Exact solutions of extended Boussinesq equations *

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Abstract. The problem addressed in this paper is the verification of numerical solutions of nonlinear dispersive wave equations such as Boussinesq-like system of equations. A practical verification tool is to compare the numerical solution to an exact solution if available. In this work, we derive some exact solitary wave solutions and several invariants of motion for a wide range of Boussinesq-like equations using Maple software. The exact solitary wave solutions can be used to specify initial data for the incident waves in the Boussinesq numerical model and for the verification of the associated computed solution. When exact solutions are not available, four invariants of motions can be used as verification tools for the conservation properties of the numerical model.

Keywords: verification tools, exact solutions, Bousinesq equations, Invariants of motion

1. Introduction

In recent years, efforts have been made by a number of researchers to extend the range of applicability of Boussinesq equations to deeper water by improving their dispersion characteristics (see Madsen et al. [5], Nwogu [6], and Beji et al. [1]).

Although most extended Boussinesq systems of equations have equivalent linearised dispersion characteristics, similar shoaling properties and formally the same accuracy, the extended Boussinesq equations proposed by Nwogu [6] have recently generated the most interest because they are easier to solve numerically in the case of variable depth [8]. Several numerical schemes have been proposed to solve the Nwogu's extended Boussinesq equations. Recently, Wei and Kirby [9] developed a numerical code, that is fourth-order accurate in time and space, for solving the Nwogu's extended Boussinesq equations. The numerical solutions of Wei and Kirby [9] are more accurate than those obtained by Nwogu [6] since their solution technique is based on a higher order finite difference discretisation scheme coupled with a high-order

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predictor-corrector time integration method. More recently, Walkley and Berzins [8] implemented a high order accurate method of lines solution using a Galerkin finite element spatial discretisation technique coupled with the adaptive time integration package SPRINT [8]

Both, Wei and Kirby [9] and Walkley and Berzins [8] investigated solitary wave propagation over a long and flat bottom in order to assess the accuracy, the stability and the conservation properties of their numerical schemes. To apply this important test problem, Wei and Kirby [9] derived approximate analytical solitary wave solutions following a procedure described in Schember [7]. The approximate analytical solutions are used to specify initial data for the incident waves in their numerical models and also to assess the accuracy of the numerical solution. The numerical results of Wei and Kirby [9] indicate that a slightly higher amplitude solitary wave is formed together with a small dispersive tail lagging behind, compared to the approximate analytical solution [7, 9]. The wave profiles show that the amplitude of the tail and the initial deviation in solitary-wave height both increase with increasing initial wave height. They also observed that the numerically predicted phase speed is somewhat smaller than the analytically predicted one, and that the difference increases with increasing wave height. Such discrepancies are explained by the fact that the analytical solution is only an approximation and does not correspond exactly to a solitary waveform as predicted by the model.

Walkley and Berzins [8] reported very similar results to those of Wei and Kirby [9] for the same solitary wave test problem. They have also observed that there is an identical slight phase error in the numerical results and a small dispersive tail. They concluded that the approximate analytical solution is only an approximation and therefore exact agreement is unlikely.

In this paper, we derive exact solitary wave solutions for the extended Nwogu Boussinesq equations. The exact solutions are obtained following an approach devised recently by Chen [2], which is more general than the procedure used in [7, 9], that leads to approximate solutions only. To overcome the problems reported by Wei et al. [9] and Walkley et al. [8], it is recommended to use the exact solitary wave solutions that we propose in this study, instead of using approximate solutions which are not accurate enough for testing high-order accurate schemes such as those given in [8, 9]. New analytical expressions of four invariants of motions (mass, momentum, energy and Hamiltonian) are also derived. These constants of motion can be used to assess the accuracy and the conservative properties of numerical schemes for Nwogu's Boussinesq models.

