesq [2] in 1871 and by Korteweg and de Vries [8] in 1895 showed that when nonlinear wave steepening, from the term uu_x, is balanced by wave dispersion, owing to the term u_{xxx}, their equation predicts a unidirectional solitary wave, (that is, a pulse which moves in one direction with a permanent shape and constant speed). They also found the explicit $\mathtt{sech}^2$ expression for its solitary wave solutions.

Because of its role as a model equation in describing a variety of physical systems, and because of its interesting mathematical properties, the KdV equation has been widely in-

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vestigated in recent decades. In the 1960's, it was discovered that the KdV equation forms a completely integrable Hamiltonian system and admits an infinite number of conservation laws and invariants of motion (see Miura et *al.* [9]). An important property of the completely integrable system is the exact interactions of its solitary wave solutions which retain their original shapes and speeds after collision and exhibit only a small overall phase shift. These special solitary waves are named solitons and their clean interactions are called elastic interactions.

More recently, similar equations to the KdV equation have been proposed. Benjamin, Bona and Mahoney [1] advocated that the PDE, $u_t + u_x + uu_x - \mu u_{xxt} = 0$, modeled the same physical phenomena equally well as the KdV equation, given the same assumptions and approximations that were originally used by Korteweg and de Vries [8]. This PDE is now often called the *regularized* long wave (RLW) equation, although it is also known as the BBM equation. The dispersive term u_{xxt} confers more expedient mathematical properties to the RLW equation and makes it a preferable model to the the KdV equation. The RLW equation also has an explicit $\texttt{sech}^2$ solitary wave solutions, but is a nonintegrable system since small dispersive effects can be observed when its solitary waves solutions collide.

Over the years, several generalizations of the KdV equation have found applications in many areas, including quantum field theory, plasma physics, solid-state physics, liquid-gas bubble mixtures, and an-harmonic crystals. For instance, Kakutani and Ono [6] proposed the generalized KdV equation $u_t + uu_x + u_{xxx} + \varepsilon u_{xxxxx} = 0$, with both a third order and fifth order derivative term, to model the dispersive effects of weak nonlinear hydromagnetic waves in cold collision-free plasma. This fifth order KdV equation is also found to be a model for long waves in liquids under ice sheets and water waves with surface tension (capillary waves). For $\varepsilon < 0$, Yamammoto and Takizawa [14], found an explicit $\texttt{sech}^4$ solitary wave solution of depression ("dark" solitary wave) for one particular positive wave speed. A similar quintic KdV (qKdV) equation, $u_t + uu_x + +\varepsilon u_{xxxxx} = 0$ was proposed by Nagashima [10] for modeling waves in nonlinear LC circuits with mutual inductance between neighboring inductors and nonlinear transmission lines. Rosenau and Hyman [12] formulated a similar quintic regularized long wave (qRLW) equation $u_t + uu_x + u_{txxxx} = 0$ for modeling the dynamics of dense discrete lattices and studied its multiplet solutions. Just as the RLW equation is preferred to the KdV equation for problems with high wave numbers, the qRLW equation is a better suited model than the qKdV equation for describing patterns involving higher gradients.

Recently, the authors [5] presented exact solitary wave solution for general types of the RLW equation with nonlinear terms of any order p, $u_t + au^p u_x - \mu u_{xxt} = 0$. The solutions were obtained by integrating a first order nonlinear ODE with symbolic computation using the mathematical software `Maple`. In this study, we show how this approach can be applied to derive the exact solutions for the classical generalized Korteweg-de Vries (gKdV) equation, which includes a nonlinear term of any order p and cubic dispersion,

$$u_t + au^p u_x + \mu u_{xxx} = 0 \ . \tag{1}$$

By coupling this gKdV equation with the quintic regularized long wave (qRLW) equation, which includes fifth order dispersion, we obtain a new evolution equation that we will call the gKdV-qRLW equation, which can model the effects of a high order singular perturbation (in the limit $\varepsilon \longrightarrow 0$) to the gKdV equation,

$$u_t + au^p u_x + \mu u_{xxx} + \varepsilon u_{txxxx} = 0 \ . \tag{2}$$

Exact solitary wave solutions will be derived for this new model equation. These analytical and explicit solutions are obtained for any p, μ and ε. The approach presented in this work is general and can also be applied for finding exact solitary wave solutions for similar nonlinear wave equations such as KdV-like and Boussinesq-like equations (see Hamdi et *al.* [4,5]). The exact solitary wave solutions can be used to specify initial data for the incident waves in the numerical model and for the verification of the associated computed solution (Hamdi et *al.* [3]). We also derive analytical expressions for three conservation laws and for three invariants of motion for solitary wave solutions of this new equation. The invariants of motion can be used as verification tools to investigate the conservation properties and performance of numerical schemes for the approximate solution of this new class of PDEs.

