Exact solutions and invariants of motion for general types of regularized long wave equations

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Abstract

New exact solitary wave solutions are derived for general types of the regularized long wave (RLW) equation and its simpler alternative, the generalized equal width wave (EW) equation, which are evolutionary partial differential equations for the simulation of one-dimensional wave propagation in nonlinear media with dispersion processes. New exact solitary wave solutions are also derived for the generalized EW-Burgers equation, which models the propagation of nonlinear and dispersive waves with certain dissipative effects. The analytical solutions for these model equations are obtained for any order of the nonlinear terms and for any given value of the coefficients of the nonlinear, dispersive and dissipative terms. Analytical expressions for three invariants of motion for solitary wave solutions of the generalized EW equation are also devised.

Key words: Equal width wave equation, nonlinearity, dispersion, exact solutions, solitary waves, invariants of motion.

1 Introduction

The well-known Korteweg-de Vries (KdV) equation, $u_t + uu_x + u_{xxx} = 0$, is a nonlinear partial differential equation (PDE) that models the time-dependent wave motion in one space dimension in media with nonlinear wave steepening and dispersion, such as shallow water waves and ion acoustic plasma waves. The pioneering study by Korteweg and

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de Vries [9] in 1895 showed that when nonlinear wave steepening, from the term uu_x, is balanced by wave dispersion, owing to the term u_{xxx}, their equation predicts a unidirectional solitary wave, that is a pulse which moves in one direction with a permanent shape and constant speed. Benjamin, Bona and Mahoney [2] advocated that the PDE $u_t + uu_x + u_x - \mu u_{xxt} = 0$ modeled the same physical phenomena equally well as the KdV equation, given the same assumptions and approximations that were originally used by Korteweg and de Vries [9]. This PDE of Benjamin et al. [2] is now often called the *regularized* long wave (RLW) equation, although it is also known as the BBM equation. Morrison et al. [10] proposed the one-dimensional PDE, $u_t+uu_x-\mu u_{xxt} = 0$, as an equally valid and accurate model for the same wave phenomena simulated by the KdV and RLW equations. This PDE is called the equal width (EW) equation because the solutions for solitary waves with a permanent form and speed, for a given value of the parameter μ, are waves with an equal width or wavelength for all wave amplitudes. The EW equation is a simpler and lesser known alternative to the RLW equation and can be easily transformed into the RLW equation by means of a simple change of the dependent variable $u \to u+1$, and therefore both equations have very similar analytical solitary wave solutions.

In this paper we present exact solitary wave solutions for the EW equation in its general form with nonlinear terms of any order p,

$$u_t + au^p u_x - \mu u_{xxt} = 0 \ , \tag{1}$$

and the generalized EW-Burgers equation with nonlinear terms of any order and dissipative effects modeled by the term δu_{xx},

$$u_t + au^p u_x - \delta u_{xx} - \mu u_{xxt} = 0 \ . \tag{2}$$

The solutions for (1) are obtained by integrating a first order nonlinear ODE with symbolic computation using the mathematical software `Maple` [1,3]. Solutions for (2) are difficult to derive by integrating directly the associated first order ODE. We used an approach devised by Zhang et al. [13] which is based on assuming a certain form of the solitary wave solution. The solutions are obtained by integrating a second order nonlinear ODE using a technique similar to that used by Kichennassamy and Olver [8] to transform a nonlinear ordinary differential equation to a homogeneous polynomial that can be solved using the mathematical software `Maple`. The approach presented in this work is general and can also be applied for finding exact solitary wave solutions for the RLW equation and other nonlinear wave equations such as Boussinesq-like equations (Hamdi et al. [7]). The exact solitary wave solutions can be used to specify initial data for the incident waves in the numerical model and for the verification of the associated computed solution [6]. We also derive analytical expressions for three invariants of motion which can be used as verification tools to investigate the conservation properties of the numerical method.