2. The one-dimensional extended Bousssinesq equations

In the case of wave propagation in the one-dimensional (1D) horizontal direction with constant depth, the extended Boussinesq equations derived by Nwogu [6] and used by Wei et al. [9] in their numerical code, reduce to the following:

$$\eta_t + hu_x + (\eta u)_x + (\alpha + 1/3)h^3 u_{xxx} = 0\,, \tag{1a}$$

$$u_t + g\eta_x + uu_x + \alpha h^2 u_{txx} = 0\,, \tag{1b}$$

with

$$\alpha = \frac{1}{2}\left(\frac{z_\alpha}{h}\right)^2 + \frac{z_\alpha}{h}\,, \tag{2}$$

where η = surface elevation; h = local water depth; $u = u(x,t)$ horizontal velocity at an arbitrary depth z_α ; and g = the gravitational acceleration. These equations are statements of conservation of mass and momentum, respectively.

Two important length scales are the characteristic water depth h_0 in the vertical direction and a typical wavelength l in the horizontal direction. The following independent, non-dimensional variables can be defined:

$$x = \frac{\tilde{x}}{l}, \qquad z = \frac{\tilde{z}}{h_0}, \qquad t = \frac{\sqrt{gh_0}}{l}\,\tilde{t}\,. \tag{3}$$

The tildes are used to connote dimensional variables as in the set of equations (1a) and (1b). For effects related to the motion of the free water surface, the typical wave amplitude a_0 is also important. The following dependent, non-dimensional variables can also be defined:

$$u = \frac{h_0}{a_0\sqrt{gh_0}}\,\tilde{u}, \qquad \eta = \frac{\tilde{\eta}}{a_0}, \qquad h = \frac{\tilde{h}}{h_0}\,. \tag{4}$$

Using the transformation (3) and (4), the Nwogu's set of equations (1a) and (1b) are rewritten in dimensionless form as follows

$$\eta_t + u_x + \delta(\eta u)_x + \mu^2(\alpha + 1/3)u_{xxx} = 0, \tag{5a}$$

$$u_t + \eta_x + \delta u u_x + \mu^2 \alpha u_{txx} = 0\,. \tag{5b}$$

The dimensionless parameters $\delta = a_0/h_0$ and $\mu = h_0/l$ are measures of nonlinearity and frequency dispersion, respectively, and are assumed to be small. The parameter α reduces to

$$\alpha = \frac{1}{2}(z_\alpha)^2 + z_\alpha\ . \tag{6}$$

3. Exact solitary wave solutions

In this section, exact solitary wave solutions are derived following an approach devised recently by Chen [2] and using `MAPLE` software. We concentrate on finding exact solitary wave solutions of the form

$$u(x,t) = u(x - x_0 - Ct)\,. \tag{7}$$

This corresponds to traveling-waves initially centered at x_0, propagating with steady velocity (or celerity) C.

We are interested in solutions depending only on the moving coordinate $\xi = x - x_0 - Ct$ as

$$u(x,t) = u(x - x_0 - Ct) \equiv u(\xi)\,, \tag{8a}$$

and,

$$\eta(x,t) = \eta(x - x_0 - Ct) \equiv \eta(\xi)\,. \tag{8b}$$

Substituting into (5a) and (5b), the functions $\eta(\xi)$ and $u(\xi)$ satisfy the third order nonlinear system of ordinary differential equations (ODEs)

$$-C\eta' + u' + \delta(\eta u)' + \mu^2(\alpha + 1/3)u''' = 0\,, \tag{9a}$$

$$-Cu' + \eta' + \delta u u' - \mu^2 \alpha C u''' = 0\,, \tag{9b}$$

in which the derivatives are performed with respect to the coordinate ξ. The solitary wave solutions are localized in space, i.e., the solution and its derivatives at large distance from the pulse are extremely small and vanish asymptotically

$$(\eta^n(\xi), u^n(\xi)) \longrightarrow 0 \text{ as } \xi \longrightarrow \pm\infty\,. \tag{10}$$

