ASSET ALLOCATION WITH TRANSACTION COSTS

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May 4, 1996

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1 Introduction

1. The asset allocation problem is of both of theoretical interest and practical importance.
2. Geometric Brownian motion and constant elasticity of variance (CEV) process models.
3. Progress we have made.
4. Problems: numerical solution of P.D.E. (or O.D.E.).

2 The Merton Problem

Assumptions:

1. **The short-term interest rate is constant and is known.**
2. **The stock price follows a geometric Brownian motion process:**
   $dS(t) = \mu S(t)dt + \sigma S(t)dB_t$
3. **The stock pays no dividends or other distributions.**
4. **There are no transaction costs.**
5. **Power utility function.**

Merton's result:

$$
\begin{aligned}
C^*(t) &= \{\frac{\rho}{1-\gamma} - \gamma[\frac{(\mu - r)^2}{2\sigma^2(1-\gamma)^2} + \frac{r}{1-\gamma}]\}W(t) \\
\pi^*(t) &= \frac{\mu - r}{\sigma^2(1-\gamma)}
\end{aligned}
\tag{1}
$$

3 CEV model

- **Maintaining all of the assumptions of Merton problem, except that the stochastic stock price is generalized to a constant elasticity of variance (CEV) process.**
- $dS(t) = \mu S(t)dt + \sigma S^{\theta}(t)dB_t$
- **Advantages and disadvantages of this model.**
- **Risk free asset follows:**
  $dX(t) = (rX(t) - C(t))dt$
- **Wealth process:** $W(t) = X(t) + S(t)$
  $dW(t) = [rX(t) - C(t) + \mu S(t)]dt + \sigma S^{\theta}(t)dB_t$
- **Proportion of amount of wealth invested in the stock:**
  $\pi(t) = \frac{S(t)}{W(t)}$

$$\begin{aligned} dW(t) &= \{[r + (\mu - r)\pi(t)]W(t) - C(t)\}dt \\ &\quad + \sigma\pi^{\theta}(t)W^{\theta}(t)dB_t \end{aligned} \tag{2}$$

- **Total expected discounted utility function:** $J(\mathbf{W_t}, C, \pi) = E\{\int_0^\infty \exp(-\rho t) U(C_s) ds\}$ **where** U **is assumed to be strictly concave in** C**.**

- **Objective:** $\max_{\{C,\pi;C\geq 0\}} J(\mathbf{W_t}, C, \pi)$ **subject to (2).**

- **Value function** V**:** $V(\mathbf{W_t}, t) = \max_{\{C.\pi,C\geq 0\}} E_t\{\int_t^\infty \exp(-\rho t) U(C_s) ds\}$

Proposition: The value function V **satisfies the following partial differential equation (H.J.B. equation):**

$$\max_{\{C,\pi;C\geq 0\}} \{\exp(-\rho t) U(C(t)) + \mathcal{L}V\} = 0 \tag{3}$$

where $\mathcal{L}$ **is a differential operator.**

$$\begin{aligned} \mathcal{L}V &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial W}\{[\pi(t)(\mu - r) + r]W(t) - C(t)\} \\ &+ \frac{1}{2}\frac{\partial^2 V}{\partial W^2}\sigma^2 \pi^{2\theta}(t) W^{2\theta}(t) \end{aligned} \tag{4}$$

Define

$$\begin{aligned} & I(W(t),t) = exp(\rho t)V(W(t),t) \\ = & \max_{\{C.\pi,C\geq 0\}} E_t\{\int_t^\infty \exp(-\rho(s-t))U(C_s)ds\} \\ = & \max_{\{C.\pi,C\geq 0\}} E\{\int_0^\infty \exp(-\rho v)U(C_v)dv\} \end{aligned} \tag{5}$$

