Superconvergent Interpolants for the Collocation Solution of BVODEs

W.H. Enright*   P.H. Muir†

Abstract

A long-standing open question associated with the use of collocation methods for boundary value ordinary differential equations is concerned with the development of a high order continuous solution approximation to augment the high order discrete solution approximation, obtained at the mesh points which subdivide the problem interval. It is well-known that the use of collocation at Gauss points leads to solution approximations at the mesh points for which the global error is $O(h^{2k})$, where k is the number of collocation points used per subinterval and h is the subinterval size. The collocation solution also yields a C^0 continuous solution approximation that has a global error of $O(h^{k+1})$. The discrete solution is thus said to be superconvergent. In this paper, we show how to efficiently augment the superconvergent discrete collocation solution to obtain C^1 continuous "superconvergent" interpolants, whose global errors are $O(h^{2k})$. The key ideas are to use the theoretical framework of continuous Runge-Kutta schemes and to augment the collocation solution with inexpensive mono-implicit Runge-Kutta stages. Specific schemes are derived for k = 1,2,3, and 4. Numerical results are provided to support the theoretical analysis.

Keywords: Collocation, Runge-Kutta methods, boundary value ordinary differential equations, interpolation.

AMS Subject Classifications: 65L05, 65L10

Abbreviated Title: Superconvergent Interpolants for Collocation

*Department of Computer Science, University of Toronto, Toronto, Ont., Canada M5S 1A4, (enright@cs.toronto.edu)

†Department of Mathematics and Computing Science, Saint Mary's University, Halifax, Nova Scotia, Canada B3H 3C3, (Paul.Muir@stmarys.ca). This work is supported by the Natural Sciences and Engineering Research Council of Canada.

1 Introduction

The method of collocation at Gauss points has enjoyed considerable success in the numerical solution of boundary value ordinary differential equations (BVODEs). The widely used COLSYS/COLNEW collocation codes [2],[4] employ, respectively, a B-spline basis and a monomial spline basis, to provide a piecewise polynomial approximation to the solution of a BVODE, which for the purposes of this paper, we will assume has the form

$$y'(t) = f(t, y(t)), \quad t \in [a, b] \qquad y : \Re \to \Re^n, \qquad f : \Re \times \Re^n \to \Re^n, \tag{1}$$

with boundary conditions,

$$g(y(a), y(b)) = 0, \quad g : \Re^n \times \Re^n \to \Re^n, \tag{2}$$

(although the particular form of the boundary conditions will not be important for this paper). We will assume appropriate conditions on f and g to guarantee the existence of a solution (see [3]). The above form is referred to as a first order system form. See [3] for a discussion on conversion of more general BVODEs to this form. While many software packages for the numerical solution of BVODEs require the system of equations to be written in the form (1), the COLSYS/COLNEW codes have the advantage that they can be used to treat mixed order systems directly.

The collocation method assumes that the problem domain is partitioned into N subintervals by a mesh, $\{t_i\}_{i=0}^{N}$, with $t_0 = a$ and $t_N = b$. Letting k be the number of collocation points per subinterval, the continuous collocation solution, on the ith subinterval, $[t_{i-1}, t_i]$, is a polynomial of degree k which can be written in the form,

$$\hat{u}_i(t) = \sum_{j=0}^{k} c_{ij} B_{ij}(t), \tag{3}$$

where each $B_{ij}(t)$ is a polynomial basis function of degree k and each $c_{ij} \in \Re^n$ is a vector of unknown coefficients to be determined by requiring $\hat{u}_i(t)$ to satisfy (1) at the collocation points, $\widehat{t}_{ir}$, i.e.

$$\hat{u}_i'(\widehat{t}_{ir}) = f(\widehat{t}_{ir}, \hat{u}_i(\widehat{t}_{ir})), \quad r = 1, \ldots, k,$$

as well as satisfying continuity conditions, $\hat{u}_i(t_{i-1}) = \hat{u}_{i-1}(t_{i-1})$, at the interior meshpoints, and the boundary conditions, (2), which will have the form, $g(\hat{u}_1(t_0), \hat{u}_N(t_N)) = 0$. The collocation points are $\widehat{t}_{ir} = t_{i-1} + \rho_r h$, where $h = t_{i-1} - t_i$ and, in COLSYS/COLNEW, the $\{\rho_r\}_{r=1}^{k}$ are the Gauss points on [0,1].

From the theory for collocation at Gauss points (see e.g. [3]) it is known that the global error at the mesh points is

$$| \hat{u}_i(t_i) - y(t_i) | \sim O(h^{2k}), \quad i = 0, \ldots, N,$$

while the global error at intermediate points is

$$| \hat{u}_i(t) - y(t) | \sim O(h^{k+1}), \quad t \in (t_{i-1}, t_i), \quad i = 0, \ldots, N.$$

While the presence of a higher order solution approximation at the mesh points is certainly a positive feature of collocation methods, it has long been recognized that it is inconvenient that the associated collocation polynomial is of substantially lower order. In [16], the author considers augmenting the high order discrete collocation solution by constructing, for each subinterval, a high order polynomial interpolant, having a global error also of $O(h^{2k})$, determined using mesh point solution approximations from several adjacent subintervals. The author reports some difficulties, one of which is that the numerical stability of the approach depends on the ratio of subinterval sizes; these ratios can be large for the nonuniform meshes which arise in the solution of difficult BVODEs.

In [17] the authors consider constructing a high order local interpolant called a superconvergent interpolant, having a global error of $O(h^{2k})$, using only information from the current subinterval. On the ith subinterval, in addition to the two endpoint solution values, y_{i-1}, y_i, extra high order information is obtained by performing a "secondary collocation". This high order interpolant, which is a polynomial, $\overline{u}_i(t)$, of degree $2k-1$ on the ith subinterval, is determined by requiring $\overline{u}_i(t)$ to match the superconvergent solution values at the mesh points, i.e.

$$\overline{u}_i(t_{i-1}) = \hat{u}_i(t_{i-1}) = y_{i-1}, \quad \overline{u}_i(t_i) = \hat{u}_i(t_i) = y_i,$$

and to satisfy $2k-2$ secondary collocation conditions,

$$\overline{u}_i'(\overline{t}_{ir}) = f(\overline{t}_{ir}, \overline{u}_i(\overline{t}_{ir})), \quad r = 1, \ldots, 2k-2, \tag{4}$$

where $\overline{t}_{ir}$ are the secondary collocation points on the ith subinterval.

While this is perhaps the natural approach to providing a high order local interpolant, there is some question, nonetheless, about the efficiency of this approach since it requires, in the general case where $f(t, y(t))$ is nonlinear, the solution of the nonlinear system (4) which is of size $O(nk)$, on each subinterval. Assuming a Newton-type iteration is used to solve these nonlinear systems, with q iterations to convergence, this leads to an overall cost of $O(Nqn^3k^3)$, which is approximately the same operation count as that associated with computing a continuous collocation solution, having a global error of $O(h^{2k})$, simply by starting over, using $2k$ collocation points per subinterval.

In this paper, we place the question of developing a superconvergent interpolant in the context of continuous Runge-Kutta (CRK) schemes, which are naturally able to yield high order interpolants using solution and derivative information of various orders. The general form of a CRK scheme, for the ith subinterval, is (see e.g. [15])

$$u_i(t_{i-1} + \theta h) = y_{i-1} + \sum_{r=1}^{s} b_r(\theta) f(\widehat{t}_r, \widehat{y}_r), \tag{5}$$

where $b_r(\theta)$ is a polynomial in $\theta \in [0,1]$, $\widehat{t}_r = t_{i-1} + c_r h$, and

$$\widehat{y}_r = y_{i-1} + h \sum_{j=1}^{s} a_{rj} f(\widehat{t}_j, \widehat{y}_j), \quad r = 1, \ldots, s. \tag{6}$$

The evaluations of the derivatives, $f(\widehat{t}_r, \widehat{y}_r)$, will be referred to as implicit Runge-Kutta (IRK) stages. It is also possible to write (6) in an equivalent "parameterized" form, [14],

$$\widehat{y}_r = (1 - v_r) y_{i-1} + v_r y_i + h \sum_{j=1}^{s} x_{rj} f(\widehat{t}_j, \widehat{y}_j), \quad r = 1, \ldots, s. \tag{7}$$

and we will refer to stages written in this form as parameterized IRK stages or PIRK stages. In the context of BVODEs it is common to employ mono-implicit Runge-Kutta (MIRK) stages, which follow from (7) by requiring the matrix X (whose elements are the x_{rj}'s) to be strictly lower triangular and thus (7) takes the form,

$$\widehat{y}_r = (1 - v_r) y_{i-1} + v_r y_i + h \sum_{j=1}^{r-1} x_{rj} f(\widehat{t}_j, \widehat{y}_j), \quad r = 1, \ldots, s. \tag{8}$$

Note that when both y_{i-1} and y_i are available, the MIRK stages can be computed quite efficiently since the calculations are explicit. Note also that any stage written in MIRK form, (8), can be transformed to IRK form, (6) (see [14]), while any IRK stage (6) can be trivially written in the form (7), with $v_r = 0$ and $x_{rj} = a_{rj}$.

The CRK scheme will be constructed to use as much information from the collocation method as possible. After the discrete collocation solution at the mesh points is computed by the collocation solver, we will use the derivative evaluations at the collocation points (which arise naturally during the collocation calculation, can be expressed in the general form of (6), and which we will call collocation stages), and derivative evaluations at the mesh points (which we will call endpoint stages and which on the ith subinterval are $f(t_{i-1}, y_{i-1})$ and $f(t_i, y_i)$), as stages for the CRK scheme. The CRK scheme will also employ additional MIRK stages, (8), in order to obtain a scheme of the desired order of accuracy. The idea is a very natural extension of the embedding of a discrete MIRK scheme within a CMIRK scheme, as employed in MIRKDC [11] and as further considered in [13]. The idea of deriving schemes which use a combination of different kinds of stages, i.e. collocation and mono-implicit stages, is also related to the SDIRK schemes [8] where singly-implicit Runge-Kutta schemes are augmented with diagonally implicit Runge-Kutta (DIRK) [1] stages.

The approach considered in this paper exploits the well-known equivalence of collocation and implicit Runge-Kutta (IRK) methods [20]. For each value of k

considered, we will first identify the collocation method with the corresponding IRK scheme and then exhibit the associated natural C^0 collocation interpolant on the ith subinterval, expressed in CRK form (5). We will then embed the collocation stages plus the endpoint stages in a general family of CRK schemes, and then derive a scheme of the desired order using extra MIRK stages. Since the extra MIRK stages and the endpoint stages can be computed explicitly, and the collocation stages are already available, the associated costs are low. The general approach used in deriving the CRK schemes is the same as in [15], to which we refer the reader for further background.

It should be emphasized that the computations for the setup of the superconvergent interpolant occur after the collocation solution has been computed. Thus the stability of the calculation of the global solution is not affected. It should also be noted that the dominant computational cost will be associated with the initial computation of the collocation solution, rather than with the subsequent construction of the superconvergent interpolant.