Integrating once, with zero boundary conditions at infinity,

$$(-C + \delta u)\eta + u + \mu^2(\alpha + 1/3)u'' = 0\,, \tag{11a}$$

$$-Cu + \eta + \frac{1}{2}\delta u^2 - \mu^2 \alpha C u'' = 0\,. \tag{11b}$$

3.1. METHOD I

This method consists of seeking functions $\eta(\xi)$ and $u(\xi)$ that are proportional

$$u(\xi) = A\,\eta(\xi)\,. \tag{12}$$

Using the relation (12) and multiplying equation (11a) by A, it follows that

$$\left(CA - A^2\right)\eta - \mu^2(\alpha + 1/3)A^2\eta'' = \delta A^2 \eta^2\,, \tag{13a}$$

$$(2CA - 2)\,\eta + 2\mu^2 \alpha CA\,\eta'' = \delta A^2 \eta^2\,. \tag{13b}$$

This system has nontrivial solitary-wave solutions if the equations (13a) and (13b) are identical. This implies that

$$CA + A^2 - 2 = 0, \tag{14a}$$

$$2\mu^2 \alpha CA + \mu^2(\alpha + 1/3)A^2 = 0, \tag{14b}$$

which is a linear system with respect to the unknowns CA and A^2.

For all optimized values of the parameter α considered by Nwogu [6], it is apparent that we always have $\alpha - 1/3 \neq 0$. It follows that the above system will be nonsingular for the values of α of interest in Nwogu's equations. The unique solution of the system is given by

$$A^2 = 4\frac{\alpha}{\alpha - 1/3}\,, \tag{15a}$$

$$CA = -2\frac{\alpha + 1/3}{\alpha - 1/3}\,. \tag{15b}$$

Note that A^2 is well defined since $\alpha/(\alpha-1/3)$ is always strictly positive for all values of interest of the parameter α.

Substituting (14a) into (13a), we readily find that the function $\eta(\xi)$ satisfies a second order nonlinear ODE,

$$2\left(1-A^2\right)\eta-\mu^2(\alpha+1/3)A^2\eta''=\delta A^2\eta^2\,. \tag{16}$$

To allow another integration, we first multiply by η'. Then each term can be integrated separately to obtain,

$$\left(1-A^2\right)\eta^2-\frac{1}{2}\mu^2(\alpha+1/3)A^2(\eta')^2=\frac{1}{3}\delta A^2\eta^3\,, \tag{17}$$

which can be written as the separable ODE.

$$\frac{\mathrm{d}\eta}{\left[\dfrac{\left(1-A^2\right)}{\frac{1}{2}\mu^2(\alpha+1/3)A^2}\eta^2-\dfrac{\delta A^2}{\frac{3}{2}\mu^2(\alpha+1/3)A^2}\eta^3\right]^{1/2}}=\mathrm{d}\xi\,. \tag{18}$$

It is important to note that all the values proposed by Nwogu for the parameter α satisfy the inequality

$$\left(1-A^2\right)(\alpha+1/3)>0\,, \tag{19}$$

Using `Maple` or a table of integration, we find that equation (18) has exact solitary wave solutions of the form

$$\eta(x,t)=\eta_0\,\mathrm{sech}^2\left(\kappa(x-x_0-Ct)\right)\,. \tag{20}$$

The wave has a single hump of amplitude η_0 initially centered at x_0 (which is a constant of integration of (18)). The wave has a wave number κ and travels without change of shape at a steady speed C.

The horizontal velocity $u(x,t)$ is given by

$$u(x,t)=A\,\eta_0\,\mathrm{sech}^2\left(\kappa(x-x_0-Ct)\right)\,, \tag{21}$$

where

$$A=\pm\sqrt{\frac{3}{\delta\,\eta_0+3}}=\pm 2\sqrt{\frac{\alpha}{\alpha-1/3}}\,, \tag{22}$$

from which we obtain the following explicit expressions for the wave speed,

$$\begin{aligned} C &= \frac{2-A^2}{A} && (23)\\ &= \pm\frac{2\,\delta\,\eta_0+3}{\sqrt{3\,(\delta\,\eta_0+3)}} && (24)\\ &= \pm\frac{\alpha+1/3}{\sqrt{\alpha(\alpha-1/3)}}\,, && (25)\end{aligned}$$