$I(W(t),t)$ **independent of explicit time. We can write** $I(W(t),t) = I(W(t))$**. From H.J.B equation, we have**

$$\begin{aligned} & \max_{\{C,\pi;C\geq 0\}}\{U(C(t)) - \rho I(W) + \frac{\partial I}{\partial W}\{[\pi(t)(\mu - r) \\ & +r]W(t) - C(t)\} + \frac{1}{2}\frac{\partial^2 I}{\partial W^2}\sigma^2\pi^{2\theta}(t)W^{2\theta}(t)\} \\ = & \ 0 \end{aligned} \tag{6}$$

We now assume that the utility function U is given by:

$$U(C) = \frac{C^\gamma}{\gamma}$$

The case of $\theta < 0$ or $\theta > \frac{1}{2}$ (For other cases, the reults are similar).

The optimal policy is given by:

$$\begin{aligned}
C^* &= [\frac{dI}{dW}]^{\frac{1}{\gamma-1}} && (7)\\
\pi^* &= [\frac{(r-\mu)\frac{dI}{dW}}{\theta\sigma^2\frac{d^2I}{dW^2}}]^{\frac{1}{2\theta-1}}\frac{1}{W} && (8)\\
\frac{1-\gamma}{\gamma} & [\frac{dI}{dW}]^{\frac{\gamma}{\gamma-1}} - \rho I(W) + rW\frac{dI}{dW} \\
& -[\theta^{\frac{1}{1-2\theta}} - \frac{1}{2}\theta^{\frac{2\theta}{1-2\theta}}]\sigma^{-\frac{2}{2\theta-1}}(r-\mu) \\
& [\frac{dI}{dW}]^{\frac{\theta}{2\theta-1}}[\frac{r-\mu}{\frac{d^2I}{dW^2}}]^{\frac{1}{2\theta-1}} = 0 && (9)
\end{aligned}$$

subject to the boundary condition:

$$\begin{aligned}
I(0) &= 0\\
I(\infty) &= \infty
\end{aligned}$$

Alternative formulation:

Introduce S as state variable and solve for $J(W, S)$. Useing results from CIR (1985), we get some new intuition.

$$\begin{aligned} dW(t) &= \{[r + (\mu - r)\pi(t)]W(t) - C(t)\}dt \\ &\quad + \sigma\pi(t)W(t)S^{\theta-1}(t)dB_t \end{aligned} \tag{10}$$

The optimal policy:

$$\begin{aligned} C^* &= [f(S)]^{\frac{1}{\gamma-1}} W \\ \pi^* &= \frac{1}{1-\gamma}\frac{\mu - r}{\sigma^2}S^{2-2\theta} + \frac{f_s}{f}\frac{1}{1-\gamma}S \end{aligned}$$

and f is defined by $J(W, S) = f(S)\frac{W^\gamma}{\gamma}$ and satisfies the following O.D.E.:

$$\begin{aligned} &\frac{1}{2\gamma}\sigma^2 S^{2\theta} f'' + [\frac{\mu - r}{1-\gamma} + \frac{\mu}{\gamma}]Sf' \\ &+\frac{\sigma^2}{2(1-\gamma)}S^{2\theta}\frac{f'^2}{f} + \frac{1-\gamma}{\gamma}[f(S)]^{\frac{\gamma}{\gamma-1}} \\ &+[\frac{(\mu - r)^2}{2(1-\gamma)\sigma^2}S^{2-2\theta} - \frac{\rho}{\gamma} + r]f = 0 \end{aligned}$$

Subject to the boundary condition:

$$f(0) = [\frac{\rho - r\gamma}{1-\gamma}]^{\gamma-1}$$

Summary Remarks:

Two different formulations give some intuition to the problem. Both need to solve an O.D.E. which does not have a closed form solution. Numerical solutions are needed.