2 Derivation of Interpolants

It is possible to consider the question of deriving or constructing superconvergent interpolants based on CRK schemes from at least two perspectives. One approach involves an examination of certain sets of equations which depend on the coefficients of the scheme. These equations are called order conditions and they characterize all schemes of a given order of accuracy. An advantage of this approach is its generality; any scheme of the desired order can be obtained in this way provided one is able to choose the coefficients of the scheme to satisfy the order conditions. The idea is to choose the minimum number of stages for the scheme such that there will be a sufficient number of free parameters to allow the order conditions to be satisfied. A disadvantage is that the process of solving the order conditions while using as few free parameters as possible is not straightforward and heuristics must frequently be used to help in the derivation, including making certain simplifying assumptions which limit, to some extent, the generality of the result. One can obtain an upper bound on the number of stages that would be required for a CRK scheme of a given order derived in this way, by referring to theoretical results which give the number of order conditions that would need to be satisfied. However, as will be demonstrated in subsequent sections, it is often possible to develop derivation strategies leading to schemes with substantially fewer stages than are given by the theoretical upper bounds.

The second approach involves viewing the derivation of interpolants based on CRK schemes within the context of a "boot strapping" algorithm [9] which we now briefly describe, for the current context. As indicated previously, we assume that the discrete superconvergent collocation solution approximations at the mesh points and the lower order collocation stages are available. As well, we assume that the evaluations of the corresponding endpoint stages have already been performed. The boot strapping algorithm begins with this information and produces a sequence of interpolants, each one order higher than the previous one, until an interpolant of the desired order is obtained. The key idea is that new stages of the appropriate order required for the each new interpolant are obtained by using the high order endpoint information plus evaluations of the previous interpolant, which is one order lower. We will describe the first few steps of this process and then present a table characterizing the number of stages the final interpolant would require for k ranging from 1 to 7. (This is the range of k values available in the COLSYS/COLNEW code.)

The natural interpolant from the collocation method, is a polynomial of degree k having local error $O(h^{k+1})$, on each subinterval. We will refer to it as $u^{(0)}(t)$. The next interpolant, which we will refer to as $u^{(1)}(t)$, is a polynomial of degree $k+1$ having local error $O(h^{k+2})$ on each subinterval and is obtained by requiring it to interpolate the endpoint solution values and by requiring its derivative to interpolate the endpoint stages, as well as $k-2$ additional stages, $f(t_j^{(1)}, u^{(0)}(t_j^{(1)}))$, $j = 1, \ldots, k-2$. These latter stages have a local error of only $O(h^{k+1})$ but since each is multiplied by h when it appears in the CRK scheme, the contribution to the local error of $u^{(1)}(t)$ is $O(h^{k+2})$. Thus $u^{(1)}(t)$ is uniquely determined by $k+2$ interpolation requirements involving information having a local error of at least $O(h^{k+2})$. The next interpolant, $u^{(2)}(t)$, a polynomial of degree $k+2$, having local error $O(h^{k+3})$, and is defined by interpolation conditions based on the endpoint values and endpoint stages plus $k-1$ additional stages, having a local error of $O(h^{k+2})$, of the form $f(t_j^{(2)}, u^{(1)}((t_j^{(2)}))$. When multiplied by h, these latter stages, plus the endpoint information, provide interpolation conditions having a local error of at least $O(h^{k+3})$ and uniquely determine $u^{(2)}(t)$ with a local error of $O(h^{k+3})$. This process continues until an interpolant having the desired local error is obtained.

In Table 1, for each value of k, we give the number of extra stages that will need to be computed in order to obtain interpolants having local errors of $O(h^{2k})$ and $O(h^{2k+1})$. In the next section of this paper, we will explain how both of these interpolants will have a global error of $O(h^{2k})$ and thus be superconvergent. In addition to giving data for interpolants based on the boot strapping approach (which we shall refer to in Table 1 as the Boot Strap method), we will also include the corresponding information for interpolants using the first approach described above (which we shall refer to in Table 1 as the Runge-Kutta approach). Further details for the results associated with the Runge-Kutta approach will be provided in subsequent sections of this paper. We will assume that the endpoint stages and collocation stages are already available. Each table entry will give the number of extra stages that must be computed by each approach in order to obtain a superconvergent interpolant with the indicated local error ($l.e.$). The entry, 'HC', will mean that no extra stages are required since the Hermite cubic interpolant, based on the endpoint solution and derivative information, y_{i-1}, $f(t_{i-1}, y_{i-1})$, y_i, $f(t_i, y_i)$, and having

k	Boot Strap	Boot Strap	Runge Kutta	Runge Kutta
l.e.	$O(h^{2k})$	$O(h^{2k+1})$	$O(h^{2k})$	$O(h^{2k+1})$
1	HC	HC	HC	HC
2	HC	1	HC	1
3	3	6	1	3
4	9	14	3	8
5	18	25	≤ 13	≤ 29
6	30	39	≤ 29	≤ 61
7	45	56	≤ 61	≤ 125

Table 1: Number of extra stages required to compute interpolant

local error $O(h^4)$, can be used.

For general k, the boot strapping approach requires $\frac{3k}{2}(k-3)+3$ extra stages to obtain the interpolant having local error $O(h^{2k})$ and $\frac{k}{2}(3k-5)$ extra stages to obtain the interpolant having local error $O(h^{2k+1})$. For general k the theoretical upper bounds giving the total number of order conditions (and thus the maximum number of stages of the CRK scheme) minus the number of available stages, $k+2$, gives an upper bound on the number of extra stages required when the Runge-Kutta approach is employed. For the interpolant having local error $O(h^{2k})$ at most $2^{k-1}-3$ extra stages are needed and for the interpolant having local error $O(h^{2k+1})$ at most 2^k-3 extra stages are needed. As we shall see in subsequent sections, it is usually possible to derive schemes of a desired order which use substantially less than the maximum number of stages.

3 Background

When the CRK scheme, (5), uses PIRK stages, (7), the coefficients are usually written in a tableau

$$\begin{array}{c|c|c} c & v & X \\ \hline & & b(\theta)^T \end{array},$$

where $c=[c_1,\ldots,c_s]^T$, $v=[v_1,\ldots,v_s]^T$, $b=[b_1(\theta),\ldots,b_s(\theta)]^T$, and the i,jth component of the s by s matrix X is x_{ij}. A CRK scheme is of order p if its local error, on a subinterval of length h, is $O(h^{p+1})$, i.e. $|u_i(t)-y_i(t)| \sim O(h^{p+1})$, where $y_i(t)$ is the true solution to the local problem (14). The rth weight polynomial, $b_r(\theta)$, is a polynomial of degree p. A CRK scheme of order p is determined by requiring its coefficients to satisfy equations, called continuous order conditions. For example, a CRK scheme with PIRK stages, (7), of order 3, would have to satisfy the conditions

$$b^T(\theta)e = \theta, \quad b^T(\theta)c = \frac{\theta^2}{2}, \quad b^T(\theta)c^2 = \frac{\theta^3}{3}, \quad b^T(\theta)(Xc + \frac{v}{2}) = \frac{\theta^3}{6}, \tag{9}$$

where e is a vector of 1's of length s and $c^l = [c_1^l, \ldots, c_s^l]^T$. An important attribute of a CRK scheme is its stage order. A CRK scheme with PIRK stages has stage order $q(\leq p)$ if it satisfies

$$Xc^{l-1} + \frac{v}{l} = \frac{c^l}{l}, \quad l = 1, \ldots, q. \tag{10}$$

The collocation polynomial obtained from the collocation method can be expressed in an alternative form to that given in (3) which allows it to be viewed within the framework of the continuous Runge-Kutta schemes. On the ith subinterval, the collocation polynomial, $\hat{u}_i(t)$, can be written in the form

$$\hat{u}_i(t) = \hat{u}_i(t_{i-1} + \theta h) = y_{i-1} + h \sum_{r=1}^{k} \hat{b}_r(\theta) f(\hat{t}_r, \hat{y}_r),) \tag{11}$$

where $t \in [t_{i-1}, t_i]$, $\theta \in [0, 1]$, y_{i-1} is the discrete collocation solution at t_{i-1},

$$\hat{t}_r = t_{i-1} + \rho_r h, \quad \hat{y}_r = y_{i-1} + h \sum_{j=1}^{s} a_{rj} f(\hat{t}_j, \hat{y}_j),$$

and the coefficients, a_{rj}, characterize the implicit Runge-Kutta scheme which is equivalent to the collocation scheme (see [20]). The collocation stages, $f(\hat{t}_r, \hat{y}_r)$, are the evaluations of the right hand side of the BVODE that occur during the calculation of the collocation solution. The collocation stages have stage order k. The weight polynomials, $\hat{b}_r(\theta)$, have derivatives that are the Lagrange interpolating polynomials for the abscissa set $\{\rho_r\}_{r=1}^k$. The collocation polynomial is conveniently represented in the tableau of a CRK scheme with IRK stages as follows,

$$\begin{array}{c|ccc} \rho_1 & a_{11} & \ldots & a_{1k} \\ \vdots & \vdots & & \vdots \\ \rho_k & a_{k1} & \ldots & a_{kk} \\ \hline & \hat{b}_1(\theta) & \ldots & \hat{b}_k(\theta). \end{array} \tag{12}$$

A superconvergent interpolant is obtained from a CRK scheme which, on the ith subinterval, will use the superconvergent endpoint values, y_{i-1} and y_i, the corresponding stages, $f(t_{i-1}, y_{i-1})$ and $f(t_i, y_i)$, the collocation stages, and as many extra MIRK stages as necessary to achieve the desired order. Thus the first $k+2$ rows of the tableau of the CRK scheme, with the stages expressed in PIRK form (7), will be

$$
\begin{array}{c|c|cccccccc}
0 & 0 & 0 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\
1 & 1 & 0 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\
\rho_1 & 0 & 0 & 0 & a_{11} & \dots & a_{1k} & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\rho_k & 0 & 0 & 0 & a_{k1} & \dots & a_{kk} & 0 & \dots & 0
\end{array}
\qquad (13)
$$

Note that the collocation stages, (12), originally expressed in IRK form, (6), can be written in PIRK form, (7), by setting the extra free parameters $v_3, \dots, v_{k+2}$, to zero, as indicated in (13).