2 Derivation of the Exact Solutions

We concentrate on finding an exact solitary wave solution of the form

$$u(x,t) = u(x - x_0 - Ct) \ . \tag{3}$$

This corresponds to a traveling-wave propagating with steady celerity C. We are interested in solutions depending only on the moving coordinate $\xi = x - x_0 - Ct$ as,

$$u(x,t) = u(x - x_0 - Ct) \equiv u(\xi)\ . \tag{4}$$

Substituting into (2), the function $u(\xi)$ satisfies a third order nonlinear ordinary differential equation (ODE),

$$-Cu' + au^p u' - \delta u'' + \mu C u''' = 0\ , \tag{5}$$

where the derivatives are performed with respect to the coordinate ξ.

Integrating once, we obtain

$$-Cu + \frac{a}{p+1}u^{p+1} - \delta u' + \mu C u'' = k_1\ . \tag{6}$$

where k_1 is a constant of integration. If we assume that the solitary wave solution and its derivatives have the following asymptotic values,

$$u(\xi) \longrightarrow u_\pm \ \text{ as }\ \xi \longrightarrow \pm\infty\ , \tag{7}$$

and for $n \geq 1$

$$u^{(n)}(\xi) \longrightarrow 0 \ \text{ as }\ \xi \longrightarrow \pm\infty\ , \tag{8}$$

where the superscript denotes differentiation to the order n, with respect to ξ, and if we also assume that $u_\pm$ satisfies the following algebraic equation

$$-Cu_\pm + \frac{a}{p+1}u_\pm^{p+1} = 0\ , \tag{9}$$

then the constant of integration k_1 is equal to zero and (6) reduces to

$$-Cu + \frac{a}{p+1}u^{p+1} - \delta u' + \mu C u'' = 0\ . \tag{10}$$

From (9) we also have the relation

$$\frac{a}{p+1}\left(u_+^{p+2} - u_-^{p+2}\right) = C\left(u_+^2 - u_-^2\right)\ . \tag{11}$$

To allow another integration, we first multiply (10) by $2u'$. Then each term can be integrated separately to obtain,

$$-Cu^2 + \frac{2a}{(p+1)(p+2)}u^{p+2} - 2\delta\int_{-\infty}^{\xi}(u')^2\,\mathrm{d}\xi + \mu C(u')^2 = k_2\ , \tag{12}$$

where k_2 is a second constant of integration.

Using the asymptotic boundary conditions (7) and (8) at infinity we have

$$-Cu_+^2 + \frac{2a}{(p+1)(p+2)}u_+^{p+2} - 2\delta\int_{-\infty}^{+\infty}(u')^2\,\mathrm{d}\xi = k_2,\ \text{ as }\ \xi \longrightarrow +\infty\ , \tag{13}$$

and

$$-Cu_-^2 + \frac{2a}{(p+1)(p+2)} u_-^{p+2} = k_2, \text{ as } \xi \longrightarrow -\infty \, . \tag{14}$$

By substituting (14) into (13) and using the relation (11), we obtain the analytical expression of the following important square integral,

$$\int_{-\infty}^{+\infty} (u')^2 \, \mathrm{d}\xi = \frac{C}{2\delta} \frac{p}{p+2} \left(u_-^2 - u_+^2 \right) \, . \tag{15}$$

In constructing solitary wave solutions for (1) and (2) with the assumptions (7), (8) and (9) the following important results should be noted.