2. DERIVATION OF THE EXACT SOLUTIONS

We concentrate on finding an exact solitary wave solution of (2) of the form

$$u(x,t) = u(x - x_0 - Ct) \equiv u(\xi) \ . \tag{3}$$

This corresponds to a traveling-wave propagating with steady celerity C. We are interested in solutions depending only on the moving coordinate $\xi = x - x_0 - Ct$. Substituting into (2), the function $u(\xi)$ satisfies a fifth order nonlinear ordinary differential equation (ODE),

$$-Cu' + a\, u^p u' + \mu\, u''' - \varepsilon\, C u''''' = 0 \ , \tag{4}$$

where the derivatives are defined with respect to the coordinate ξ.

Integrating once, we obtain

$$-Cu + \frac{a}{p+1} u^{p+1} + \mu\, u'' - \varepsilon\, C u'''' = k_1 \ , \tag{5}$$

where k_1 is a constant of integration. If we assume that the solitary wave solution and its derivatives have the following asymptotic values,

$$u(\xi) \longrightarrow u_\pm \ \text{ as } \ \xi \longrightarrow \pm\infty \ , \ \text{ and for } \ \mathrm{n} \geq 1, \ \ \mathrm{u}^{(\mathrm{n})}(\xi) \longrightarrow 0 \ \text{ as } \ \xi \longrightarrow \pm\infty \ , \tag{6}$$

(where the superscript denotes differentiation to the order n, with respect to ξ), and if we also assume that $u_\pm$ satisfies the following algebraic equation

$$-Cu_\pm + \frac{a}{p+1} u_\pm^{p+1} = 0 \ , \tag{7}$$

then the constant of integration k_1 is equal to zero and (5) reduces to

$$-Cu + \frac{a}{p+1} u^{p+1} + \mu\, u'' - \varepsilon\, C u'''' = 0\,. \tag{8}$$

From (7) we also have the relation

$$\frac{a}{p+1}\left(u_+^{p+2} - u_-^{p+2}\right) = C\left(u_+^2 - u_-^2\right) \quad . \tag{9}$$

To allow another integration, we first multiply (8) by $2u'$. Then each term can be integrated separately to obtain,

$$-Cu^2 + \frac{2a}{(p+1)(p+2)} u^{p+2} + \mu\,(u')^2 - \varepsilon\, C(u''')^2 = k_2\,, \tag{10}$$

where k_2 is a second constant of integration.

Using the asymptotic boundary conditions (6) at infinity we have

$$-Cu_+^2 + \frac{2a}{(p+1)(p+2)} u_+^{p+2} = k_2, \text{ as } \xi \longrightarrow +\infty\,, \tag{11}$$

and

$$-Cu_-^2 + \frac{2a}{(p+1)(p+2)} u_-^{p+2} = k_2, \text{ as } \xi \longrightarrow -\infty\,. \tag{12}$$

By equating (11) and (12) we obtain,

$$\frac{2a}{(p+1)(p+2)}\left(u_+^{p+2} - u_-^{p+2}\right) = C\left(u_+^2 - u_-^2\right) \quad . \tag{13}$$

From relations (9) and (13) we conclude that $\left(u_+^2 - u_-^2\right) = 0\,.$

Therefore, the traveling wave solutions for (2) that satisfy the assumptions (6) and (7) should also satisfy the condition $|u_-| = |u_+|$. This condition suggests seeking bell shaped solitary wave solutions that can be expressed as a polynomial of the hyperbolic secant. Moreover, these traveling wave solutions could not have kink-profiles since such profiles require $|u_-| \neq |u_+|$.