When a k-point Gaussian collocation scheme is used in the numerical solution of a BVODE, the discrete collocation scheme will have a local error of $O(h^{2k+1})$ and a global error of $O(h^{2k})$ [3]. In order to examine the global error of the collocation polynomial, we begin by letting $y_i(t)$ be the true solution to the local problem (on the ith subinterval),

$$y'(t) = f(t, y(t)), \quad y(t_{i-1}) = y_{i-1}, \qquad (14)$$

where y_{i-1} is the approximate (but superconvergent) solution value at t_{i-1}. Since the global error in the initial condition of the local problem is $O(h^{2k})$, it follows that

$$|y_i(t) - y(t)| \quad \sim \quad O(h^{2k}),$$

i.e. the true local solution agrees with the true global solution to within a (global) error of $O(h^{2k})$. The collocation polynomial on the ith interval, $\hat{u}_i(t)$, agrees with the true local solution, $y_i(t)$, to within a local error of $O(h^{k+1})$, i.e.

$$|y_i(t) - \hat{u}_i(t)| \quad \sim \quad O(h^{k+1}).$$

Thus the local error of $\hat{u}_i(t)$ dominates the global error of $y_i(t)$ and we have

$$|\hat{u}_i(t) - y(t)| \quad \sim \quad O(h^{k+1}),$$

i.e. the global error of the collocation polynomial is $O(h^{k+1})$.

We now consider the CRK interpolants, similarly. Since the global error of the true local solution, $y_i(t)$, is $O(h^{2k})$, it is sufficient for the CRK scheme to agree with $y_i(t)$ to within a local error of $O(h^{2k})$. Then the local error of the CRK scheme agrees with the global error of the true local solution and the global error of the CRK scheme will also be $O(h^{2k})$. On the other hand, it is sometimes useful, e.g. if the defect of the solution is important, to have an interpolant whose local error is dominated by the global error of the discrete solution. Thus we will also consider CRK schemes whose local error is $O(h^{2k+1})$ (but with a global error is $O(h^{2k})$). In the subsequent sections of this paper, for $k = 1, 2, 3, 4$, we will derive two CRK schemes, each having a global error of $O(h^{2k})$. The first scheme will have a local error of $O(h^{2k})$ while the second will have a local error of $O(h^{2k+1})$.

We conclude this section with a brief comment on the two approaches described in the previous section for deriving CRK schemes and the continuity of the interpolants obtained from these CRK schemes. An interpolant based on a CRK scheme will have C^1 continuity provided the CRK scheme interpolates y_{i-1} and y_i and its derivative interpolates ${y'}_{i-1} = f(t_{i-1}, y_{i-1})$ and ${y'}_i = f(t_i, y_i)$. The general form of the scheme derived in the boot-strapping approach includes explicit dependence on all of these quantities. For the ith subinterval, it is

$$u_i(t_{i-1} + \theta h) = d_0(\theta) y_{i-1} + d_1(\theta) y_i + \sum_{r=1}^{s} b_r(\theta) f(\widehat{t}_r, \widehat{y}_r), \tag{15}$$

where $d_0(\theta)$, $d_1(\theta)$, and $b_r(\theta)$, $r = 1, \ldots, s$, are polynomials in $\theta \in [0, 1]$, and the first two of the stages are $f(t_{i-1}, y_{i-1})$ and $f(t_i, y_i)$ The weight polynomials are determined by Hermite-Birkhoff interpolation. Thus $u_i(t_{i-1} + \theta h)$ interpolates the solution values, y_{i-1} and y_i, at the endpoints, and its derivative interpolates the derivatives at the endpoints, ${y'}_{i-1} = f(t_{i-1}, y_{i-1})$ and ${y'}_i = f(t_i, y_i)$, and the corresponding interpolant will have C^1 continuity.

For the CRK scheme, (5), assuming PIRK stages (7), there will be explicit dependence on y_{i-1} and on the derivative values at the endpoints of the subinterval, and the weight polynomials derived from the order conditions, will lead to a scheme which interpolates the solution value, y_{i-1}, at the left endpoint, and whose derivative interpolates ${y'}_{i-1} = f(t_{i-1}, y_{i-1})$ and ${y'}_i = f(t_i, y_i)$. Since there is no explicit dependence in (5) itself on y_i the conditions for interpolation of y_i are slightly more complicated (see [19]). In this setting they are,

$$b_1(1) = b_2(1) = 0, \quad b_r(1) = \hat{b}_{r-2}, r = 3, \ldots, k+2, \quad b_r(1) = 0, r = k+3, \ldots, s, \tag{16}$$

where $\hat{b}_r = \hat{b}_r(1)$ in (11). Under these conditions, the evaluation of the CRK scheme at the righthand endpoint gives y_i in terms of the (discrete) implicit Runge-Kutta method equivalent to the collocation method. Then the CRK scheme interpolates y_i and the corresponding interpolant will have C^1 continuity.

In subsequent sections we shall demonstrate that although (16) is naturally satisfied by the higher order CRK schemes, some of the lower order schemes do not satisfy it. (This happens because the schemes of the desired order can be obtained without explicit dependence on all of the collocation stages.) However the value from the CRK scheme will agree with the true solution to within a global error of $O(h^{2k})$, as does the discrete collocation solution, y_i. Thus the CRK scheme leads to an interpolant, with a continuous first derivative, that itself is continuous to order $O(h^{2k})$. That is, in general, there may be jump discontinuities in the interpolant, at the mesh points, of size $O(h^{2k})$. However, it is possible to derive related CRK schemes (including explicit dependence on all collocation stages) so that (16) is satisfied. For lower order CRK schemes, we will derive schemes using the minimum number of stages and having continuity to order ($O(h^{2k})$, as well as schemes having an extra stage that allows them to have "exact" continuity. Note that in either case the derivative will be continuous.

4 Collocation with $k = 1$

A collocation scheme with collocation at one Gauss point per subinterval is equivalent to the one stage implicit Runge-Kutta scheme called the midpoint scheme. Its local error is $O(h^3)$; its global error is $O(h^2)$. The associated C^0 collocation polynomial written in CRK form has the tableau

$$\begin{array}{c|c} \frac{1}{2} & \frac{1}{2} \\ \hline & \theta. \end{array}$$

Its local and global error are both $O(h^2)$. Thus, in this case, the collocation polynomial is already "superconvergent".

A two-stage CRK scheme can be employed to provide an interpolant which provides a continuous first derivative (unlike the collocation polynomial above). We construct this scheme using the endpoint stages, $f(t_{i-1}, y_{i-1})$ and $f(t_i, y_i)$. This determines the c, v and x coefficients of the scheme. When the weight polynomials are determined by the continuous Runge-Kutta order conditions up to order two,

$$b(\theta)^T e = \theta, \quad \text{and} \quad b(\theta)^T c = \frac{\theta^2}{2},$$

this scheme has local error $O(h^3)$ and global error is $O(h^2)$. Its tableau is

$$\begin{array}{c|c|cc} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline & & \frac{\theta(2-\theta)}{2} & \frac{\theta^2}{2}. \end{array}$$

This scheme is the continuous version of the trapezoidal scheme. It does not use the collocation stage generated during the calculation of the collocation solution. Therefore the associated interpolant has only $O(h^2)$ continuity although its derivative is continuous.

A C^1 continuous interpolant is obtained from the unique cubic CRK scheme determined such that it interpolates y_{i-1} and its derivative interpolates the two endpoint stages and the collocation stage. Its tableau is

$$\begin{array}{c|c|ccc} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \hline & & \theta(\theta-1)^2 & \theta^2(\theta-1) & \theta^2(3-2\theta). \end{array}$$

This scheme has local error $O(h^3)$ and global error is $O(h^2)$.

Note that since endpoint stages can be shared between adjacent subintervals, the CRK schemes are obtained at the cost of only one extra function evaluation per subinterval.

5 Collocation with $k = 2$

The implicit Runge-Kutta scheme corresponding to collocation at two Gauss points per subinterval is given in [7]. Its local error is $O(h^5)$ and its global error is $O(h^4)$. The stage order of each stage and thus of the whole scheme is 2. The associated C^0 collocation polynomial, written in CRK form, has the tableau,

$$\begin{array}{c|cc}
\frac{1}{2}-\frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4}-\frac{\sqrt{3}}{6} \\
\frac{1}{2}+\frac{\sqrt{3}}{6} & \frac{1}{4}+\frac{\sqrt{3}}{6} & \frac{1}{4} \\
\hline
 & (\frac{1+\sqrt{3}}{2})\theta - \frac{\sqrt{3}}{2}\theta^2 & (\frac{1-\sqrt{3}}{2})\theta + \frac{\sqrt{3}}{2}\theta^2.
\end{array}$$

The local and global errors for this scheme are both $O(h^3)$.

5.1 A superconvergent interpolant with local error $O(h^4)$

We now derive a CRK scheme with local and global errors of $O(h^4)$. The interpolant is based on a CRK scheme whose first two stages are the endpoint stages and whose third stage is one of the collocation stages. This determines the c, v, and x coefficients of the scheme. The three weight polynomials are determined by requiring the scheme to satisfy the continuous order conditions up to order 3, of which there are four. However, since the stage order of the first two stages is 3 and the stage order of the third stage is 2, the stage order of the whole scheme is 2 and the order conditions reduce to the three conditions,

$$b(\theta)^T e = \theta, \quad b(\theta)^T c = \frac{\theta^2}{2}, \quad \text{and} \quad b(\theta)^T c^2 = \frac{\theta^3}{3}. \tag{17}$$

The tableau of the resultant scheme is

$$\begin{array}{c|c|cccc}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{2}-\frac{\sqrt{3}}{6} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4}-\frac{\sqrt{3}}{6} \\
\frac{1}{2}+\frac{\sqrt{3}}{6} & 0 & 0 & 0 & \frac{1}{4}+\frac{\sqrt{3}}{6} & \frac{1}{4} \\
\hline
 & & b_1(\theta) & b_2(\theta) & b_3(\theta) & b_4(\theta)
\end{array}, \tag{18}$$

where

$$b_1(\theta) = \frac{(3+\sqrt{3})}{12}\theta(4\theta^2 - (9-\sqrt{3})\theta + 6 - 2\sqrt{3}),$$

$$b_2(\theta) = \frac{(3-\sqrt{3})}{12}\theta^2(4\theta - 3 + \sqrt{3}), \quad b_3(\theta) = \theta^2(3-2\theta),$$

and $b_4(\theta) \equiv 0$. The scheme is written as a four stage scheme to clarify the representation of the collocation stages. Since the fourth weight is identically zero, the scheme can be implemented as a three stage scheme. However, since both collocation stages are not included, the associated interpolant will have a continuous derivative but itself will have only continuity to $O(h^4)$.

A interpolant with C^1 continuity is obtained by including the second collocation stage in (18), i.e. the $b_4(\theta)$ must not be identically zero. Using the same tableau as in (18), the weight polynomials are determined so that the CRK scheme interpolates y_{i-1}, its derivative interpolates the first two stages (the derivatives at the endpoints), and so that the conditions (16) are satisfied. The weight polynomials are

$$b_1(\theta) = \theta(\theta-1)^2, \quad b_2(\theta) = \theta^2(\theta-1),$$

and

$$b_3(\theta) = b_4(\theta) = \frac{1}{2}\theta^2(3-2\theta).$$

This scheme satisfies the order conditions (17) and has local and global errors of $O(h^4)$.