(1) The integral $\int_{-\infty}^{+\infty} (u')^2 \, \mathrm{d}\xi$ is well defined and the function $u'(\xi)$ is square integrable over the domain $[-\infty, +\infty]$.
(2) The generalized EW equation (1) with no dissipation ($\delta = 0$) has bell-profile solitary wave solutions, which satisfy the condition $|u_-| = |u_+|$.
(3) The generalized EW-Burgers equation (2) with dissipative effects ($\delta > 0$) has only kink-profile solitary wave solutions satisfying $|u_-| \neq |u_+|$. It does not have bell-profile solitary wave solutions. According to (15) solitary waves solutions exist if and only if $|u_-| > |u_+|$ because $\int_{-\infty}^{+\infty} (u')^2 \, \mathrm{d}\xi > 0$. The trivial case $|u_-| = |u_+|$ does not lead to solitary wave solutions because the solution is constant ($u'(\xi) = 0$) since $\int_{-\infty}^{+\infty} (u')^2 \, \mathrm{d}\xi = 0$. The EW-Burgers equation (2) with ($\delta > 0$) has no solutions if $|u_-| < |u_+|$.

These theoretical results and soliton theory [4,12] suggest to seek bell shaped and kink shaped solitary wave solutions, for the generalized EW equation and the generalized EW-Burgers equation, that can be expressed as polynomials of the hyperbolic secant and tangent functions, respectively.

2.1 Solitary Wave Solutions of the Generalized EW Equation

Setting $\delta = 0$ (no dissipation) in (12) leads to the following nonlinear ODE,

$$-Cu^2 + \frac{2a}{(p+1)(p+2)} u^{p+2} + \mu C (u')^2 = k_2 \, . \tag{16}$$

If we also assume that $|u_-| = |u_+| = 0$ then the second constant of integration k_2 is equal to zero. There are several analytical techniques for integrating this type of nonlinear ODEs [7]. These ODEs can also be solved using symbolic computation. First we make the following change of the dependent variable in (16)

$$u(\xi) = \phi^{1/p}(\xi) \tag{17}$$

in order to reduce the power of the nonlinear term u^{p+2} in (16) . Using the above transformation we obtain,

$$-C\phi^2 + \frac{2a}{(p+1)(p+2)} \phi^3 + \frac{\mu C}{p^2} (\phi')^2 = 0 \, . \tag{18}$$

Let $\alpha = C, \beta = \dfrac{2a}{(p+1)(p+2)}$ and $\gamma = \dfrac{\mu C}{p^2}$, then we have

$$-\alpha\phi^2 + \beta\phi^3 + \gamma(\phi')^2 = 0 \; , \tag{19}$$

which can be solved using the following script of `Maple` software

```
> ode:=-alpha*phi(xi)^2+beta*phi(xi)^3+gamma*diff(phi(xi),xi)^2;
> dsolve(ode);
```

which returns the following exact solution

$$\phi(\xi) = \frac{\alpha}{\beta}\left[1 - \tanh^2\left(\frac{1}{2}\sqrt{\frac{\alpha}{\gamma}}\,(\xi - x_0)\right)\right] = \frac{4\alpha}{\beta}\frac{e^{\sqrt{\frac{\alpha}{\gamma}}(\xi - x_0)}}{\left[e^{\sqrt{\frac{\alpha}{\gamma}}(\xi - x_0)} + 1\right]^2} \; , \tag{20}$$

in which x_0 is a constant of integration. As $\mathrm{sech}^2 = 1 - \tanh^2$, it follows that

$$\phi(\xi) = \frac{\alpha}{\beta}\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{\alpha}{\gamma}}\,(\xi - x_0)\right) . \tag{21}$$

After substitutions ($\xi = x - C\,t$, $\alpha = C, \beta = 2a/(p+1)(p+2)$, $\gamma = \mu C/p^2$ and $u(\xi) = \phi^{1/p}(\xi)$), we obtain an explicit expression of the exact solution for the generalized EW equation:

$$u(x,t) = \left[\frac{(p+1)(p+2)}{2a}\,C\,\mathrm{sech}^2\left(\frac{1}{2}p\sqrt{\frac{1}{\mu}}\,(x - C\,t - x_0)\right)\right]^{\dfrac{1}{p}} . \tag{22}$$

This analytical solution is a bell-profile solitary wave. It is a single pulse of amplitude $[C(p+1)(p+2)/2a]^{1/p}$ initially centered at x_0 and traveling without change of shape at a steady celerity C and a wave number $\frac{1}{2}p\sqrt{1/\mu}$. The wave number also characterizes the width of the solitary wave. Unlike solitary waves of the KdV, the wave number of the EW equation is independent of the wave amplitude and the wave celerity. For a given value of the parameter μ all solitary waves of the EW equation have an equal width for all wave amplitudes and wave celerities. This is the special feature for which the Equal Width equation was named.