If $\varepsilon = 0$ (no quintic dispersion) equation (2) reduces to the classical generalized KdV equation (1). In this case, equation (10) reduces to the following nonlinear ODE,

$$-Cu^2 + \frac{2a}{(p+1)(p+2)} u^{p+2} + \mu\,(u')^2 = k_2\,. \tag{14}$$

If we also assume that $|u_-| = |u_+| = 0$, then the second constant of integration k_2 is equal to zero. There are several approaches for integrating this type of nonlinear ODEs [4]. These ODEs can also be solved using symbolic computation as described in Hamdi et al. [5]. Without making any assumptions regarding the mathematical form of the solution, the ODE (14) is integrated using `Maple` to obtain an analytical expression of an exact solution of the gKdV equation,

$$u(x,t) = \left[\frac{(p+1)(p+2)}{2a} C \operatorname{sech}^2 \left(\frac{1}{2} p \sqrt{\frac{C}{\mu}} \, (x - C\,t - x_0) \right) \right]^{\frac{1}{p}} . \tag{15}$$

This solution is a bell-profile solitary wave of amplitude $[C(p+1)(p+2)/2a]^{1/p}$ initially centered at x_0 and traveling without change of shape at a steady celerity C and a wave number $\frac{1}{2}p\sqrt{C/\mu}$. The amplitude of this solitary wave is proportional to the wave speed C.

If $\varepsilon \neq 0$, the exact solution of (2) based on the direct integration of the resulting high order dispersive and nonlinear ODEs (8) or (10) is not straightforward. The solutions that we will derive are obtained by assuming a certain form of the solitary wave in order to transform the fourth order ODE (8) into a fourth order polynomial that can be solved with `Maple`.

Since we are seeking a bell-profile solitary wave solution that satisfies the assumptions (6) and (7) and also the condition $|u_-| = |u_+|$, we will assume that such a solution can be expressed as a monomial of the hyperbolic secant function (`sech` = `1/cosh`). That is,

$$u(\xi) = u_0(\operatorname{sech}(\kappa\xi))^m = u_0 s^m \,, \qquad \text{where} \quad s = \operatorname{sech}(\kappa\xi) \,. \tag{16}$$

This particular form is suggested by the previous solitary wave solution of the generalized KdV equation (1) which is a $\texttt{sech}^{2/p}$ function given by (15). This approach is similar to the technique used by Kichennassamy and Olver [7] and in general is similar to sech-based methods for transforming nonlinear ODEs into polynomials. *A priori* the exponent m, the wave amplitude u_0, the wave celerity C and the wave number κ are unknown and will be determined in the sequel.

The hyperbolic secant function `sech` has an interesting recursive property: All even derivatives of `sech` are polynomials in sech. Using this recursive property, we can express the even derivative u'' and u'''' which appear in (8) as homogeneous polynomials of the dependent variable $\mathtt{s}(\xi)$ only. That is,

$$u'' = u_0 m^2 \kappa^2 \, s^m + (-u_0 m^2 \kappa^2 - m u_0 \kappa^2) \, s^{(m+2)} \,, \tag{17}$$

and

$$\begin{aligned} u'''' = {} & u_0 m^4 \kappa^4 \, s^m + (-2u_0 m^4 \kappa^4 - 8u_0 m^2 \kappa^4 - 6m^3 u_0 \kappa^4 - 4m\kappa^4 u_0) \, s^{(m+2)} \\ & + (11u_0 m^2 \kappa^4 + u_0 m^4 \kappa^4 + 6m\kappa^4 u_0 + 6m^3 u_0 \kappa^4) \, s^{(m+4)} \,. \end{aligned} \tag{18}$$

Now, substituting (17) and (18) into (8) leads to a homogeneous polynomial in s on its left-hand side. After simplification by the factor $u_0\, s^m$, the ODE (8) reduces to

$$\frac{a\, {u_0}^p}{(p+1)}\, s^{(m\, p)} + (-11\,\varepsilon\, C\, m^2\, \kappa^4 - \varepsilon\, C\, m^4\, \kappa^4 - 6\,\varepsilon\, C\, m\, \kappa^4 - 6\,\varepsilon\, C\, m^3\, \kappa^4)\, s^4$$

$$+ (-\mu\, m^2\, \kappa^2 - \mu\, m\, \kappa^2 + 2\,\varepsilon\, C\, m^4\, \kappa^4 + 8\,\varepsilon\, C\, m^2\, \kappa^4 + 6\,\varepsilon\, C\, m^3\, \kappa^4 + 4\,\varepsilon\, C\, m\, \kappa^4)\, s^2$$

$$- C + \mu\, m^2\, \kappa^2 - \varepsilon\, C\, m^4\, \kappa^4 \ = 0\,. \tag{19}$$

