Robust Numerical Methods for PDE Models of Asian Options

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Results are preliminary, please do not quote.

Motivation

- Barraquand and Pudet (1996) state that PDE approaches to pricing Asian and other path-dependent options are limited.

- They correctly note that fully-implicit methods introduce additional numerical diffusion.

Objectives

- Speed

- Accuracy

- Generality

PDE Pricing Models

Advantages

- Handle early exercise features
- Compute a range of option values
- A range of sensitivities are directly computable

Disadvantages

- Initial development
- Stability/Convergence issues

Initial Analysis

We used the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} = rV,$$

with r large relative to σ.

Discretization

- Non-uniform grid spacing

We examined three schemes to handle convection.

- Central weighting
- First-order upstream weighting
- The van Leer flux limiter

Central Weighting

- Second-order accurate

To ensure monotonicity, we must meet both the

- Peclet condition

- Courant condition

Note the Peclet condition

$$\frac{(rS_i)\Delta S_i}{\frac{1}{2}\sigma^2 S_i^2} < 2$$

Prohibitively small grid spacing is required for convection dominated problems.

First-Order Upstream Weighting

- First-order accurate
- Adds numerical diffusion
- No Peclet condition
- Generates excessively "smeared" solutions

The van Leer Flux Limiter

- Second-order accurate
- Total variation diminishing (TVD)
- Non-linear

The non-linear system was solved using full Newton iteration.

Summary

- The van Leer flux limiter works well for the Black-Scholes equation when it is convection dominated.

- Need to use the Crank-Nicolson method with the van Leer limiter when the problem is convection dominated.

Asian Options

Options where the payoff depends on some average of the underlying asset.

- Discrete and continuous arithmetic Asian options

One-Dimensional Continuous Models

Obtained by performing a similarity reduction on the two-dimensional model (Ingersoll, 1987; Wilmott et al, 1993).

- Floating-strike

- Continuous dividend yield

Rogers and Shi (1995) derived

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial x^2} - (\rho_t + rx)\frac{\partial f}{\partial x} = 0.$$

- Floating and fixed-strike
- Cannot handle the early exercise feature
- Cannot handle discrete dividends

Note that it becomes harder to solve for shorter maturities.

Discrete Arithmetic Asian Options

The value of a discretely sampled arithmetic Asian option can be modeled by

$$\frac{\partial V}{\partial t} + \sum_{k=1}^{N} \delta(t - t_k) S \frac{\partial V}{\partial I} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} = rV$$

(Wilmott et al, 1993). At sampling dates the equation can be approximated by

$$\frac{\partial V}{\partial t} + \delta(t - t_k) S \frac{\partial V}{\partial I} = 0.$$

We can solve the equation across sampling dates by ensuring that

$$V(S_i, I_j, t^{before}) = V(S_i, I_j + S_i, t^{after}).$$

We used quadratic Lagrangian interpolation for non-uniform grids.

Consistent with Douglas and Russel (1982).

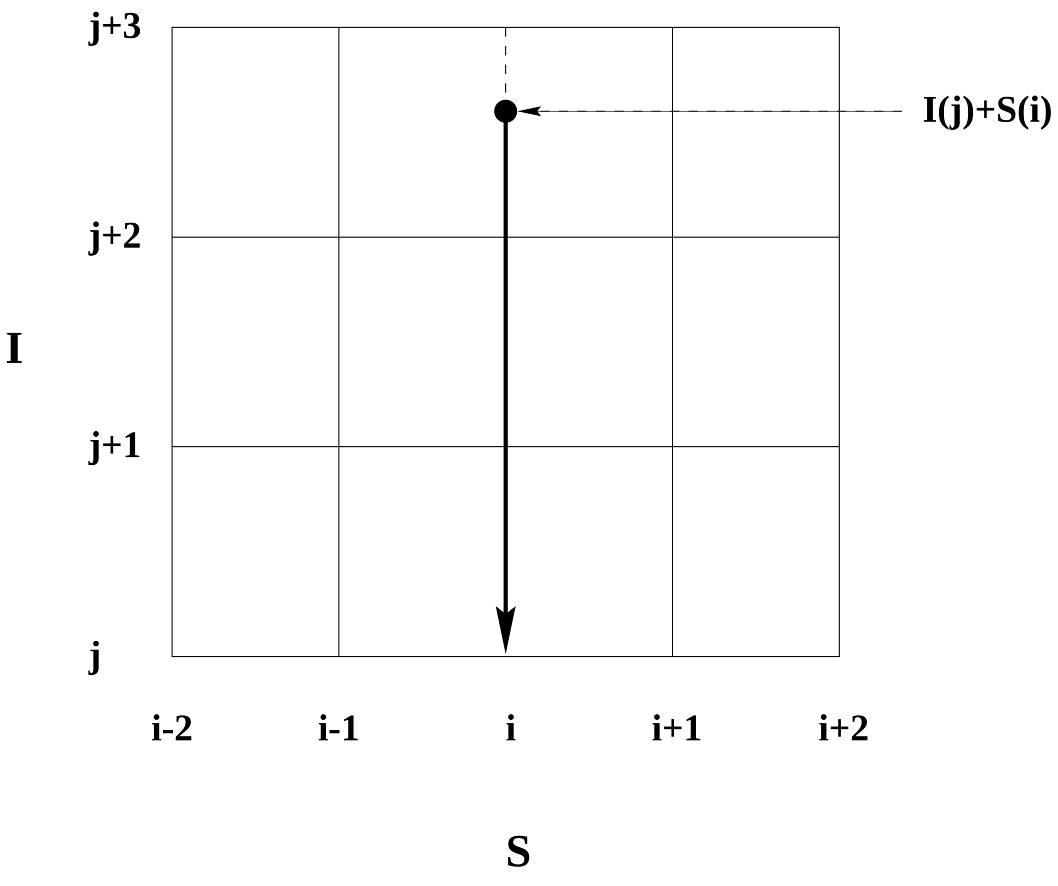

Figure: Propagation along a characteristic across a sampling date.