5.2 A superconvergent interpolant with local error $O(h^5)$

We conclude this section by deriving a C^1 interpolant based on a CRK scheme with global error $O(h^4)$ and local error $O(h^5)$. The first two stages of the scheme will be the endpoint stages and the third and fourth stages will be the collocation stages. Since the stage order of the whole scheme is only 2, the eight continuous order conditions associated with order four can be reduced to five independent conditions. The first four of these are,

$$b(\theta)^T e = \theta, \quad b(\theta)^T c = \frac{\theta^2}{2}, \quad b(\theta)^T c^2 = \frac{\theta^3}{3}, \quad b(\theta)^T c^3 = \frac{\theta^4}{4},$$

which we shall refer to as quadrature conditions. The fifth condition, a non-quadrature condition, is

$$b(\theta)^T \left(Xc^2 + \frac{v}{3} - \frac{c^3}{3} \right) = 0.$$

Since these conditions are independent, the CRK scheme must employ five stages; thus we add a fifth MIRK stage, to which we apply the stage order 4 conditions. The embedding of the endpoint and collocation stages defines the first four components of c and v and the first four rows of x. The application of the stage order 4 conditions to the fifth stage defines x_{51}, x_{52}, x_{53}, and x_{54}, in terms of c_5 and v_5, and since the stage is mono-implicit, $x_{55} = 0$.

The weight polynomials are determined by solving the above order conditions and we obtain a two parameter family of CRK schemes, with c_5 and v_5 as the free parameters, with the restriction that $c_5 \neq 0, \frac{1}{2}, 1$. Since the expressions for the coefficients of the fifth stage and the weight polynomials are somewhat complicated, we do not give them here. Rather we present a specific CRK scheme obtained by choosing $c_5 = v_5 = \frac{2}{5}$. The tableau of the resultant scheme is

$$
\begin{array}{c|c|ccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2}-\frac{\sqrt{3}}{6} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4}-\frac{\sqrt{3}}{6} & 0 \\
\frac{1}{2}+\frac{\sqrt{3}}{6} & 0 & 0 & 0 & \frac{1}{4}+\frac{\sqrt{3}}{6} & \frac{1}{4} & 0 \\
\frac{2}{5} & \frac{2}{5} & \frac{36}{625} & -\frac{6}{625} & -\frac{3}{125}+\frac{54\sqrt{3}}{625} & -\frac{3}{125}-\frac{54\sqrt{3}}{625} & 0 \\
\hline
 & & b_1(\theta) & b_2(\theta) & b_3(\theta) & b_4(\theta) & b_5(\theta),
\end{array}
$$

where

$$b_1(\theta) = \frac{1}{4}\theta(\theta-1)^2(5\theta+4), \quad b_2(\theta) = \frac{1}{3}\theta^2(\theta-1)(10\theta-7),$$

$$b_3(\theta) = -\frac{1}{2}\theta^2(3\theta-2)(5\theta-6), \quad b_4(\theta) = b_3(\theta),$$

$$b_5(\theta) = \frac{125}{12}\theta^2(\theta-1)^2.$$

6 Collocation with $k = 3$

The implicit Runge-Kutta scheme that is equivalent to collocation with three Gauss points is given in [7]. It has a local error of $O(h^7)$ and a global error of $O(h^6)$. Its stage order is 3. The corresponding C^0 collocation polynomial has local and global errors of $O(h^4)$. When it is written in CRK form its tableau is

$$
\begin{array}{c|ccc}
\frac{1}{2}-\frac{\sqrt{15}}{10} & \frac{5}{36} & \frac{2}{9}-\frac{\sqrt{15}}{15} & \frac{5}{36}-\frac{\sqrt{15}}{30} \\
\frac{1}{2} & \frac{5}{36}+\frac{\sqrt{15}}{24} & \frac{2}{9} & \frac{5}{36}-\frac{\sqrt{15}}{24} \\
\frac{1}{2}+\frac{\sqrt{15}}{10} & \frac{5}{36}+\frac{\sqrt{15}}{30} & \frac{2}{9}+\frac{\sqrt{15}}{15} & \frac{5}{36} \\
\hline
 & b_1(\theta) & b_2(\theta) & b_3(\theta),
\end{array}
$$

where

$$b_1(\theta) = \frac{\theta\left(20\,\theta^2 - 30\,\theta - 3\,\theta\sqrt{15} + 15 + 3\sqrt{15}\right)}{18},$$

$$b_2(\theta) = -\frac{2\,\theta\left(10\,\theta^2 - 15\,\theta + 3\right)}{9},$$

and

$$b_3(\theta) = \frac{\theta\left(20\,\theta^2 - 30\,\theta + 3\,\theta\sqrt{15} + 15 - 3\,\sqrt{15}\right)}{18}.$$

6.1 A superconvergent interpolant with local error $O(h^6)$

The first of the two superconvergent interpolants we will derive is based on a CRK scheme with local and global errors of $O(h^6)$. The scheme must satisfy the continuous order conditions up to order 5. Since the scheme will include the three collocation stages, its stage order will be 3. This leads to simplifications in the continuous order conditions; the seventeen order conditions for order 5 reduce to the following six independent conditions; the five quadrature conditions,

$$b(\theta)^T e = \theta, \quad b(\theta)^T c = \frac{\theta^2}{2}, \quad b(\theta)^T c^2 = \frac{\theta^3}{3},$$

$$b(\theta)^T c^3 = \frac{\theta^4}{4}, \quad b(\theta)^T c^4 = \frac{\theta^5}{5},$$

and the nonquadrature condition,

$$b(\theta)^T \left(X c^3 + \frac{v}{4} - \frac{c^4}{4} \right) = 0.$$

In addition to the two endpoint stages and the three collocation stages, we will need one extra MIRK stage, which we will require to have stage order 5. The nonquadrature condition, after some simplification, can be shown to lead to the requirement that

$$b_5(\theta) = \frac{5}{4} b_4(\theta) - b_3(\theta).$$

This leaves $b_1(\theta)$, $b_2(\theta)$, $b_3(\theta)$, $b_4(\theta)$, and $b_6(\theta)$ to satisfy the remaining five quadrature conditions.

The two endpoint stages and the three collocation stages define the first five components of c and v and the first five rows of x. The imposition of stage order 5 on the 6th stage gives expressions for $x_{61}, \ldots, x_{65}$ in terms of c_6 and v_6 and since it is a MIRK stage, $x_{66} = 0$. The solution of the order conditions gives expressions for $b_1(\theta)$, $b_2(\theta)$, $b_3(\theta)$, $b_4(\theta)$, and $b_6(\theta)$ in terms of the free parameters c_6 and v_6 with the restrictions that $c_6 \neq 0, 1$, or $\frac{1}{2} \pm \frac{\sqrt{5}}{10}$. Since the expressions for x_{61}, x_{62}, x_{63}, x_{64}, and x_{65}, and for the weight polynomials are somewhat complicated we do not present them here. For the particular choice of $c_6 = v_6 = \frac{3}{5}$, we get the CRK scheme

$$
\begin{array}{c|c|cccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2}-\frac{\sqrt{15}}{10} & 0 & 0 & 0 & \frac{5}{36} & \frac{2}{9}-\frac{\sqrt{15}}{15} & \frac{5}{36}-\frac{\sqrt{15}}{30} & 0 \\
\frac{1}{2} & 0 & 0 & 0 & \frac{5}{36}+\frac{\sqrt{15}}{24} & \frac{2}{9} & \frac{5}{36}-\frac{\sqrt{15}}{24} & 0 \\
\frac{1}{2}+\frac{\sqrt{15}}{10} & 0 & 0 & 0 & \frac{5}{36}+\frac{\sqrt{15}}{30} & \frac{2}{9}+\frac{\sqrt{15}}{15} & \frac{5}{36} & 0 \\
\frac{3}{5} & \frac{3}{5} & -\frac{78}{3125} & \frac{72}{3125} & \frac{6\sqrt{15}}{125}-\frac{47}{1875} & \frac{488}{9375} & -\frac{6\sqrt{15}}{125}-\frac{47}{1875} & 0 \\
\hline
 & & b_1(\theta) & b_2(\theta) & b_3(\theta) & b_4(\theta) & b_5(\theta) & b_6(\theta)
\end{array}
$$

where

$$b_1(\theta) = \frac{\theta\,(\theta-1)^2\,(5\,\theta^2-10\,\theta+3)}{3}, \quad b_2(\theta) = \frac{\theta^2\,(\theta-1)\,(30\,\theta^2-35\,\theta+9)}{4},$$

$$b_3(\theta) = \frac{(-5+\sqrt{15})\,(60\,\theta^3-15\,\sqrt{15}\theta^2-195\,\theta^2+220\,\theta+32\,\theta\sqrt{15}-18\,\sqrt{15}-90)\,\theta^2}{36},$$

$$b_4(\theta) = -\frac{4\,\theta^2\,(-18+62\,\theta-75\,\theta^2+30\,\theta^3)}{9},$$

$$b_5(\theta) = -\frac{(5+\sqrt{15})\,(60\,\theta^3-195\,\theta^2+15\,\sqrt{15}\theta^2-32\,\theta\sqrt{15}+220\,\theta+18\,\sqrt{15}-90)\,\theta^2}{36},$$

and

$$b_6(\theta) = \frac{125\,\theta^2\,(2\,\theta-1)\,(\theta-1)^2}{12}.$$

6.2 A superconvergent interpolant with local error $O(h^7)$

We conclude this section by considering a superconvergent C^1 interpolant based on a CRK scheme with a local error of $O(h^7)$ and a global error of $O(h^6)$. Its stage order is 3 and it must satisfy ten continuous order conditions; the six quadrature conditions,

$$b(\theta)^T e = \theta, \quad b(\theta)^T c = \frac{\theta^2}{2}, \quad b(\theta)^T c^2 = \frac{\theta^3}{3},$$

$$b(\theta)^T c^3 = \frac{\theta^4}{4}, \quad b(\theta)^T c^4 = \frac{\theta^5}{5}, \quad b(\theta)^T c^5 = \frac{\theta^6}{6},$$

and the four nonquadrature conditions,

$$b(\theta)^T \left(X c^3 + \frac{v}{4} - \frac{c^4}{4}\right) = 0, \quad b(\theta)^T c \left(X c^3 + \frac{v}{4} - \frac{c^4}{4}\right) = 0,$$

$$b(\theta)^T \left(Xc^4 + \frac{v}{5} - \frac{c^5}{5} \right) = 0, \quad b(\theta)^T \left(X \left(Xc^3 + \frac{v}{4} \right) + \frac{v}{20} - \frac{c^5}{20} \right) = 0.$$

In the analysis to follow, we will show that it is possible to satisfy the ten order conditions using a CRK scheme with eight stages. The resultant scheme will contain the endpoint stages and collocation stages embedded within it and will include three extra MIRK stages assumed to have at least stage order 5.