In order to obtain a nontrivial solution, the power mp of the first term of equation (19) must be equal to 4 and all the coefficients of the polynomial in s on the left-hand side must be zero. Setting the coefficients to zero yields a nonlinear algebraic system of three equations for the three unknowns C, κ and u_0,

$$\begin{cases} (-176\,\dfrac{\varepsilon\, C\, \kappa^4}{p^2} - \dfrac{256\,\varepsilon\, C\, \kappa^4}{p^4} - \dfrac{24\,\varepsilon\, C\, \kappa^4}{p} - \dfrac{384\,\varepsilon\, C\, \kappa^4}{p^3} + \dfrac{a\, {u_0}^p}{p+1}) = 0\,, \\[2ex] -16\,\dfrac{\mu}{p^2} - \dfrac{4\,\mu}{p} + \dfrac{512\,\varepsilon\, C\, \kappa^2}{p^4} + \dfrac{128\,\varepsilon\, C\, \kappa^2}{p^2} + \dfrac{384\,\varepsilon\, C\, \kappa^2}{p^3} + \dfrac{16\,\varepsilon\, C\, \kappa^2}{p} = 0\,, \\[2ex] -256\,\dfrac{\varepsilon\, C\, \kappa^4}{p^4} - C + \dfrac{16\,\mu\, \kappa^2}{p^2} = 0\,. \end{cases} \tag{20}$$

The analytical solution of this nonlinear system (20) can be obtained using `Maple`. After simplifications, we get the following exact and real expressions for the celerity C, wave number κ and wave amplitude u_0 as functions of the given parameters and coefficients p, a, μ and ε of the generalized quintic regularized long wave equation (2).

$$C = \frac{\mu}{4\,\varepsilon}\, \frac{\sqrt{256\,\varepsilon + 64\,\varepsilon\, p^2 + 256\,\varepsilon\, p}}{(p^2 + 4\,p + 8)}\,, \tag{21}$$

$$\kappa = \pm\, \frac{p}{\sqrt[4]{256\,\varepsilon + 64\,\varepsilon\, p^2 + 256\,\varepsilon\, p}}\,, \tag{22}$$

$$u_0 = \left[\frac{2\mu}{a}\, \frac{(25\,p^3 + 70\,p^2 + 80\,p + 32 + 3\,p^4)}{(p^2 + 4\,p + 8)\, \sqrt{256\,\varepsilon + 64\,\varepsilon\, p^2 + 256\,\varepsilon\, p}} \right]^{\dfrac{1}{p}}\,. \tag{23}$$

From (16) and the previous relation for the celerity C, wave number κ and wave amplitude u_0 , we finally deduce an explicit expression for this exact solution,

$$u(x,t) = u_0\, [\operatorname{sech}\, (\kappa(x - x_0 - Ct))]^{\dfrac{4}{p}}\,. \tag{24}$$

It is easy to verify that (24) satisfies the generalized quintic regularized long wave equation (2) and also satisfies the assumptions (6) and (7). Unlike solitary wave solution of the gKdV equation (1), which are possible for all wave speeds, the solitary wave (24) exists for only one particular wave speed (21), for any specified values of the parameters and coefficients p, a, μ and ε. Moreover, the wave number κ is independent of the coefficient μ and depends only on p and ε. Although equation (2) can be considered for modeling the effects of a high order singular perturbation (in the limit $\varepsilon \longrightarrow 0$) to the gKdV equation, its solitary wave solution (24) is fairly localized and definitely is a nonperturbative solution in the sense that this solution does not converge to the gKdV solitary wave solution as given by (15).

3. CONSERVATION LAWS AND INVARIANTS OF MOTION

A conservation law in differential equation form can be written as $T_t + X_x = 0$, in which the "density" T and the "flux" X are polynomials in the solution u and its x-derivatives. If both T and X_x are integrable over the domain $(-\infty, +\infty)$, then the assumption that $X \longrightarrow 0$ as $|x| \longrightarrow \infty$, implies that the conservation law can be integrated over all x to yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{-\infty}^{+\infty} T \mathrm{d}x\right) = 0 \qquad \text{or} \qquad \int_{-\infty}^{+\infty} T \mathrm{d}x = \text{constant}. \tag{25}$$

The intergral of T, over the entire spatial domain is therefore invariant with time and usually called an *invariant of motion* or a *constant of motion*. The KdV equation itself is already in conservation form (i.e. $T = u, \;\; X = u_{xx} + \frac{1}{2}u^2$). In the late 1960's Miura et *al.* [9] discovered that the KdV equation admits an infinite number of conservation laws and forms a completely integrable Hamiltonian system. Olver [11] has shown that the RLW equation, $(u_t + u_x + uu_x - \mu u_{xxt} = 0)$, has only three non-trivial conservation laws and therefore is not an integrable PDE. These conservation laws for water waves are the equivalents of the conservation of mass, momentum and energy. Recently, Hamdi et *al.* [5]) identified three conservation laws for the generalized equal width (EW) wave equation, $u_t + au^p u_x - \mu u_{xxt} = 0$, and derived analytical expressions for the corresponding invariants of motion.