4 Asset Allocation with Transaction Costs under CEV Model

Let $X(t)$ be the number of dollars invested in the money market at time t and $Y(t)$ be the number of dollars invested in the stock at time t. (X, Y) follows the following stochastic differential equations:

$$\begin{aligned} dX(t) &= (rX(t) - C(t))dt - dL(t) \\ &\quad +(1-\lambda_1)dM(t) \qquad (11) \\ dY(t) &= \mu Y(t)dt + \sigma Y^{\theta}(t)dB_t \\ &\quad +(1-\lambda_2)dL(t) - dM(t) \qquad (12) \end{aligned}$$

where $C(t)$ is the consumption process which is adapted nonnegative process and integrable, that is,

$$C(t) \geq 0, \qquad \int_0^t C(s)ds < \infty \quad \textbf{for any} \quad t \geq 0, \quad a.s.$$

The process L denotes the cumulative dollar value of all money market shares sold for the purpose of buying stock and M is the cumulative sale of stock for the purpose of investment in the money market. The processes L and M

are adapted right continuous with left hand limits, nonnegative and nondecreasing.

The constants $0 \leq \lambda_1 < 1$ **and** $0 \leq \lambda_2 < 1$ **denote the proportional transaction costs.**

Following Shreve and Soner (1994) or Davis and Norman (1990), we define the solvency region:

$$\varphi \equiv \{(x,y); x + \frac{y}{1-\lambda_2} > 0, x + (1-\lambda_1)y > 0\} \quad (13)$$

The boundary can partition into:

$$\begin{aligned} \partial_1 \ \ \varphi &\equiv \{(x,y); y \leq 0, x + \frac{y}{1-\lambda_2} = 0\} \\ \partial_2 \ \ \varphi &\equiv \{(x,y); y > 0, x + (1-\lambda_1)y = 0\} \end{aligned}$$

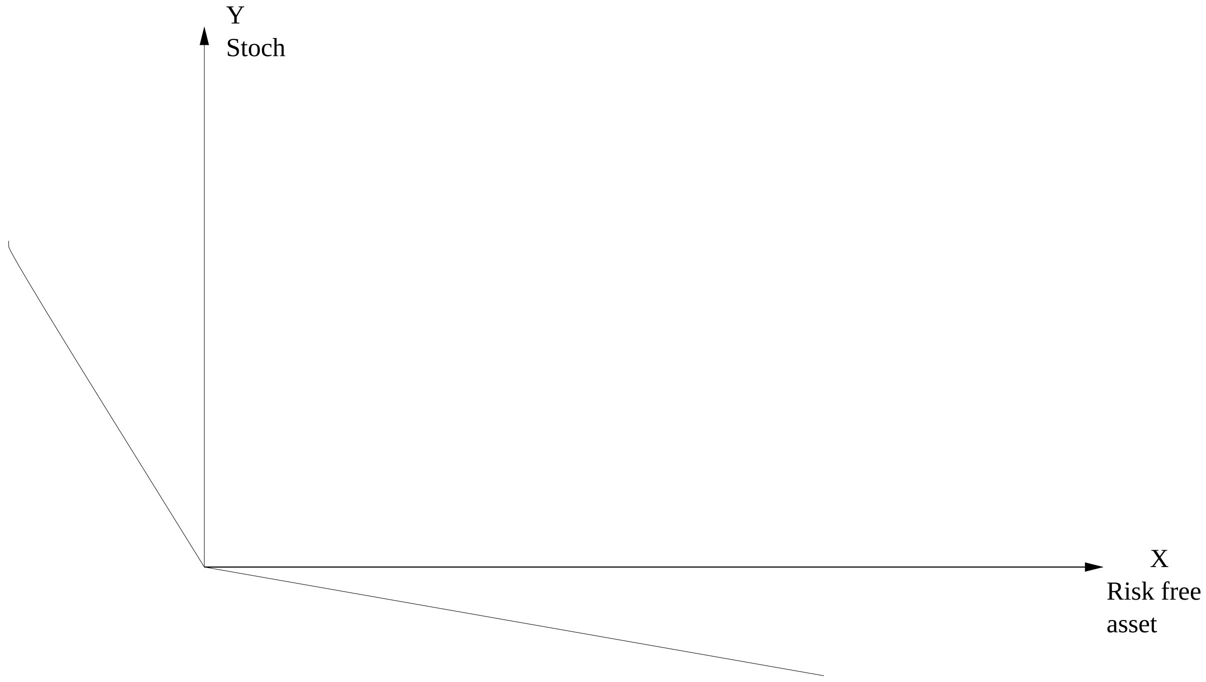

Figure 1: The solvency region

Proposition: Let ϕ be a C^2 function of X, Y and a C^1 function of t, then the H.J.B. equation for the problem with transaction costs is given by:

$$\max_{\{C,M,L\}}\{u(C(t)) \;+\; \phi_t + \mathcal{L}\phi - C\phi_x, (1-\lambda_1)\frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial s}, \\ -\frac{\partial \phi}{\partial x} + (1-\lambda_2)\frac{\partial \phi}{\partial s}\} = 0 \qquad (14)$$

where

$$\mathcal{L}\phi \;=\; \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial X}[rX(t) - C(t)] + \frac{\partial \phi}{\partial Y}\mu Y \\ + \frac{1}{2}\frac{\partial^2 \phi}{\partial Y^2}\sigma^2 Y^{2\theta}(t) \qquad (15)$$

Following Davis and Norman (1990), we can get some idea of the transaction region by take L and M to be absolutely continuous with bounded derivatives functions, i.e.

$$L_t = \int_0^t l_s ds, \qquad M_t = \int_0^t m_s ds,$$
$$0 \leq l_s, m_s \leq \kappa.$$

In this case the H.J.B. equation can be written as:

$$\begin{aligned}\max_{\{C,M,L\}}\{u(C(t)) \; + \; & \phi_t + \mathcal{L}\phi - C\phi_x \\ & +[(1-\lambda_1)\frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial y}]m \\ & +[-\frac{\partial \phi}{\partial x} + (1-\lambda_2)\frac{\partial \phi}{\partial y}]l\} = 0\end{aligned}$$

Notice that both $\frac{\partial\phi}{\partial y}$ and $\frac{\partial\phi}{\partial x}$ are positive since extra wealth will provide increased utility. The maxima are achieved as follows:

$$
\begin{aligned}
C &= [\frac{\partial\phi}{\partial x}]^{\frac{1}{p-1}} && && (16)\\
l &= \kappa && \textbf{if} && \frac{\partial\phi}{\partial y} \geq \frac{1}{1-\lambda_2}\frac{\partial\phi}{\partial x}\\
&\ 0 && \textbf{if} && \frac{\partial\phi}{\partial y} < \frac{1}{1-\lambda_2}\frac{\partial\phi}{\partial x}\\
m &= 0 && \textbf{if} && \frac{\partial\phi}{\partial y} > (1-\lambda_1)\frac{\partial\phi}{\partial x}\\
&\ \kappa && \textbf{if} && \frac{\partial\phi}{\partial y} \leq (1-\lambda_1)\frac{\partial\phi}{\partial x} && (17)
\end{aligned}
$$