2.2 Solitary Wave Solutions of the Generalized EW-Burgers Equation

The dissipation effects in the generalized EW-Burgers equation are modeled by the term δu_{xx}. Solutions based on the integration of the first order ODE (12) with $\delta > 0$ are difficult to obtain because the square integral $\int_{-\infty}^{\xi}(u')^2\,\mathrm{d}\xi$ cannot be expressed in explicit form. The solutions will be obtained using the second order ODE (10) instead, and by assuming a certain form for the solitary wave to transform this nonlinear ODE to a fourth order homogeneous polynomial that can be solved using the mathematical software `Maple`. This approach is very similar to the technique used by Kichennassamy and Olver [8] and to `tanh`-based methods for transforming nonlinear ODEs to homogeneous polynomials.

First we choose a particular change for the dependent variable $u(\xi) = \phi^{2/p}(\xi)$ to reduce the power of the nonlinear term u^{p+1} in (10). The change of variable $u(\xi) = \phi^{1/p}(\xi)$ used previously for the derivation of exact solutions for the EW equation leads to a nonlinear ODE that we could not solve analytically. After transformations and simplifications we obtain

$$-C\phi^2 + \frac{a}{(p+1)}\phi^4 - \frac{2\delta}{p}\phi\phi' + \frac{2\mu\, C}{p}\phi\phi'' + \frac{2\mu\, C(2-p)}{p^2}(\phi')^2 = 0\ . \tag{23}$$

The integration of the above ODE is not straightforward because all terms are nonlinear.

Since we are seeking a kink-profile solitary wave solution, which satisfies the condition $|u_-| > |u_+|$ and the assumptions (7), (8) and (9), we will assume that such solution can be expressed as a polynomial of the hyperbolic tangent function,

$$\phi(\xi) = \frac{\phi_0}{2}\left[1 - \tanh\left(\,\kappa\,(\xi - x_0)\right)\right] = \frac{\phi_0}{e^{2\,\kappa\,(\xi - x_0)} + 1}\ . \tag{24}$$

This particular form corresponds to the initial profile of a shock wave or a shallow water bore which can be modeled by the equal width wave equation (see Hamdi et al. [6]).

The hyperbolic tangent function has an interesting recursive property: The derivative of `tanh` is a polynomial of `tanh`, i.e. $\texttt{tanh}' = 1 - \texttt{tanh}^2$. Consequently, all derivatives of `tanh` are also polynomials of `tanh`. Using this recursive property of the function `tanh`, we can express the derivative terms ϕ', ϕ'' and $(\phi')^2$ which appear in ODE (23) as homogeneous polynomials of the dependent variable ϕ only, that is,

$$\phi' = -2\kappa\phi + 2\frac{\kappa}{\phi_0}\phi^2\ , \tag{25}$$

$$\phi'' = 4\kappa^2\phi - 12\frac{\kappa^2}{\phi_0}\phi^2 - 8\frac{\kappa^2}{\phi_0^2}\phi^3\ , \tag{26}$$

$$(\phi')^2 = 4\kappa^2\phi^2 - 8\frac{\kappa^2}{\phi_0}\phi^3 + 4\frac{\kappa^2}{\phi_0^2}\phi^4\ . \tag{27}$$