the maximum wave amplitude

$$\eta_0 = \frac{3(1-A^2)}{\delta A^2} = -\frac{1}{4}\,\frac{9\,\alpha+1}{\alpha\,\delta}\,, \tag{26}$$

and the wave number

$$\kappa \;=\; \frac{1}{2}\sqrt{\frac{2(1-A^2)}{\mu^2(\alpha+1/3)A^2}} \tag{27}$$

$$\tag{28}$$

$$\;=\; \frac{1}{2}\sqrt{\frac{2\,\eta_0\,\delta}{3\mu^2(\alpha+1/3)}} \tag{29}$$

$$\tag{30}$$

$$\;=\; \frac{\sqrt{6}}{12}\sqrt{\frac{9\,\alpha+1}{(\alpha+1/3)\,\mu^2\,(-\alpha)}}\quad . \tag{31}$$

3.2. METHOD II

The previous method is less general because the solution procedure was simplified by seeking functions $\eta(\xi)$ and $u(\xi)$ that are proportional. In this section, we present a more general approach that can be used without imposing assumptions on the unknowns $\eta(\xi)$ and $u(\xi)$, and which can be applied to a wide class of Boussinesq-like equations. We write the system of equations (11a) and (11b) as a single ODE

$$\delta\,\mu^2\,\alpha\,C\,uu'' + (-\mu^2\,\alpha C^2 + \mu^2\,(\alpha+\frac{1}{3}))\,u'' - \frac{1}{2}\,\delta^2\,u^3 + \frac{3}{2}C\,\delta\,u^2 + (-C^2+1)\,u = 0\,. \tag{32}$$

The coefficients of this ODE depend on the unknown C. To solve this equation, we adopt a technique similar to that used by Kichennassamy and Olver [4]. This technique is easier than solving $u(\xi)$ directly from (32) and also more general than the procedure used in [7, 9], which leads to approximate solutions only. First, we assume that the function $u(\xi)$ can be expressed as the solution of a single first order ordinary differential equation

$$\phi(u) = (u')^2\;. \tag{33}$$

Solutions can also be reconstructed using functions $\phi(u)$ in different forms (see for example Yang et *al*. [10]). From equation (33), for $u' \neq 0$, we have

$$u'' = \frac{1}{2}\phi'\,, \tag{34}$$

where the prime over ϕ denotes a differentiation with respect to the dependent variable u. If we assume that the solitary wave solution has the form

$$u(\xi) = u_0\,\mathrm{sech}^2(\kappa\xi)\,,\quad \kappa > 0\;, \tag{35}$$

where u_0 is the maximum velocity amplitude, then the function ϕ defined by equation (33) has to be the cubic polynomial:

$$\phi(u) = 4\kappa^2 \left(u^2 - \frac{1}{u_0} u^3 \right) = \lambda\, u^2 + \rho\, u^3 \ , \tag{36}$$

where

$$\lambda = 4\kappa^2 \quad \text{and} \quad \rho = -4\,\frac{\kappa^2}{u_0} . \tag{37}$$

Now substituting the expression (36) for ϕ into (34), we can write the second derivative u'' as a second order polynomial in u.

$$u'' = \lambda\, u + \frac{3}{2}\rho\, u^2 . \tag{38}$$

Substituting the resulting expression into equation (32), we obtain on its left-hand side a degree three homogeneous polynomial in u,

$$\begin{aligned} &\left[(-C\,(C + \mu^2\,\alpha C\lambda) + \mu^2\,(\alpha + \frac{1}{3})\,\lambda + 1) \right] u \\ &+ \left[(\delta\,(C + \mu^2\,\alpha C\lambda) - C\,(\frac{3}{2}\mu^2\,\alpha C\rho - \frac{1}{2}\delta) + \frac{3}{2}\mu^2\,(\alpha + \frac{1}{3})\,\rho) \right] u^2 \\ &\qquad + \left[\delta\,(-\frac{1}{2}\delta + \frac{3}{2}\mu^2\,\alpha C\rho) \right] u^3 \,=\, 0 \quad . \end{aligned} \tag{39}$$