We continue with our analysis of the order conditions, following a strategy similar to that of [15]. We focus on the four nonquadrature conditions. Since all stages except the third, fourth, and fifth have at least stage order 5, the first three nonquadrature conditions reduce to

$$b_3(\theta) \left(\frac{1}{400} \right) + b_4(\theta) \left(-\frac{1}{320} \right) + b_5(\theta) \left(\frac{1}{400} \right) = 0,$$

$$b_3(\theta) \left(\frac{1}{800} - \frac{\sqrt{15}}{4000} \right) + b_4(\theta) \left(-\frac{1}{640} \right) + b_5(\theta) \left(\frac{1}{800} + \frac{\sqrt{15}}{4000} \right) = 0,$$

$$b_3(\theta) \left(\frac{1}{200} - \frac{3\sqrt{15}}{10000} \right) + b_4(\theta) \left(-\frac{1}{160} \right) + b_5(\theta) \left(\frac{1}{200} + \frac{3\sqrt{15}}{10000} \right) = 0.$$

These are all satisfied provided

$$b_4(\theta) = \frac{8}{5} b_3(\theta) \quad \text{and} \quad b_5(\theta) = b_3(\theta).$$

We now return to the last of the nonquadrature conditions, namely,

$$b(\theta)^T \left(X \left(Xc^3 + \frac{v}{4} \right) + \frac{v}{20} - \frac{c^5}{20} \right) = 0.$$

Since all stages except the third, fourth, and fifth are chosen to have at least stage order 5, this condition reduces to

$$b_3(\theta) \left(\frac{\sqrt{15}}{8000} \right) + b_5(\theta) \left(-\frac{\sqrt{15}}{8000} \right) + b_6(\theta) \left(\frac{x_{63}}{400} - \frac{x_{64}}{320} + \frac{x_{65}}{400} \right) +$$

$$b_7(\theta) \left(\frac{x_{73}}{400} - \frac{x_{74}}{320} + \frac{x_{75}}{400} \right) + b_8(\theta) \left(\frac{x_{83}}{400} - \frac{x_{84}}{320} + \frac{x_{85}}{400} \right) = 0.$$

Since $b_3(\theta) = b_5(\theta)$, this condition is satisfied if we choose,

$$x_{r5} = \frac{5}{4} x_{r4} - x_{r3}, \quad r = 6, 7, 8. \tag{19}$$

Then the last nonquadrature condition is satisfied, and we can determine the six weight polynomials, $b_1(\theta)$, $b_2(\theta)$, $b_3(\theta)$, $b_6(\theta)$, $b_7(\theta)$, $b_8(\theta)$, by requiring them to satisfy the remaining six quadrature conditions.

The above strategy requires us to be able to impose the stage order 5 conditions as well as the conditions, (19), on the last three stages. For the 6th stage, these requirements force $c_6 = \frac{1}{2}$ and the parameter v_6 is left free. When we apply the stage order 5 conditions plus (19) to the 7th stage, we are left with c_7 and v_7 as free parameters. For the 8th stage, we apply stage order 6 (there are sufficient free parameters to allow higher stage order to be imposed) and (19), which leaves c_8 and v_8 free.

The weight polynomials are expressed in terms of the free parameters, v_6, c_7, v_7, c_8, v_8, which must satisfy the restrictions,

$$c_7 \neq 0, 1, \frac{1}{2}, c_8, 1 - c_8, \frac{1}{2} \pm \frac{\sqrt{15}}{10}, \quad c_8 \neq 0, 1, \frac{1}{2}. \tag{20}$$

Since the expressions for the coefficients of the last three stages and for the weight polynomials are somewhat complicated we do not give them here. For the particular choice of the free parameters, $v_6 = \frac{1}{2}, c_7 = \frac{1}{3}, v_7 = \frac{1}{3}, c_8 = \frac{2}{3}, v_8 = \frac{2}{3}$, we get the scheme with tableau

$$\begin{array}{c|c|cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} - \frac{\sqrt{15}}{10} & 0 & 0 & 0 & \frac{5}{36} & \frac{2}{9} - \frac{\sqrt{15}}{15} & \frac{5}{36} - \frac{\sqrt{15}}{30} & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & \frac{5}{36} + \frac{\sqrt{15}}{24} & \frac{2}{9} & \frac{5}{36} - \frac{\sqrt{15}}{24} & 0 & 0 & 0 \\
\frac{1}{2} + \frac{\sqrt{15}}{10} & 0 & 0 & 0 & \frac{5}{36} + \frac{\sqrt{15}}{30} & \frac{2}{9} + \frac{\sqrt{15}}{15} & \frac{5}{36} & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & x_{71} & x_{72} & x_{73} & x_{74} & x_{75} & x_{76} & 0 & 0 \\
\frac{3}{5} & \frac{3}{5} & x_{81} & x_{82} & x_{83} & x_{84} & x_{85} & x_{86} & x_{87} & 0 \\
\hline
 & & b_1(\theta) & b_2(\theta) & b_3(\theta) & b_4(\theta) & b_5(\theta) & b_6(\theta) & b_7(\theta) & b_8(\theta)
\end{array}$$

where

$$x_{61} = -\frac{1}{32}, \quad x_{62} = \frac{1}{32}, \quad x_{63} = \frac{5\sqrt{15}}{96}, \quad x_{64} = 0, \quad x_{65} = -\frac{5\sqrt{15}}{96},$$

$$x_{71} = -\frac{2}{243}, \quad x_{72} = \frac{4}{243}, \quad x_{73} = \frac{25}{729} + \frac{10\sqrt{15}}{243},$$

$$x_{74} = \frac{40}{729}, \quad x_{75} = \frac{25}{729} - \frac{10\sqrt{15}}{243}, \quad x_{76} = -\frac{32}{243},$$

$$x_{81} = -\frac{1038}{15625}, \quad x_{82} = \frac{684}{15625}, \quad x_{83} = \frac{131}{20625} + \frac{2298\sqrt{15}}{34375},$$

$$x_{84} = \frac{1048}{103125}, \quad x_{85} = \frac{131}{20625} - \frac{2298\sqrt{15}}{34375}, \quad x_{86} = \frac{26208}{171875}, \quad x_{87} = -\frac{26244}{171875},$$

and the weight polynomials are,

$$b_1(\theta) = \frac{\theta(\theta-1)^2(75\theta^3 - 63\theta^2 + 17\theta + 3)}{3},$$

$$b_2(\theta) = \frac{\theta^2(\theta-1)(125\theta^3 - 234\theta^2 + 149\theta + 36)}{4},$$

$$b_3(\theta) = -\frac{5}{18}\theta^2(300\theta^4 - 876\theta^3 + 945\theta^2 - 460\theta + 90),$$

$$b_4(\theta) = \frac{8}{5}b_3(\theta), \quad b_5(\theta) = b_3(\theta),$$

$$b_6(\theta) = -16\theta^2(\theta-1)^2(25\theta^2 - 26\theta + 9), \quad b_7(\theta) = \frac{81}{8}\theta^2(\theta-1)^2(25\theta^2 - 25\theta + 9),$$

$$b_8(\theta) = \frac{3125}{24}\theta^2(\theta-1)^2(3\theta^2 - 3\theta + 1).$$

7 Collocation with $k = 4$

In this section we consider collocation at four Gauss points per subinterval. The equivalent implicit Runge-Kutta scheme is given in [7]. This scheme has a local error of $O(h^9)$ and a global error of $O(h^8)$. The corresponding C^0 collocation polynomial written in CRK form has the tableau

$$\begin{array}{c|cccc}
\frac{1}{2} - w_2 & w_1 & w'_1 - w_3 + w'_4 & w'_1 - w_3 - w'_4 & w_1 - w_5 \\
\frac{1}{2} - w'_2 & w_1 - w'_3 + w_4 & w'_1 & w'_1 - w'_5 & w_1 - w'_3 - w_4 \\
\frac{1}{2} + w'_2 & w_1 + w'_3 + w_4 & w'_1 + w'_5 & w'_1 & w_1 + w'_3 - w_4 \\
\frac{1}{2} + w_2 & w_1 + w_5 & w'_1 + w_3 + w'_4 & w'_1 + w_3 - w'_4 & w_1 \\
\hline
 & b_1(\theta) & b_2(\theta) & b_3(\theta) & b_4(\theta)
\end{array},$$

where,

$$w_1 = \frac{1}{8} - \frac{\sqrt{30}}{144}, \quad w'_1 = \frac{1}{8} + \frac{\sqrt{30}}{144}, \quad w_2 = \frac{\sqrt{525 + 70\sqrt{30}}}{70}, \tag{21}$$

$$w'_2 = \frac{\sqrt{525 - 70\sqrt{30}}}{70}, \quad w_3 = \frac{(4 + \sqrt{30})\sqrt{525 + 70\sqrt{30}}}{1680}, \tag{22}$$

$$w'_3 = -\frac{(-4 + \sqrt{30})\sqrt{525 - 70\sqrt{30}}}{1680}, \quad w_4 = \frac{(8 + 5\sqrt{30})\sqrt{525 + 70\sqrt{30}}}{11760}, \tag{23}$$

$$w'_4 = -\frac{\left(-8+5\sqrt{30}\right)\sqrt{525-70\sqrt{30}}}{11760}, \tag{24}$$

$$w_5 = \frac{\sqrt{525+70\sqrt{30}}}{105} - \frac{\sqrt{30}\sqrt{525+70\sqrt{30}}}{840}, \tag{25}$$

$$w'_5 = \frac{\sqrt{525-70\sqrt{30}}}{105} + \frac{\sqrt{30}\sqrt{525-70\sqrt{30}}}{840}, \tag{26}$$

and the weight polynomials are

$$b_1(\theta) = -\frac{1}{88200}\alpha_1\left(\sqrt{30}-4\right)\theta(3675\,\theta^3 - 7350\,\theta^2 - 70\,\theta^2\alpha_1 + 4725\,\theta + 105\,\theta\alpha_1 + 105\,\theta\sqrt{30} - 1050 - 105\sqrt{30} - 30\,\alpha_1 - 3\,\alpha_1\sqrt{30}),$$

$$b_2(\theta) = \frac{1}{88200}\alpha_2\left(\sqrt{30}+4\right)\theta(3675\,\theta^3 - 7350\,\theta^2 - 70\,\theta^2\alpha_2 + 4725\,\theta - 105\,\theta\sqrt{30} + 105\,\theta\alpha_2 - 1050 - 30\,\alpha_2 + 105\sqrt{30} + 3\,\alpha_2\sqrt{30}),$$

$$b_3(\theta) = -\frac{1}{88200}\alpha_2\left(\sqrt{30}+4\right)\theta(3675\,\theta^3 - 7350\,\theta^2 + 70\,\theta^2\alpha_2 + 4725\,\theta - 105\,\theta\sqrt{30} - 105\,\theta\alpha_2 - 1050 + 30\,\alpha_2 + 105\sqrt{30} - 3\,\alpha_2\sqrt{30}),$$

$$b_4(\theta) = \frac{1}{88200}\alpha_1\left(\sqrt{30}-4\right)\theta(3675\,\theta^3 - 7350\,\theta^2 + 70\,\theta^2\alpha_1 + 4725\,\theta - 105\,\theta\alpha_1 + 105\,\theta\sqrt{30} - 1050 - 105\sqrt{30} + 30\,\alpha_1 + 3\,\alpha_1\sqrt{30}),$$

where

$$\alpha_1 = \sqrt{525+70\sqrt{30}} \quad \text{and} \quad \alpha_2 = \sqrt{525-70\sqrt{30}}.$$

The local and global errors of this scheme are $O(h^5)$ and its stage order is 4.