Now substituting (25), (26) and (27) into (23) leads to a quartic homogeneous polynomial in u on its left-hand side,

$$\begin{aligned}
&\left(-C + \frac{4\,\delta\,\kappa}{p} + \frac{8\,\mu\,C\,(2-p)\,\kappa^2}{p^2} + \frac{8\,\mu\,C\,\kappa^2}{p}\right)\phi^2 \\
&\quad + \left(-24\,\frac{\mu\,C\,\kappa^2}{p\,\phi_0} - \frac{4\,\delta\,\kappa}{p\,\phi_0} - \frac{16\,\mu\,C\,(2-p)\,\kappa^2}{p^2\,\phi_0}\right)\phi^3 \\
&\qquad + \left(\frac{a}{p+1} + \frac{16\,\mu\,C\,\kappa^2}{p\,\phi_0^2} + \frac{8\,\mu\,C\,(2-p)\,\kappa^2}{p^2\,\phi_0^2}\right)\phi^4\ .
\end{aligned} \tag{28}$$

It follows from equation (28) that all the coefficients of this polynomial must be zero in order to obtain a nontrivial solution ϕ. Setting the coefficients to zero yields a nonlinear algebraic system of three equations for the three unknowns C, κ and ϕ_0,

$$\begin{cases} -C + \dfrac{4\,\delta\,\kappa}{p} + \dfrac{8\,\mu\,C\,(2-p)\,\kappa^2}{p^2} + \dfrac{8\,\mu\,C\,\kappa^2}{p} = 0\,, \\[2ex] -24\,\dfrac{\mu\,C\,\kappa^2}{p\,\phi_0} - \dfrac{4\,\delta\,\kappa}{p\,\phi_0} - \dfrac{16\,\mu\,C\,(2-p)\,\kappa^2}{p^2\,\phi_0} = 0\,, \\[2ex] \dfrac{a}{p+1} + \dfrac{16\,\mu\,C\,\kappa^2}{p\,\phi_0^2} + \dfrac{8\,\mu\,C\,(2-p)\,\kappa^2}{p^2\,\phi_0^2} = 0\,. \end{cases} \tag{29}$$

The analytical solution of the system (29) can be obtained using `Maple` software.

If we consider the generalized EW-Burgers equation $u_t + au^p u_x \pm \delta u_{xx} \pm \mu u_{xxt} = 0$, with arbitrary signs for the dissipative and dispersive parameters then the solution of a more general form of the system (29) provides the following exact expressions for the celerity C, wave number κ and wave amplitude ϕ_0.

$$C = \pm\,\frac{2}{\mu\,(p+4)}\,\frac{\delta}{\sqrt{\pm\lambda}}\,, \tag{30}$$

$$\kappa = \pm\,\frac{p}{4}\,\sqrt{\pm\lambda}\,, \tag{31}$$

$$\phi_0 = \pm 2\,\sqrt{\pm\frac{(\,p^2+3\,p+2\,)\,\delta\,\sqrt{\pm\lambda}}{4\,a\,\,(p+4)}}\,, \tag{32}$$

where

$$\lambda = \frac{2}{\mu\,(p+2)}\,. \tag{33}$$

From (24) and the relation ($u(\xi) = \phi^{2/p}(\xi)$) we finally deduce the explicit expressions for the exact solution

$$\phi(\xi) = \left\{\frac{\phi_0}{2}\,\left[1 - \tanh\left(\,\kappa\,(\xi - x_0)\right)\right]\right\}^{2/p} = \left[\frac{\phi_0}{e^{2\,\kappa\,(\xi - x_0)}+1}\right]^{2/p}\,. \tag{34}$$

Using the identity $\mathrm{sech}^2 = 1 - \tanh^2$, the solution can also be written as

$$u(x,t) = \left\{\phi_0\,\left[1 - \tanh\left(\,\kappa\,(\xi - C\,t - x_0)\right) - \frac{1}{2}\mathrm{sech}^2\left(\,\kappa\,(\xi - C\,t - x_0)\right)\right]\right\}^{1/p}\,. \tag{35}$$

This monotonic function is a kink-profile solitary wave. Similar profiles for modeling the propagation of undular bores were studied numerically by Hamdi et al. [6] using the EW equation. It is easy to verify that the solution (35) satisfies (2) and also satisfies the assumptions (7), (8) and (9).