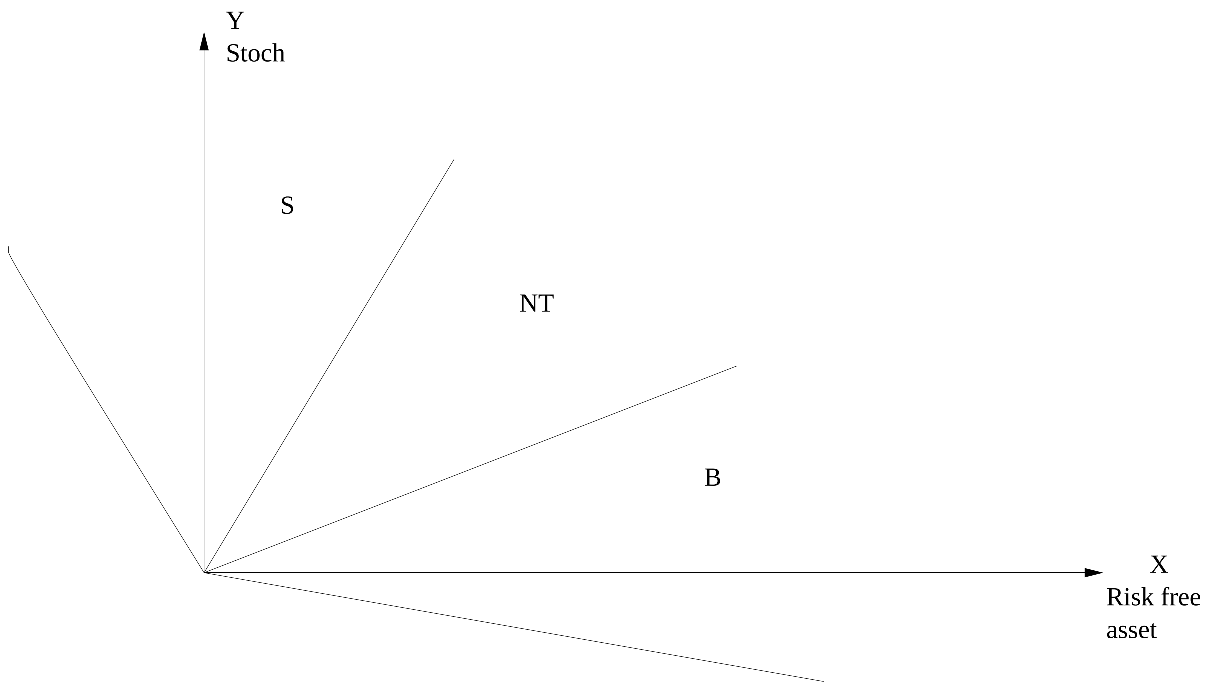

Figure 2: Transactions Region

The following properties are easy to see or same argument to Shreve and Soner (1994) can be applied.

Property 1: The value function V is concave.

Property 2: The value function V is continuous on $\bar{\varphi}$.

Property 3: The value function V has the homotheticity property

Theorem: The value function is the unique viscosity solution of H.J.B. equation.

5 Asset allocation with time variation in expected returns

Let X_t be the amount of wealth invested at time t in the riskfree asset, S denote the stock price, r be the short term interest rate, l be the yield on a zero coupon bond which has a fixed time to maturity, δ be the dividend yield on the stock.

The equations governing the state variables are:

$$\begin{aligned}
dS &= \mu_s S dt + \sigma_{01} S dB_1 + \sigma_{02} S dB_2 + \sigma_{03} S dB_3 \\
&\quad + \sigma_{04} S dB_4 + (1-\lambda) dM_t - dN_t \\
dX_t &= \mathbf{r}_t X_t dt - C_t dt - dM_t + (1-\mu) dN_t \\
dY_t &= Y_t[\mu_r A_1 + \mu_l A_2 - A_0'(T-t) \\
&\quad - A_1'(T-t)\mathbf{r}_t - A_2'(T-t)\mathbf{l}_t \\
&\quad + \frac{1}{2} A_1^2(\sigma_{11}^2 + \sigma_{12}^2) + \frac{1}{2} A_2^2(\sigma_{21}^2 + \sigma_{22}^2) \\
&\quad + A_1 A_2(\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22})] dt \\
&\quad + (A_1\sigma_{11} + A_2\sigma_{21}) Y_t dB_1 \\
&\quad + (A_1\sigma_{12} + A_2\sigma_{22}) Y_t dB_2 \\
d\mathbf{r} &= \mu_r dt + \sigma_{11} dB_1 + \sigma_{12} dB_2
\end{aligned}$$

$$
\begin{aligned}
d\mathbf{l} &= \mu_l dt + \sigma_{21} dB_1 + \sigma_{22} dB_2 \\
d\delta &= \mu_\delta dt + \sigma_{31} dB_1 + \sigma_{32} dB_2 + \sigma_{33} dB_3 + \sigma_{34} dB_4
\end{aligned}
$$

where M_t, N_t are $\mathcal{F}_t$ measurable, right continuous and nondecreasing processes, $1-\lambda$ and $1-\mu$ are the rate of transaction cost, M and N are interpreted as cumulative purchases and sales.