It follows from (39) that all the coefficients of this polynomial must be zero in order to obtain a nontrivial solution u. Setting the coefficients to zero yields a nonlinear algebraic system of three equations for the three unknowns C, λ, and ρ :

$$\begin{cases} \delta\,(-\frac{1}{2}\delta + \frac{3}{2}\mu^2\,\alpha C\rho) = 0\,, \\ (\delta\,(C + \mu^2\,\alpha C\lambda) - C\,(\frac{3}{2}\mu^2\,\alpha C\rho - \frac{1}{2}\delta) + \frac{3}{2}\mu^2\,(\alpha + \frac{1}{3})\,\rho) = 0\,, \\ (-C\,(C + \mu^2\,\alpha C\lambda) + \mu^2\,(\alpha + \frac{1}{3})\,\lambda + 1) = 0\,. \end{cases} \tag{40}$$

Solving this nonlinear system analytically, we obtain the following exact expressions for the unknowns C, λ, and ρ.

$$C = \pm \frac{\alpha + 1/3}{\sqrt{\alpha(\alpha - 1/3)}}\,, \tag{41}$$

$$\lambda = -\frac{1}{2}\,\frac{9\,\alpha + 1}{(1 + 3\,\alpha)\,\mu^2\,\alpha}\,, \tag{42}$$

$$\rho = \pm \frac{\sqrt{3}}{3}\,\frac{\delta\,(3\,\alpha - 1)}{\sqrt{\alpha(\alpha - 1/3)}\,\mu^2\,(1 + 3\,\alpha)}\ . \tag{43}$$

Therefore, from (37) we deduce the analytical expressions for the wave number

$$\kappa = \frac{\sqrt{6}}{12}\sqrt{\frac{9\,\alpha+1}{(\alpha+1/3)\,\mu^2\,(-\alpha)}}\,, \tag{44}$$

and the peak amplitude of the wave velocity

$$u_0 = \frac{\sqrt{3}}{2}\,\frac{(9\,\alpha+1)\,\sqrt{\alpha\,(3\,\alpha-1)}}{\alpha\,\delta\,(3\,\alpha-1)}\,, \tag{45}$$

which yields the exact solitary wave solution for $u(x,t)$

$$u(x,t) = u_0\,\mathrm{sech}^2\left(\kappa(x-x_0-Ct)\right)\,. \tag{46}$$

The exact solution for the surface elevation η can be obtained by substituting the solution (46) into (11b).

After simplifications and collecting all the terms in the right-hand side, we obtain the exact expression for the solitary wave solution $\eta\,(x,t)$.

$$\eta\,(x,t) = \eta_0\,\mathrm{sech}^2\left(\kappa(x-x_0-Ct)\right)\,, \tag{47}$$

where the maximum wave amplitude η_0 is given by

$$\eta_0 = -\frac{1}{4}\,\frac{9\,\alpha+1}{\alpha\,\delta}\,. \tag{48}$$

It is worth confirming that the exact solutions u and η are proportional

$$u(x,t) = u_0\,\mathrm{sech}^2\left(\kappa(x-x_0-Ct)\right) = A\,\eta_0\,\mathrm{sech}^2\left(\kappa(x-x_0-Ct)\right) = A\,\eta\,(x,t)\,, \tag{49}$$

where the constant A is given by

$$A = \frac{u_0}{\eta_0} = \pm 2\sqrt{\frac{\alpha}{\alpha-1/3}}\quad. \tag{50}$$

It is easy to check that the exact solitary wave solutions obtained by either *Method I* or *Method II* are identical.

4. Invariants of motion of solitary wave solutions

In this section, we derive exact expressions for four constants of motion corresponding to solitary wave solutions. The analytical expressions are obtained using MAPLE software. The Nwogu system of equations seems to have at least four conservation laws.