3 Invariants of motion of solitary wave solutions

Olver [11] has shown that solutions of the EW equation ($u_t + uu_x - \mu u_{xxt} = 0$) have only three non-trivial conservation laws that can be written in the general form $T_t + X_x = 0$. These laws are the equivalents of the conservation of mass, momentum and energy in fluid mechanics. Olver showed that the three laws lead directly to three so-called invariants of motion given by

$$C_1 = \int_{-\infty}^{+\infty} u \, \mathrm{d}x, \quad C_2 = \int_{-\infty}^{+\infty} \left(u^2 + \mu u_x u_x\right) \mathrm{d}x, \quad C_3 = \int_{-\infty}^{+\infty} u^3 \, \mathrm{d}x, \tag{36}$$

provided that the integrals converge. These invariants of motion for the EW equation need to be extended for the general case $u_t + au^p u_x - \mu u_{xxt} = 0$ and also generalized for the case of a finite spatial domain with asymptotic boundary conditions (7) and (8). This is done by multiplying the generalized EW equation, $u_t + au^p u_x - \mu u_{xxt} = 0$, by 1, u and $((2a/p+1)\, u^{p+1} - 2\mu u_{xt})$, and then the resulting three equations can each be expressed in the form $T_t + X_x = 0$ as

$$(u)_t + \left(\frac{a}{(p+1)} u^{p+1} - \mu u_{xt}\right)_x = 0\,,$$

$$\left(u^2 + \mu u_x u_x\right)_t + \left(2\frac{a}{(p+2)} u^{p+2} - 2\mu u u_{xt}\right)_x = 0\,,$$

$$\left(\frac{2\, a\, u^{p+2}}{(p+1)(p+2)}\right)_t + \left(\frac{a^2\, u^{2(p+1)}}{(p+1)^2} - \mu u_t u_t - \frac{2\mu\, a\, u^{p+1} u_{xt}}{(p+1)} + \mu^2 u_{xt} u_{xt}\right)_x = 0\,.$$

These three conservation laws can now be integrated easily with respect to x over a large but finite spatial domain $[x_L, x_U]$ instead of $[-\infty, +\infty]$ to obtain the intermediate results

$$\begin{aligned} \frac{\partial}{\partial t}\int_{x_L}^{x_U} u \, \mathrm{d}x + \frac{a}{(p+1)}\left(u_U^{(p+1)} - u_L^{(p+1)}\right) &= 0\,, \\ \frac{\partial}{\partial t}\int_{x_L}^{x_U} \left[u^2 + \mu u_x u_x\right] \mathrm{d}x + \frac{2\, a}{(p+2)}\left(u_U^{(p+2)} - u_L^{(p+2)}\right) &= 0\,, \\ \frac{\partial}{\partial t}\int_{x_L}^{x_U} \left(\frac{2\, a}{(p+2)}\, u^{p+2}\right) \mathrm{d}x + \frac{a^2}{(p+1)}\left(u_U^{2(p+1)} - u_L^{2(p+1)}\right) &= 0\,, \end{aligned} \tag{40}$$

after simplifications. In these equations, $u_L = u(x_L, t)$ and $u_U = u(x_U, t)$ are time-invariant constants at the domain boundaries. In the simplifications, the terms $[u_{xt}]_{x_L}^{x_U}$, $[uu_{xt}]_{x_L}^{x_U}$, $[u_t u_t]_{x_L}^{x_U}$, $[u^2 u_{xt}]_{x_L}^{x_U}$ and $[u_{xt} u_{xt}]_{x_L}^{x_U}$ are zero at the boundaries (assumptions (7) and (8)). Equation (40) can now be integrated with respect to t to yield