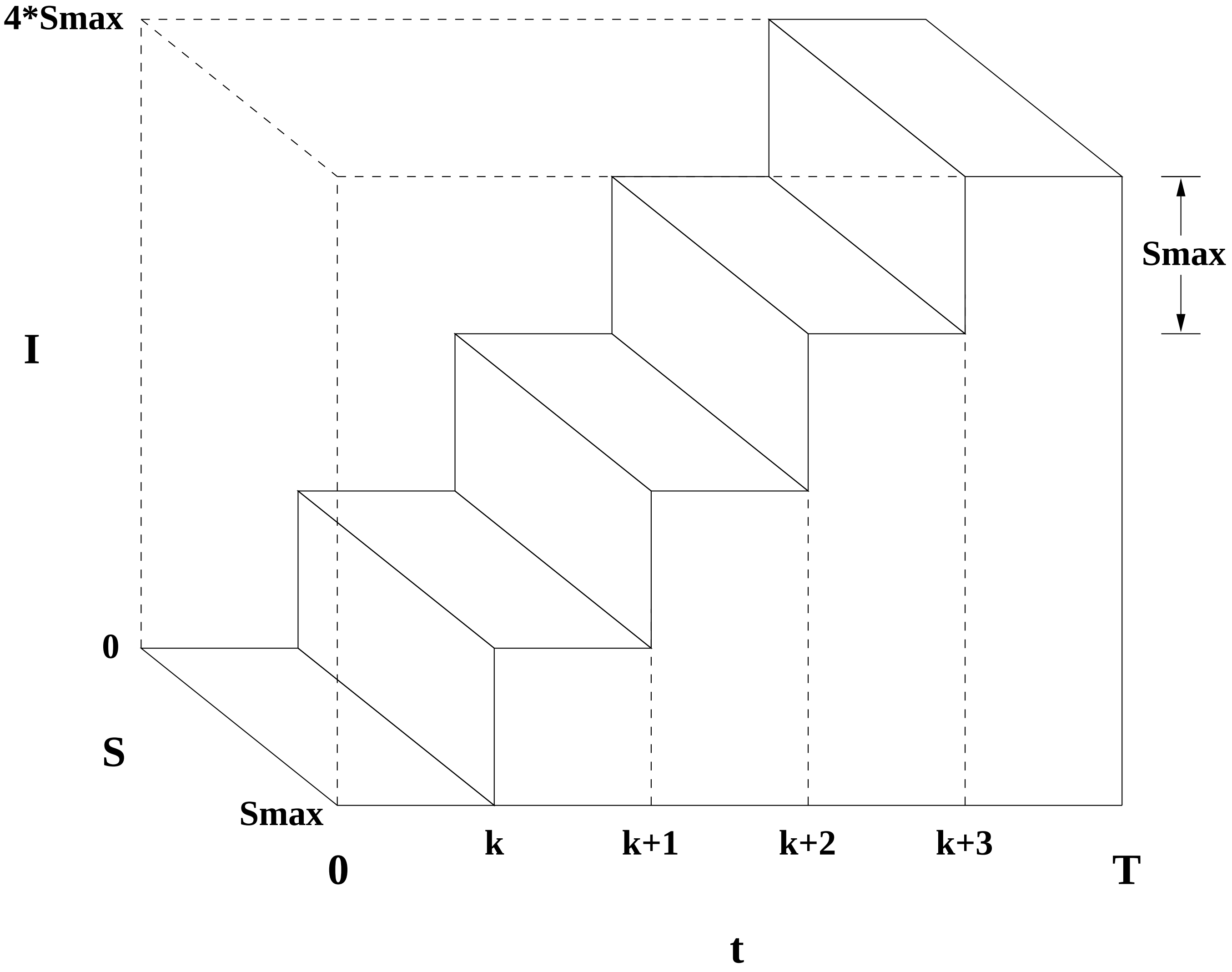

Figure: The numerical domain for the discrete Asian option.

Continuous Arithmetic Asian Options

The value of a continuously sampled arithmetic Asian option can be modeled by

$$\frac{\partial V}{\partial t} + S\frac{\partial V}{\partial I} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} = rV.$$

Cannot ensure monotonicity if central weighting is used.

Discretization

- Roe's "superbee" limiter was used
- TVD schemes in 2D are at most first-order accurate (Goodman & LeVeque, 1985)

ILU and CGSTAB were used to solve the resulting Jacobian.

Model becomes harder to solve for short maturities.

Conclusions

The following discrete and continuous Asian options can be valued accurately and quickly using PDE methods

- European floating-strike
- American floating-strike
- European fixed-strike

Fixed-strike early exercise Asian options

- Slow to achieve desirable accuracy

Alternatives

- Hull & White (1993)
- Barraquand & Pudet (1996)