7.1 A superconvergent interpolant with local error $O(h^8)$

We next derive a C^1 interpolant based on a CRK scheme having local and global errors of $O(h^8)$. We will assume the two endpoint stages and the four collocation stages are to be embedded and that any extra stages are required to have at least stage order four so that the stage order of the whole scheme will be four. With this latter assumption, the desired scheme must satisfy eleven order conditions; the seven quadrature conditions,

$$b(\theta)^T e = \theta, \quad b(\theta)^T c = \frac{\theta^2}{2}, \quad b(\theta)^T c^2 = \frac{\theta^3}{3}, \quad b(\theta)^T c^3 = \frac{\theta^4}{4}, \tag{27}$$

$$b(\theta)^T c^4 = \frac{\theta^5}{5}, \quad b(\theta)^T c^5 = \frac{\theta^6}{6}, \quad b(\theta)^T c^6 = \frac{\theta^7}{7}, \tag{28}$$

and the four nonquadrature conditions,

$$b(\theta)^T\left(Xc^4+\frac{v}{5}-\frac{c^5}{5}\right)=0,\quad b(\theta)^T\left(Xc^5+\frac{v}{6}-\frac{c^6}{6}\right)=0, \tag{29}$$

$$b(\theta)^T c\left(Xc^4+\frac{v}{5}-\frac{c^5}{5}\right)=0,\quad b(\theta)^T X\left(Xc^4+\frac{v}{5}-\frac{c^5}{5}\right)=0. \tag{30}$$

In order to simplify the analysis, we make the further assumption that any new stages will have stage order six. We now focus on the first three of the nonquadrature conditions. Since all stages except the third through sixth have stage order six, these conditions simplify to

$$\sum_{j=3}^{6} b_j(\theta)\left[Xc^4+\frac{v}{5}-\frac{c^5}{5}\right]_j=0, \tag{31}$$

$$\sum_{j=3}^{6} b_j(\theta)\left[Xc^5+\frac{v}{6}-\frac{c^6}{6}\right]_j=0, \tag{32}$$

$$\sum_{j=3}^{6} b_j(\theta)c_j\left[Xc^4+\frac{v}{5}-\frac{c^5}{5}\right]_j=0, \tag{33}$$

where the notation $[w]_j$, for a vector, w, means the jth component of w. In the above equations, (31), (32), (33),

$$\left[Xc^4+\frac{v}{5}-\frac{c^5}{5}\right]_3=\left(\frac{3\alpha_1}{171500}-\frac{\sqrt{30}\alpha_1}{171500}\right),$$

$$\left[Xc^4+\frac{v}{5}-\frac{c^5}{5}\right]_4=\left(\frac{3\alpha_2}{171500}+\frac{\sqrt{30}\alpha_2}{171500}\right),$$

$$\left[Xc^4+\frac{v}{5}-\frac{c^5}{5}\right]_5=\left(-\frac{3\alpha_2}{171500}-\frac{\sqrt{30}\alpha_2}{171500}\right),$$

$$\left[Xc^4+\frac{v}{5}-\frac{c^5}{5}\right]_6=\left(-\frac{3\alpha_1}{171500}+\frac{\sqrt{30}\alpha_1}{171500}\right),$$

$$\left[Xc^5+\frac{v}{6}-\frac{c^6}{6}\right]_3=\left(\frac{3\alpha_1}{68600}+\frac{\sqrt{30}}{34300}-\frac{\sqrt{30}\alpha_1}{68600}+\frac{1}{20580}\right),$$

$$\left[Xc^5+\frac{v}{6}-\frac{c^6}{6}\right]_4=\left(\frac{3\alpha_2}{68600}-\frac{\sqrt{30}}{34300}+\frac{\sqrt{30}\alpha_2}{68600}+\frac{1}{20580}\right),$$

$$\left[Xc^5+\frac{v}{6}-\frac{c^6}{6}\right]_5=\left(-\frac{3\alpha_2}{68600}-\frac{\sqrt{30}}{34300}-\frac{\sqrt{30}\alpha_2}{68600}+\frac{1}{20580}\right),$$

and

$$\left[Xc^5+\frac{v}{6}-\frac{c^6}{6}\right]_6=\left(-\frac{3\alpha_1}{68600}+\frac{\sqrt{30}}{34300}+\frac{\sqrt{30}\alpha_1}{68600}+\frac{1}{20580}\right).$$

With these substitutions, the conditions (31), (32), (33), can be seen to be dependent; it is possible to satisfy them by solving for $b_5(\theta)$ and $b_6(\theta)$ in terms of $b_3(\theta)$ and $b_4(\theta)$. We get

$$b_5(\theta)=\left(\frac{2}{7}-\frac{5\sqrt{30}}{28}-\frac{3\alpha_1\alpha_2}{1715}-\frac{59\sqrt{30}\alpha_1\alpha_2}{102900}\right)b_3(\theta)+$$

$$\left(\frac{\sqrt{30}\alpha_1\alpha_2}{2100}+\frac{\sqrt{30}}{4}\right)b_4(\theta), \tag{34}$$

$$b_6(\theta)=\left(\frac{2}{7}+\frac{5\sqrt{30}}{28}-\frac{3\alpha_1\alpha_2}{1715}+\frac{59\sqrt{30}\alpha_1\alpha_2}{102900}\right)b_4(\theta)$$

$$-\left(\frac{\sqrt{30}\alpha_1\alpha_2}{2100}+\frac{\sqrt{30}}{4}\right)b_3(\theta). \tag{35}$$

We next consider the last of the nonquadrature conditions, namely,

$$b(\theta)^TX\left(Xc^4+\frac{v}{5}-\frac{c^5}{5}\right)=0.$$

Since all stages except the third through sixth have stage order six, this condition simplifies substantially. Due to the zero structure of X, the coefficients of $b_1(\theta)$ and $b_2(\theta)$ are both zero. The coefficients of $b_3(\theta)$, $b_4(\theta)$, $b_5(\theta)$, and $b_6(\theta)$ have the form,

$$\sum_{j=3}^{6}x_{rj}\left[Xc^5+\frac{v}{6}-\frac{c^6}{6}\right]_j,\quad r=3,\ldots,6.$$

Upon simplification of these expressions the part of this order condition involving the third through sixth weight polynomials reduces to,

$$b_3(\theta)\left(-\frac{1}{68600}-\frac{3\sqrt{30}}{343000}\right)+b_4(\theta)\left(-\frac{1}{68600}+\frac{3\sqrt{30}}{343000}\right)+$$

$$b_5(\theta)\left(-\frac{1}{68600}+\frac{3\sqrt{30}}{343000}\right)+b_6(\theta)\left(-\frac{1}{68600}-\frac{3\sqrt{30}}{343000}\right).$$

Upon substitution of (34) and (35) for $b_5(\theta)$ and $b_6(\theta)$, this expression further reduces to 0. This leaves only the coefficients of the weight polynomials from $b_7(\theta)$ through $b_s(\theta)$ to be considered, (where s is the total number of stages of the scheme). Each such coefficient has the form

$$\sum_{j=3}^{6} x_{rj}\left[Xc^5+\frac{v}{6}-\frac{c^6}{6}\right]_j, \quad r=7,\ldots,s.$$

We will require each of these coefficients to be 0. Assuming this is possible, it will follow that the last order condition can then be ignored since the sum of the terms involving $b_3(\theta)$ through $b_6(\theta)$ is zero, and the coefficient of every other weight polynomial is zero.

The implication of the above analysis is that with $b_5(\theta)$ and $b_6(\theta)$ expressed in terms of $b_3(\theta)$ and $b_4(\theta)$, the remaining weight polynomials can be determined by the remaining seven independent quadrature order conditions. Thus nine weight polynomials will be required, and the CRK scheme will require nine stages altogether and thus three extra MIRK stages. The three new stages will have to satisfy the stage order conditions up to order 6 and the additional requirement

$$\sum_{j=3}^{6} x_{rj}\left[Xc^5+\frac{v}{6}-\frac{c^6}{6}\right]_j = 0, \quad r=7,8,9.$$

For the seventh stage this forces $c_7=\frac{1}{2}\pm\frac{\sqrt{7}}{14}$ (we choose $c_7=\frac{1}{2}+\frac{\sqrt{7}}{14}$) and leaves v_7 free. For the eighth and ninth stages we can get expressions for the coefficients of each stage in terms of the free coefficients c_8 and v_8, and c_9, v_9, and x_{98}, respectively. It is then possible to determine the weight polynomials from the remaining order conditions. The expressions for the coefficients and weight polynomials are somewhat complicated and are not given here.

We conclude this case by presenting a specific CRK scheme obtained by setting $c_8=\frac{1}{5}$ and $c_9=\frac{4}{5}$, $v_8=c_8$, $v_9=c_9$, and $x_{98}=0$. The tableau is

$$\begin{array}{c|c|ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
c_3 & 0 & 0 & 0 & x_{33} & x_{34} & x_{35} & x_{36} & 0 & 0 & 0\\
c_4 & 0 & 0 & 0 & x_{43} & x_{44} & x_{45} & x_{46} & 0 & 0 & 0\\
c_5 & 0 & 0 & 0 & x_{53} & x_{54} & x_{55} & x_{56} & 0 & 0 & 0\\
c_6 & 0 & 0 & 0 & x_{63} & x_{64} & x_{65} & x_{66} & 0 & 0 & 0\\
c_7 & v_7 & x_{71} & x_{72} & x_{73} & x_{74} & x_{75} & x_{76} & 0 & 0 & 0\\
\frac{1}{5} & \frac{1}{5} & x_{81} & x_{82} & x_{83} & x_{84} & x_{85} & x_{86} & x_{87} & 0 & 0\\
\frac{4}{5} & \frac{4}{5} & x_{91} & x_{92} & x_{93} & x_{94} & x_{95} & x_{96} & x_{97} & 0 & 0\\
\hline
 & & b_1(\theta) & b_2(\theta) & b_3(\theta) & b_4(\theta) & b_5(\theta) & b_6(\theta) & b_7(\theta) & b_8(\theta) & b_9(\theta)
\end{array}$$

where

$$c_3 = \frac{1}{2} - w_2, \quad c_4 = \frac{1}{2} - {w'}_2, \quad c_5 = \frac{1}{2} + {w'}_2, \quad c_6 = \frac{1}{2} + w_2, \tag{36}$$

$$x_{33} = w_1, \quad x_{34} = {w'}_1 - w_3 + {w'}_4, \quad x_{35} = {w'}_1 - w_3 - {w'}_4, \quad x_{36} = w_1 - w_5, \tag{37}$$