$$\begin{aligned} C_1 &= \int_{x_L}^{x_U} u \, \mathrm{d}x + \frac{a}{(p+1)}\left(u_U^{(p+1)} - u_L^{(p+1)}\right) t\,, \\ C_2 &= \int_{x_L}^{x_U} \left[u^2 + \mu u_x u_x\right] \mathrm{d}x + \frac{2\, a}{(p+2)}\left(u_U^{(p+2)} - u_L^{(p+2)}\right) t\,, \\ C_3 &= \int_{x_L}^{x_U} \left(\frac{2\, a}{(p+2)}\, u^{p+2}\right) \mathrm{d}x + \frac{a^2}{(p+1)}\left(u_U^{2(p+1)} - u_L^{2(p+1)}\right) t\,. \end{aligned} \tag{41}$$

These invariants of motion are generalizations of those given by equation (36), extended for the case of a large but finite length spatial domain when the solution of $u(x,t)$ of the generalized EW equation is constant but not necessarily zero at the domain boundaries (assumptions (7) and (8)). The extra terms stem directly from the convection of mass, momentum and energy into and out of the lower and upper boundaries of the spatial domain. During the whole period of time in the course of which the waves propagate inside the domain $[x_L, x_U]$, these invariants of motion remain conserved and equal to their original values that are well determined initially at $t = 0$. Note that during numerical computations that provide solutions to the generalized EW equation, C_1, C_2 and C_3 can be calculated after each successive time step over the entire spatial domain $x_L \leq x \leq x_U$ that contains the wave motion, such that the conservation properties of the numerical algorithm can be monitored and thereby assessed. This approach for the derivation of these invariants of motion (41) is general and can be easily applied for finding similar invariants of motion for other general types of regularized long wave (RLW) equations and generalized EW-Burgers equation.

In this section we will derive analytic expressions for the three invariants of motion corresponding to the solitary wave solutions of the generalized EW equation $u_t + au^p u_x - \mu u_{xxt} = 0$. For simplicity, the invariants are given for $p = 1$. However, the evaluation of these invariants for any given value of the order p ($p \geq 1$) is straightforward (simple change of the value of p in the script of `Maple` software). These invariants, also called constants of motion, are independent of time and hence have the same value at any time T for solitary waves initially located at any position x_0. Therefore, these invariants will be evaluated at the initial state $T = 0$. We also suppose that the solitary wave is initially located at $x_0 = 0$ and $[x_L, x_U] = [-L, L]$.

The first invariant of motion corresponds to the conservation of mass,

$$I_1 = \int_{-\infty}^{+\infty} u \, \mathrm{d}x \quad . \tag{42}$$

The conservation of mass associated with the spatial domain $[-L, L]$ is given by

$$I_1(L) = \int_{-L}^{+L} u \, \mathrm{d}x \quad . \tag{43}$$

Integrating (22) using `Maple` software we have,

```
> u:=(((p+1)*(p+2)/(2*a))*C* \
     (sech((1/2)*p*sqrt(1/mu)*(x-X0-C*T)))^2)^(1/p):
> p:=1:
> X0:=0:
> T:=0:
> I_1:=int(u,x=-L..L):
```

$$I_1(L) = \int_{-L}^{L} u(x,t) \, \mathrm{d}x = 12 \, \frac{\sinh(\frac{1}{2}\sqrt{\frac{1}{\mu}}\,L)}{a\sqrt{\frac{1}{\mu}}\cosh(\frac{1}{2}\sqrt{\frac{1}{\mu}}\,L)} \quad . \tag{44}$$

The conservation of mass over the whole domain $[-\infty, +\infty]$ is then given by,

```
> simplify(limit(I_1, L=infinity)):
```

$$\int_{-\infty}^{\infty} u(x,t)\,\mathrm{d}x = \lim_{L\to+\infty} I_1(L) = 12\,\frac{\sqrt{C}\,\sqrt{\mu}}{a} \quad . \tag{45}$$