In this section, we first derive three conservation laws for the generalized quintic regularized long wave equations (2) . The first conservation law is obtained directly from (2) by rewriting it in the form $T_t + X_x = 0$ as follows,

$$(u)_t + \left(\frac{a}{(p+1)}u^{p+1} + \mu u_{xx} + \varepsilon u_{txxx}\right)_x = 0\,. \tag{26}$$

The second conservation law is derived by multiplying the equation $(u_t + au^p u_x + \mu u_{xxx} + \varepsilon u_{txxxx} = 0)$ by $2\,u$. After performing several integrations by parts and simplifications we have,

$$\left(u^2 + \varepsilon u_{xx}^2\right)_t + \left(\frac{2\,a}{(p+2)}u^{p+2} + 2\mu u u_{xx} - \mu u_{xx}^2 + 2\varepsilon u\, u_{txxx} - 2\varepsilon u_x\, u_{txx}\right)_x = 0\,. \tag{27}$$

The third conservation law is more complicated to derive as it requires less obvious substitutions. After several integrations by parts and tedious manipulations the resulting quantity can be expressed in conservation form $T_t + X_x = 0$ as,

$$\left(\frac{2\,a\,u^{p+2}}{(p+1)(p+2)} - \mu u_x^2\right)_t + \left(\frac{a^2\,u^{2(p+1)}}{(p+1)^2} + \frac{2\,a\,u^{p+1}}{(p+1)}(\varepsilon\,u_{txxx} - \mu\,u_{xx}) \right.$$
$$\left. -\ 2\mu^2\,u_x\,u_{xxx} + \mu^2 u_{xx}^2 + 2\mu a\,u^p\,u_x^2 - 2\mu\varepsilon\,u_x\,u_{txxxx} + 2\mu\varepsilon\,u_{xx}\,u_{txxx} + \varepsilon^2\,u_{txxx}\right)_x = 0\,. \quad (28)$$

These three conservation laws can now be integrated easily with respect to x over a large but finite spatial domain $[x_L, x_U]$ instead of $[-\infty, +\infty]$ to obtain the intermediate results

$$\frac{\partial}{\partial t}\int_{x_L}^{x_U} u\,\mathrm{d}x + \frac{a}{(p+1)}\left(u_U^{(p+1)} - u_L^{(p+1)}\right) = 0\,,$$
$$\frac{\partial}{\partial t}\int_{x_L}^{x_U} (u^2 + \varepsilon\,u_{xx}^2)\mathrm{d}x + \frac{2\,a}{(p+2)}\left(u_U^{(p+2)} - u_L^{(p+2)}\right) = 0\,, \quad (29)$$
$$\frac{\partial}{\partial t}\int_{x_L}^{x_U} \left(\frac{2\,a\,u^{p+2}}{(p+1)(p+2)} - \mu u_x^2\right)\mathrm{d}x + \frac{a^2}{(p+1)^2}\left(u_U^{2(p+1)} - u_L^{2(p+1)}\right) = 0\,,$$

after simplifications. In these equations, $u_L = u(x_L, t)$ and $u_U = u(x_U, t)$ are time-invariant constants at the domain boundaries. In the simplifications, the terms $[u_{xx}]_{x_L}^{x_U}$, $[u_{txxx}]_{x_L}^{x_U}$, $[uu_{xx}]_{x_L}^{x_U}$, $[u^2u_{xx}]_{x_L}^{x_U}$, $[uu_{txxx}]_{x_L}^{x_U}$, $[u_xu_{txx}]_{x_L}^{x_U}$, $[u_xu_{xxx}]_{x_L}^{x_U}$, $[u^2u_x]_{x_L}^{x_U}$, $[u^pu_x^2]_{x_L}^{x_U}$, $[u_xu_{txxxx}]_{x_L}^{x_U}$ and $[u_{xx}u_{txxx}]_{x_L}^{x_U}$ are zero at the boundaries (assumptions (6)). Equation (29) can now be integrated with respect to t to yield

$$C_1 = \int_{x_L}^{x_U} u\,\mathrm{d}x + \frac{a}{(p+1)}\left(u_U^{(p+1)} - u_L^{(p+1)}\right)t\,,$$
$$C_2 = \int_{x_L}^{x_U} (u^2 + \varepsilon\,u_{xx}^2)\mathrm{d}x + \frac{2\,a}{(p+2)}\left(u_U^{(p+2)} - u_L^{(p+2)}\right)t\,, \quad (30)$$
$$C_3 = \int_{x_L}^{x_U} \left(\frac{2\,a\,u^{p+2}}{(p+1)(p+2)} - \mu u_x^2\right)\mathrm{d}x + \frac{a^2}{(p+1)^2}\left(u_U^{2(p+1)} - u_L^{2(p+1)}\right)t\,.$$