The first invariant of motion corresponds to the conservation of mass

$$I_1 = \int_{-\infty}^{+\infty} \eta\,(x,t)\,\mathrm{d}x\,. \tag{51}$$

Using (20) for $\eta(x,t)$, we obtain

$$I_1 = 2\,\frac{\eta_0}{\kappa}\,. \tag{52}$$

Substituting (26) and (31) into (52), it follows that

$$I_1 = \frac{\sqrt{6}}{\delta\,\alpha}\sqrt{-\alpha\,\mu^2\,(\alpha+1/3)\,(9\,\alpha+1)}\,. \tag{53}$$

Another, obvious invariant of motion is the integral

$$I_2 = \int_{-\infty}^{+\infty} u(x,t)\,\mathrm{d}x\,. \tag{54}$$

Substituting the expression (21) of the horizontal velocity in the above integral we obtain

$$I_2 = A\,I_1 = \frac{2\,\eta_0}{\kappa}A\,, \tag{55}$$

and, in explicit form,

$$I_2 = \pm\frac{2\,\eta_0}{\kappa}\sqrt{\frac{3}{\delta\,\eta_0+3}} \tag{56}$$

$$= \pm\frac{2}{\delta}\sqrt{\frac{-6\,\mu^2\,(\alpha+1/3)\,(9\,\alpha+1)}{\alpha-1/3}}\,. \tag{57}$$

The impulse functional given by

$$I_3 = I(\eta,u) = \int_{-\infty}^{+\infty} \eta(x,t)u(x,t)\,\mathrm{d}x\,, \tag{58}$$

is also a constant of motion which can be expressed explicitly

$$I_3 = A\int_{-\infty}^{+\infty} \eta^2\,\mathrm{d}x \tag{59}$$

$$= \frac{4}{3}\frac{\eta_0^2}{\kappa}A \tag{60}$$

$$= \pm\frac{1}{\alpha\delta^2}\sqrt{\frac{2\,\mu^2\,(\alpha+1/3)\,(9\,\alpha+1)^3}{3\alpha-1}}\,. \tag{61}$$

Because dissipation is ignored in the derivation of Nwogu Boussinesq model, we can define a Hamiltonian form for the system (5a) and (5b)

$$I_4 = H(\eta,u) = \int_{-\infty}^{+\infty} (\eta_x^2 + u_x^2 - \eta^2 - u^2 - u^2\eta)\,\mathrm{d}x\ . \tag{62}$$

The Hamiltonian is a constant of motion, which to say that the functional H satisfies

$$H(\eta(x,t),u(x,t)) = H(\eta(x,0),u(x,0))\,. \tag{63}$$

Substituting (20) and (21) for $\eta(x,t)$ and $u(x,t)$ respectively in (62), we obtain

$$I_4 = \frac{4}{15}\,\frac{\eta_0^2\,(4\,\kappa^2+4A^2\,\kappa^2-5-5A^2-4A^2\,\eta_0)}{\kappa}\,. \tag{64}$$

which can be written in explicit form

$$\begin{aligned} I_4 = -\frac{\sqrt{2}}{20}&\left[\sqrt{-\frac{\alpha\,(3\,\alpha+1)}{(9\,\alpha+1)\,\mu^2}}\,\frac{(9\,\alpha+1)^2}{\alpha^3\,\delta^3\,(9\,\alpha^2-1)}\right] \\ &\times\left[135\,\alpha^2\,\delta+6\,\alpha\,\delta-\delta+450\,\alpha^3\,\mu^2\,\delta-10\,\alpha\,\mu^2\,\delta\right. \\ &\qquad \left.+120\,\alpha^2\,\mu^2\,\delta-648\,\mu^2\,\alpha^3-288\,\alpha^2\,\mu^2-24\,\mu^2\,\alpha\right]\,. \end{aligned} \tag{65}$$

5. Conclusion

New exact solitary wave solutions for the Nwogu's one-dimensional extended Boussinesq equations were developed. New analytical expressions for four invariants of motions were also derived. The accuracy of extended Boussinesq models can be assessed using the new exact solutions and their conservation properties can also be verified using the analytical expressions of the constants of motion. These verification tools are implemented in a method of lines solver for Boussinesq equations which is available from the authors [3]. The methods presented in this work are general and can be used for a wide class of nonlinear dispersive wave equations such as Bousinesq-like system of equations.

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