The second invariant of motion represents the conservation of energy,

$$I_2 = \int_{-\infty}^{+\infty} \left(u^2 + \mu u_x u_x\right)\,\mathrm{d}x \quad . \tag{46}$$

The conservation of energy over the spatial domain $[-L, L]$ is given by

$$I_2(L) = \int_{-L}^{L} \left(u^2 + \mu u_x u_x\right)\,\mathrm{d}x. \tag{47}$$

Using `Maple` software, we obtain,

```
> I_2:=int(u^2 + mu*(u_x)^2,x=-L..L):
```

$$I_2(L) = \frac{36}{5}\,\frac{C^2 \sinh(\frac{1}{2}\sqrt{\frac{1}{\mu}}\,L)\,(2\cosh(\frac{1}{2}\sqrt{\frac{1}{\mu}}\,L)^2 + 4\cosh(\frac{1}{2}\sqrt{\frac{1}{\mu}}\,L)^4 - 1)}{a^2\sqrt{\frac{1}{\mu}}\cosh(\frac{1}{2}\sqrt{\frac{1}{\mu}}\,L)^5}\,, \tag{48}$$

The conservation of energy over the whole domain $[-\infty, +\infty]$ is then given by

```
> simplify(limit(I_2, L=infinity)):
```

$$\int_{-\infty}^{+\infty} \left(u^2 + \mu u_x u_x\right)\,\mathrm{d}x \;\; = \lim_{L\to+\infty} I_2(L) = \frac{144}{5}\,\frac{C^2\sqrt{\mu}}{a^2} \quad . \tag{49}$$

The third invariant, which is similar to the invariant discovered by Whitham [12], is given by

$$I_3 = \int_{-\infty}^{+\infty} u^3\,\mathrm{d}x \quad . \tag{50}$$

This invariant considered over the spatial domain $[-L, L]$ reduces to

$$I_3(L) = \int_{-L}^{+L} u^3\,\mathrm{d}x \quad . \tag{51}$$

Using `Maple` software, we obtain,

```
> I_3:=int(u^3,x=-L..L):
```

$$I_3(L) = \frac{36}{5}\,\frac{C^3 \sinh(\frac{1}{2}\sqrt{\frac{1}{\mu}}\,L)\,(3 + 4\cosh(\frac{1}{2}\sqrt{\frac{1}{\mu}}\,L)^2 + 8\cosh(\frac{1}{2}\sqrt{\frac{1}{\mu}}\,L)^4)}{a^3\sqrt{\frac{1}{\mu}}\cosh(\frac{1}{2}\sqrt{\frac{1}{\mu}}\,L)^5} \tag{52}$$

The third invariant evaluated over the whole domain $[-\infty, +\infty]$ is then given by

```
> simplify(limit(I_3, L=infinity)):
```

$$\int_{-\infty}^{+\infty} u^3 \,\mathrm{d}x = \lim_{L\to+\infty} I_3(L) = \frac{288}{5}\frac{C^3\sqrt{\mu}}{a^3} \quad .$$

4 Conclusion

New exact solitary wave solutions are derived analytically for the first time for general types of the RLW equations such as the generalized EW equation and the generalized EW-Burgers equation which includes dissipative effects that are modeled by a generalized Burgers term. The analytical solutions for these evolutionary partial differential equations are obtained for any order of the nonlinear terms and for any given value of the coefficients of the nonlinear, dispersive and dissipative terms. New analytical expressions for three invariants of motions were also devised. The accuracy and stability of numerical schemes for the solution of these general model equations can be assessed using, as test problems, the new exact solutions and their conservation properties can also be verified using the analytical expressions of the constants of motion. These verification tools for these evolutionary partial differential equations are implemented in a method of lines solver that is available from the authors [6,7]. The methods, that we devised in this work, for finding exact solitary wave solutions, are general and can be used for a wide class of nonlinear dispersive wave equations, such as general types of regularized long wave equations and Boussinesq-like system of equations.

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