$$x_{43} = w_1 - {w'}_3 + w_4, \quad x_{44} = {w'}_1, \quad x_{45} = {w'}_1 - {w'}_5, \quad x_{46} = w_1 - {w'}_3 - w_4, \tag{38}$$

$$x_{53} = w_1 + {w'}_3 + w_4, \quad x_{54} = {w'}_1 + {w'}_5, \quad x_{55} = {w'}_1, \quad x_{56} = w_1 + {w'}_3 - w_4, \tag{39}$$

$$x_{63} = w_1 + w_5, \quad x_{64} = {w'}_1 + w_3 + {w'}_4, \quad x_{65} = {w'}_1 + w_3 - {w'}_4, \quad x_{66} = w_1, \tag{40}$$

where the values $w_1, w_2, w_3, w_4, w_5, {w'}_1, {w'}_2, {w'}_3, {w'}_4$, and ${w'}_5$ are given by equations (21)-(26), and

$$c_7 = v_7 = \frac{1}{2} + \frac{\sqrt{7}}{14}, \quad x_{71} = \frac{3\sqrt{7}}{392} - \frac{3}{392}, \quad x_{72} = \frac{3\sqrt{7}}{392} + \frac{3}{392},$$

$$x_{73} = -\frac{5\sqrt{30}\sqrt{7}}{1176} - \frac{3\sqrt{30}\alpha_1}{10976} - \frac{3\sqrt{7}}{784} + \frac{27\alpha_1}{5488},$$

$$x_{74} = \frac{5\sqrt{30}\sqrt{7}}{1176} + \frac{3\sqrt{30}\alpha_2}{10976} - \frac{3\sqrt{7}}{784} + \frac{27\alpha_2}{5488},$$

$$x_{75} = \frac{5\sqrt{30}\sqrt{7}}{1176} - \frac{3\sqrt{30}\alpha_2}{10976} - \frac{3\sqrt{7}}{784} - \frac{27\alpha_2}{5488},$$

$$x_{76} = -\frac{5\sqrt{30}\sqrt{7}}{1176} + \frac{3\sqrt{30}\alpha_1}{10976} - \frac{3\sqrt{7}}{784} - \frac{27\alpha_1}{5488},$$

$$x_{81} = -\frac{992}{140625} + \frac{152\sqrt{7}}{140625}, \quad x_{82} = -\frac{628}{140625} + \frac{152\sqrt{7}}{140625},$$

$$x_{83} = \frac{452\,\alpha_1}{109375} + \frac{193\sqrt{30}}{18750} + \frac{152\sqrt{30}\sqrt{7}}{140625} - \frac{76\sqrt{7}}{9375} - \frac{226\sqrt{30}\,\alpha_1}{984375} + \frac{9}{3125},$$

$$x_{84} = \frac{452\,\alpha_2}{109375} - \frac{193\sqrt{30}}{18750} - \frac{152\sqrt{30}\sqrt{7}}{140625} - \frac{76\sqrt{7}}{9375} + \frac{226\sqrt{30}\,\alpha_2}{984375} + \frac{9}{3125},$$

$$x_{85} = -\frac{452\,\alpha_2}{109375} - \frac{193\sqrt{30}}{18750} - \frac{152\sqrt{30}\sqrt{7}}{140625} - \frac{76\sqrt{7}}{9375} - \frac{226\sqrt{30}\,\alpha_2}{984375} + \frac{9}{3125},$$

$$x_{86} = -\frac{452\,\alpha_1}{109375} + \frac{193\sqrt{30}}{18750} + \frac{152\sqrt{30}\sqrt{7}}{140625} - \frac{76\sqrt{7}}{9375} + \frac{226\sqrt{30}\,\alpha_1}{984375} + \frac{9}{3125},$$

$$x_{87} = \frac{4256\sqrt{7}}{140625}, \quad x_{91} = \frac{628}{140625} + \frac{152\sqrt{7}}{140625}, \quad x_{92} = \frac{992}{140625} + \frac{152\sqrt{7}}{140625},$$

$$x_{93} = -\frac{9}{3125} + \frac{452\,\alpha_1}{109375} - \frac{193\sqrt{30}}{18750} - \frac{76\sqrt{7}}{9375} - \frac{226\sqrt{30}\alpha_1}{984375} + \frac{152\sqrt{30}\sqrt{7}}{140625},$$

$$x_{94} = -\frac{9}{3125} + \frac{452\,\alpha_2}{109375} + \frac{193\sqrt{30}}{18750} - \frac{76\sqrt{7}}{9375} + \frac{226\sqrt{30}\alpha_2}{984375} - \frac{152\sqrt{30}\sqrt{7}}{140625},$$

$$x_{95} = -\frac{9}{3125} - \frac{452\,\alpha_2}{109375} + \frac{193\,\sqrt{30}}{18750} - \frac{76\,\sqrt{7}}{9375} - \frac{226\,\sqrt{30}\alpha_2}{984375} - \frac{152\,\sqrt{30}\sqrt{7}}{140625},$$
$$x_{96} = -\frac{9}{3125} - \frac{452\,\alpha_1}{109375} - \frac{193\,\sqrt{30}}{18750} - \frac{76\,\sqrt{7}}{9375} + \frac{226\,\sqrt{30}\alpha_1}{984375} + \frac{152\,\sqrt{30}\sqrt{7}}{140625},$$
$$x_{97} = \frac{4256\,\sqrt{7}}{140625}.$$

The expressions for the weight polynomials, even with the use of specific values for the free parameters, are very complicated. Thus we provide them in a simpler form by reducing the coefficients to 16 significant digits. This gives,

$$b_1(\theta) = 118.2155391766658t^7 - 450.2127204516691t^6 + 677.9074502047085t^5$$
$$-506.6326577159307t^4 + 190.3161507410786t^3 - 30.59376195485317t^2 + 1.0t,$$
$$b_2(\theta) = -60.92387250999991t^7 + 249.6918871183337t^6 - 401.4282835380396t^5$$
$$+316.7368243825975t^4 - 122.3369840744121t^3 + 18.26042862151984t^2,$$
$$b_3(\theta) = -312.8902724924551t^7 + 1182.484831564545t^6 - 1760.013679899522t^5$$
$$+1290.052895837388t^4 - 465.9005706172412t^3 + 66.44072302985349t^2,$$
$$b_4(\theta) = -305.5922221060167t^7 + 1134.240232880773t^6 - 1652.285224203485t^5$$
$$+1181.297756609701t^4 - 415.3918067568098t^3 + 58.05733615326755t^2,$$
$$b_5(\theta) = 61.0377890325620t^7 - 278.2997171236728t^6 + 497.0100823644751t^5$$
$$-432.9611914049309t^4 + 182.2499138934502t^3 - 28.71080418445299t^2,$$
$$b_6(\theta) = 182.4447055659099t^7 - 725.9253473216322t^6 + 1143.788821738517t^5$$
$$-890.8894610421607t^4 + 341.5424634806017t^3 - 50.78725499866808t^2,$$
$$b_7(\theta) = 214.9122807017543t^7 - 752.1929824561483t^6 + 1032.438596491236t^5$$
$$-700.6140350877189t^4 + 238.1228070175432t^3 - 32.66666666666666t^2,$$
$$b_8(\theta) = 429.1380138467125t^7 - 1603.256196611661t^6 + 2349.773462964071t^5$$
$$-1685.448258473616t^4 + 588.9108452680820t^3 - 79.11786699358911t^2,$$
$$b_9(\theta) = -326.3419612151336t^7 + 1243.470012401131t^6 - 1887.191226121963t^5$$
$$+1428.458126894669t^4 - 537.5128189522927t^3 + 79.11786699358910t^2.$$

7.2 A superconvergent interpolant with local error $O(h^9)$

We conclude this section by deriving a C^1 interpolant based on a CRK scheme having local error $O(h^9)$ and global error $O(h^8)$. This scheme will include the two endpoint stages and the four collocation stages. We will choose any extra required stages to have stage order 7; this reduces the complexity of the order conditions that must be satisfied in order to derive the CRK scheme. With this assumption the stage order of the CRK scheme will be 4 and it must satisfy 19 order conditions; the eight quadrature conditions,

$$b(\theta)^T e = \theta, \quad b(\theta)^T c = \frac{\theta^2}{2}, \quad b(\theta)^T c^2 = \frac{\theta^3}{3}, \quad b(\theta)^T c^3 = \frac{\theta^4}{4},$$

$$b(\theta)^T c^4 = \frac{\theta^5}{5}, \quad b(\theta)^T c^5 = \frac{\theta^6}{6}, \quad b(\theta)^T c^6 = \frac{\theta^7}{7}, \quad b(\theta)^T c^7 = \frac{\theta^8}{8},$$

and the eleven nonquadrature conditions,

$$b(\theta)^T \left(Xc^4 + \frac{v}{5} - \frac{c^5}{5}\right) = 0, \quad b(\theta)^T \left(Xc^5 + \frac{v}{6} - \frac{c^6}{6}\right) = 0,$$

$$b(\theta)^T c \left(Xc^4 + \frac{v}{5} - \frac{c^5}{5}\right) = 0, \quad b(\theta)^T X \left(Xc^4 + \frac{v}{5} - \frac{c^5}{5}\right) = 0.$$

$$b(\theta)^T \left(Xc^6 + \frac{v}{7} - \frac{c^7}{7}\right) = 0, \quad b(\theta)^T c \left(Xc^5 + \frac{v}{6} - \frac{c^6}{6}\right) = 0,$$

$$b(\theta)^T X \left(Xc^5 + \frac{v}{6} - \frac{c^6}{6}\right) = 0, \quad b(\theta)^T c^2 \left(Xc^4 + \frac{v}{5} - \frac{c^5}{5}\right) = 0,$$

$$b(\theta)^T Xc \left(Xc^4 + \frac{v}{5} - \frac{c^5}{5}\right) = 0, \quad b(\theta)^T cX \left(Xc^4 + \frac{v}{5} - \frac{c^5}{5}\right) = 0,$$

$$b(\theta)^T X^2 \left(Xc^4 + \frac{v}{5} - \frac{c^5}{5}\right) = 0.$$

We begin by focusing on six of the nonquadrature conditions, namely,

$$b(\theta)^T \left(Xc^4 + \frac{v}{5} - \frac{c^5}{5}\right) = 0, \quad b(\theta)^T \left(Xc^5 + \frac{v}{6} - \frac{c^6}{6}\right) = 0,$$

$$b(\theta)^T c \left(Xc^4 + \frac{v}{5} - \frac{c^5}{5}\right) = 0, \quad b(\theta)^T \left(Xc^6 + \frac{v}{7} - \frac{c^7}{7}\right) = 0,$$

$$b(\theta)^T \left(c(Xc^5 + \frac{v}{6} - \frac{c^6}{6})\right) = 0, \quad b(\theta)^T \left(c^2(Xc^4 + \frac{v}{5} - \frac{c^5}{5})\right) = 0.$$