Proposition: Let ϕ be a C^2 function of x and a C^1 function of t, then the H.J.B. equation for the problem with transaction costs is given by:

$$
\max_{\{C,M,N\}} \{u(C(t)) \;+\; \phi_t + \mathcal{L}\phi - C\phi_x, (1-\mu)\frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial s}, -\frac{\partial \phi}{\partial x} + (1-\lambda)\frac{\partial \phi}{\partial s}\} = 0 \tag{18}
$$

where

$$
\begin{aligned}
\mathcal{L} &= \mu_{\mathbf{X}} D_{\mathbf{X}} + \frac{1}{2} tr G D^2_{\mathbf{X}} - \beta \\
G &= \sigma_{\mathbf{X}} {\sigma_{\mathbf{X}}}^T \\
\mu_{\mathbf{X}} &= \begin{bmatrix} \mu_s S \\ \mathbf{r} X \\ Y_t \mu_y \\ \mu_r \\ \mu_l \\ \mu_\delta \end{bmatrix}
\end{aligned}
$$

$$(19)$$

and

$$\begin{aligned}
\mu_y &= \mu_r A_1 + \mu_l A_2 - A_0'(T-t) - A_1'(T-t)\mathbf{r}_t \\
&\quad - A_2'(T-t)\mathbf{l}_t + \frac{1}{2}A_1^2(\sigma_{11}^2 + \sigma_{12}^2) \\
&\quad + \frac{1}{2}A_2^2(\sigma_{21}^2 + \sigma_{22}^2) + A_1 A_2(\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22})
\end{aligned}$$

$$\sigma_{\mathbf{X}} = \begin{bmatrix} \sigma_{01}S & \sigma_{02}S & \sigma_{03}S & \sigma_{04}S \\ 0 & 0 & 0 & 0 \\ \sigma_{41}Y & \sigma_{42}Y & 0 & 0 \\ \sigma_{11} & \sigma_{12} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \end{bmatrix} \tag{20}$$

where $\sigma_{41} = A_1\sigma_{11} + A_2\sigma_{21}$ **and** $\sigma_{42} = A_1\sigma_{12} + A_2\sigma_{22}$**.**

References

[1] Brennan, M. J., Schwartz, E. S. and Lagnado, R., “Strategic asset allocation”. Working paper, UCLA, (1994).

[2] Cox, J. C., Ingersoll,Jr. J. E. and Ross, S. A., “A theory of the term structure of interest rates”, Econometrica, Vol. 53, No. 2, 385-407. (1985).

[3] Crandall, M. G., Ishii, H. and Lions, P. L., “User’s guide to viscosity solutions of second order partial differential equations”, Bulletin (new series) of the American Mathematical Society, 27 1-67 (1992).

[4] Davis, M. H. A. and Norman, A. R.., “Portfolio selection with transaction costs”, Math. Op. Res. 15, 676-713 (1990).

[5] Emanuel, D.C. and MacBeth, J.D., “Further Results on the Constant Elasticity of Variance Call Option Pricing Model”, Journal of Financial and Quantitative Analysis, Vol. 17, No. 4, 1982.

[6] Macbeth, J.D. and Merville, L.J., "Test of the Black-Scholes and Cox Call Option Valuation Models", The Journal of Finance, Vol. 35, No. 2, 1980.

[7] Merton, R. C., "Lifetime portfolio selection under uncertainty: The continuous - time case". Review of Economics and Statistics, 1969, vol. 51, 247-257.

[8] Merton R. C.(1990) Continuous Time Finance, Basil Blackwell, Cambridge ,MA.

[9] Shreve, S. E. and Soner, H. M., "Optimal investment and consumption with transaction costs". The Annals of Applied Probability, 1994, vol. 4, No. 3, 609-692.