These invariants of motion are determined over a large but finite length spatial domain when the solution $u(x,t)$ of (2) is constant but not necessarily zero at the domain boundaries (assumptions (6)). The extra terms stem directly from the convection of mass, momentum and energy into and out of the lower and upper boundaries of the spatial domain. During the whole period of time in the course of which the waves propagate inside the domain $[x_L, x_U]$, these invariants of motion remain conserved and equal to their original values that are well determined initially at $t = 0$. Note that during numerical computations that provide solutions to (2), C_1, C_2 and C_3 can be calculated after each successive time step over the entire spatial domain $x_L \leq x \leq x_U$ that contains the wave motion, such that the conservation properties of the numerical algorithm can be monitored and thereby assessed. This approach for the derivation of these invariants of motion (30) is general and can be easily applied for finding similar invariants of motion for other general types of KdV and RLW equations.

In this last section we will derive analytic expressions for the three invariants of motion corresponding to the solitary wave solutions (24) of (2). For simplicity, the invariants are given for $p=1$. However, the evaluation of these invariants for any given value of the order p $(p \geq 1)$ is straightforward (consists in a simple change of the value of p in the `Maple` script). Since these invariants are independent of time and therefore have the same value at any time t, they will be evaluated at the initial state $t=0$ for solitary waves initially located at any arbitrary position x_0. We will suppose that $x_0=0$ and $[x_L, x_U]=[-L, L]$.

The first invariant of motion corresponds to the conservation of mass. It can be computed in `Maple` for the solitary wave profile (24) over the entire spatial domain $(-\infty, +\infty)$ as described in Hamdi et al. [5],

$$\int_{-\infty}^{\infty} u(x,t)\,\mathrm{d}x = \frac{35}{13}\,\frac{\pi\sqrt{6}\,\mu}{\varepsilon^{(1/4)}\,a} \quad . \tag{31}$$

The second invariant of motion represents the conservation of energy, which can be also evaluated using `Maple`,

$$\int_{-\infty}^{+\infty} \left(u^2+\varepsilon u_{xx}^2\right)\,\mathrm{d}x = \frac{2118515}{292032}\,\frac{\mu^2\sqrt{6}}{\varepsilon^{(3/4)}\,a^2} \quad . \tag{32}$$

Using `Maple`, the third invariant is given by

$$\int_{-\infty}^{+\infty} \left(\frac{2\,a\,u^{p+2}}{(p+1)(p+2)} - \mu u_x^2\right)\,\mathrm{d}x = \frac{1225}{158184}\,\frac{(-13+105\,\pi)\sqrt{6}\,\mu^3}{\varepsilon^{(5/4)}\,a^2} \quad .$$

4. CONCLUSION

In this study, a new evolution model equation is derived by coupling the generalized Korteweg-de Vries equation, which includes nonlinear terms of any order and cubic dispersion, with the quintic regularized long wave equation, which includes fifth order dispersion. A simple and direct method using `Maple` is devised for finding exact and explicit solitary wave solutions for this new equation. The analytical construction of such solitary waves is an obvious proof of existence of solutions for this equation. These solitary waves are localized traveling waves that tend asymptotically to zero at large distances and are non-pertubative solutions (in the sense that for small perturbations they do not converge to the generalized Korteweg-de Vries equation solitary wave solutions). Analytical expressions for three conservation laws and for three invariants of motion for these solitary wave solutions are also derived. The accuracy and stability of numerical schemes for the solution of these general model equations can be assessed using, as test problems, the new exact solutions. Their conservation properties can also be verified using the analytical expressions of the constants of motion. These verification tools for these evolutionary partial differential equations are implemented in a method of lines solver that is available from the authors [3]. The approach that we introduced for finding exact solitary wave solutions and conservation laws, is general and can be used for a wide class of nonlinear dispersive wave equations, such as general types of KdV-like and RLW-like equations.

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