The vectors

$$\left(Xc^4 + \frac{v}{5} - \frac{c^5}{5}\right), \quad \left(Xc^5 + \frac{v}{6} - \frac{c^6}{6}\right), \quad \left(Xc^6 + \frac{v}{7} - \frac{c^7}{7}\right),$$

are the stage order conditions for stage order 5, 6, and 7. Since we are assuming all stages except the third through sixth will have at least stage order 7, these vectors are nonzero only in their third through sixth components, and thus the above order conditions will involve only $b_3(\theta)$, $b_4(\theta)$, $b_5(\theta)$, and $b_6(\theta)$. In fact, these conditions are not independent and are satisfied provided,

$$b_6(\theta) = b_3(\theta) \quad \text{and} \quad b_4(\theta) = b_5(\theta) = \frac{59 + 6\sqrt{30}}{49} b_3(\theta).$$

The remaining nonquadrature conditions are

$$b(\theta)^T X \left(Xc^4 + \frac{v}{5} - \frac{c^5}{5} \right) = 0, \quad b(\theta)^T X \left(Xc^5 + \frac{v}{6} - \frac{c^6}{6} \right) = 0,$$

$$b(\theta)^T \left(Xc(Xc^4 + \frac{v}{5} - \frac{c^5}{5}) \right) = 0, \quad b(\theta)^T \left(cX(Xc^4 + \frac{v}{5} - \frac{c^5}{5}) \right) = 0,$$

$$b(\theta)^T \left(X^2(Xc^4 + \frac{v}{5} - \frac{c^5}{5}) \right) = 0.$$

Assuming the above relationships between the third through sixth weight polynomials, we examine the contributions of the terms involving these weight polynomials to these order conditions. We find

$$\sum_{j=3}^{6} b_j(\theta) \left[X \left(Xc^4 + \frac{v}{5} - \frac{c^5}{5} \right) \right]_j = 0,$$

$$\sum_{j=3}^{6} b_j(\theta) \left[X \left(Xc^5 + \frac{v}{6} - \frac{c^6}{6} \right) \right]_j = 0,$$

$$\sum_{j=3}^{6} b_j(\theta) \left[Xc \left(Xc^4 + \frac{v}{5} - \frac{c^5}{5} \right) \right]_j = 0,$$

$$\sum_{j=3}^{6} b_j(\theta) c_j \left[X \left(Xc^4 + \frac{v}{5} - \frac{c^5}{5} \right) \right]_j = 0,$$

$$\sum_{j=3}^{6} b_j(\theta) \left[X^2 \left(Xc^4 + \frac{v}{5} - \frac{c^5}{5} \right) \right]_j = 0.$$

Since the structure of X implies that the contributions to the above order conditions will also not depend on the first or second weight polynomials, it follows that only weight polynomials $b_7(\theta), \ldots, b_s(\theta)$ will contribute to these order conditions. Thus the remaining nonquadrature order conditions reduce to,

$$\sum_{j=7}^{s} b_j(\theta) \left[X \left(Xc^4 + \frac{v}{5} - \frac{c^5}{5} \right) \right]_j = 0,$$

$$\sum_{j=7}^{s} b_j(\theta)\left[X\left(Xc^5+\frac{v}{6}-\frac{c^6}{6}\right)\right]_j=0,$$

$$\sum_{j=7}^{s} b_j(\theta)\left[Xc\left(Xc^4+\frac{v}{5}-\frac{c^5}{5}\right)\right]_j=0,$$

$$\sum_{j=7}^{s} b_j(\theta)c_j\left[X\left(Xc^4+\frac{v}{5}-\frac{c^5}{5}\right)\right]_j=0,$$

and

$$\sum_{j=7}^{s} b_j(\theta)\left[X^2\left(Xc^4+\frac{v}{5}-\frac{c^5}{5}\right)\right]_j=0.$$

We next consider conditions to be applied to the extra stages 7 through s of the CRK scheme. As indicated earlier we will apply stage order conditions up to and including stage order 7 to all the new stages. In addition, an inspection of the above order conditions shows that it would also be useful if any remaining free parameters of each stage could be chosen to satisfy, among others, conditions of the form,

$$\left[X\left(Xc^4+\frac{v}{5}-\frac{c^5}{5}\right)\right]_r=0,\quad \left[X\left(Xc^5+\frac{v}{6}-\frac{c^6}{6}\right)\right]_r=0,$$

and

$$\left[X^2\left(Xc^4+\frac{v}{5}-\frac{c^5}{5}\right)\right]_r=0,\quad r=7,\ldots,s,$$

since these arise as coefficients of the weight polynomials in some of the above order conditions. After some analysis, we find that it is only possible to impose these extra conditions on some of the new stages. For the seventh stage we can only impose the stage order 7 conditions and we find that we must choose $c_7=\frac{1}{2}$ or $\frac{1}{2}\pm\frac{\sqrt{15}}{10}$; v_7 is free. We make the choice $c_7=\frac{1}{2}$. For the eighth and ninth stages we can impose the stage order 7 conditions plus the first of the above extra conditions. This forces $c_8=\frac{1}{2}\pm\frac{\sqrt{7}}{14}$ and we choose $c_8=\frac{1}{2}-\frac{\sqrt{7}}{14}$. This leaves v_8, c_9, and v_9 free. For the tenth stage we can impose the stage order 7 conditions plus the first two extra conditions. This leaves c_{10} and v_{10} free. For the eleventh through sth stages we can impose the stage order 7 conditions plus all of the extra conditions. The free coefficients are c_{11}, v_{11}, c_{12}, v_{12}, $x_{12,11}$, c_{13}, v_{13}, $x_{13,11}$, $x_{13,12}$, c_{14}, v_{14}, $x_{14,11}$, $x_{14,12}$, $x_{14,13}$, ….

Returning to the remaining nonquadrature order conditions, assuming that the new stages satisfy the additional properties identified above, we see that these conditions reduce to the following,

$$b_7(\theta)\left[X\left(Xc^4+\frac{v}{5}-\frac{c^5}{5}\right)\right]_7=0,$$

$$b_7(\theta)\left[X\left(Xc^5+\frac{v}{6}-\frac{c^6}{6}\right)\right]_7+\ldots+b_9(\theta)\left[X\left(Xc^5+\frac{v}{6}-\frac{c^6}{6}\right)\right]_9=0,$$

$$b_7(\theta)\left[Xc\left(Xc^4+\frac{v}{5}-\frac{c^5}{5}\right)\right]_7+\ldots+b_9(\theta)\left[Xc\left(Xc^4+\frac{v}{5}-\frac{c^5}{5}\right)\right]_9=0,$$

$$b_7(\theta)c_7\left[X\left(Xc^4+\frac{v}{5}-\frac{c^5}{5}\right)\right]_7=0,$$

$$b_7(\theta)\left[X^2\left(Xc^4+\frac{v}{5}-\frac{c^5}{5}\right)\right]_7+\ldots+b_{10}(\theta)\left[X^2\left(Xc^4+\frac{v}{5}-\frac{c^5}{5}\right)\right]_{10}=0.$$

The first and fourth conditions require $b_7(\theta)\equiv 0$ since $\left[X\left(Xc^4+\frac{v}{5}-\frac{c^5}{5}\right)\right]_7\neq 0$. The remaining three order conditions become

$$b_8(\theta)\left[X\left(Xc^5+\frac{v}{6}-\frac{c^6}{6}\right)\right]_8+b_9(\theta)\left[X\left(Xc^5+\frac{v}{6}-\frac{c^6}{6}\right)\right]_9=0,$$

$$b_8(\theta)\left[Xc\left(Xc^4+\frac{v}{5}-\frac{c^5}{5}\right)\right]_8+b_9(\theta)\left[Xc\left(Xc^4+\frac{v}{5}-\frac{c^5}{5}\right)\right]_9=0,$$

$$b_8(\theta)\left[X^2\left(Xc^4+\frac{v}{5}-\frac{c^5}{5}\right)\right]_8+\ldots+b_{10}(\theta)\left[X^2\left(Xc^4+\frac{v}{5}-\frac{c^5}{5}\right)\right]_{10}=0.$$

An examination of the second of these conditions shows that it can be reduced to the first.

Thus, in summary, the last five nonquadrature order conditions, after we impose extra conditions on the new stages, reduce to the following three independent conditions,

$$b_7(\theta)\equiv 0,$$

$$b_8(\theta)\left[X\left(Xc^5+\frac{v}{6}-\frac{c^6}{6}\right)\right]_8+b_9(\theta)\left[X\left(Xc^5+\frac{v}{6}-\frac{c^6}{6}\right)\right]_9=0,$$

and

$$b_8(\theta)\left[X^2\left(Xc^4+\frac{v}{5}-\frac{c^5}{5}\right)\right]_8+b_9(\theta)\left[X^2\left(Xc^4+\frac{v}{5}-\frac{c^5}{5}\right)\right]_9+$$

$$b_{10}(\theta)\left[X^2\left(Xc^4+\frac{v}{5}-\frac{c^5}{5}\right)\right]_{10}=0.$$

In addition we have the three conditions

$$b_6(\theta)=b_3(\theta)\quad\text{and}\quad b_4(\theta)=b_5(\theta)=\frac{59+6\sqrt{30}}{49}b_3(\theta),$$

and the eight quadrature conditions,

$$b(\theta)^T e=\theta,\quad b(\theta)^T c=\frac{\theta^2}{2},\quad b(\theta)^T c^2=\frac{\theta^3}{3},$$

$$b(\theta)^T c^3 = \frac{\theta^4}{4}, \quad b(\theta)^T c^4 = \frac{\theta^5}{5}, \quad b(\theta)^T c^5 = \frac{\theta^6}{6},$$
$$b(\theta)^T c^6 = \frac{\theta^7}{7}, \quad b(\theta)^T c^7 = \frac{\theta^8}{8}.$$

This gives a total of fourteen independent conditions and implies that the CRK scheme must use fourteen stages, i.e. eight new MIRK stages in addition to the two endpoint stages and the four collocation stages.

The relatively large number of free parameters greatly increases the complexity of the calculations for this case. Thus, we will first choose specific values for the free parameters and then solve the above order conditions for that special case. For the free parameters, we choose $v_r = c_r, r = 7, \ldots, 14$, $x_{rj} = 0, j = 11, \ldots, r-1, r = 12, \ldots, 14$, and $c_9 = \frac{1}{8}$, $c_{10} = \frac{1}{4}$, $c_{11} = \frac{3}{4}$, $c_{12} = \frac{7}{8}$, $c_{13} = \frac{2}{5}$, $c_{14} = \frac{3}{5}$. All fourteen stages of the CRK scheme are then determined. The fourteen order conditions listed above are then used to determine the weight polynomials. The tableau of this CRK scheme is

$$\begin{array}{c|c|ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
c_3 & 0 & 0 & 0 & x_{33} & x_{34} & x_{35} & x_{36} & 0 & \cdots & 0 \\
c_4 & 0 & 0 & 0 & x_{43} & x_{44} & x_{45} & x_{46} & 0 & \cdots & 0 \\
c_5 & 0 & 0 & 0 & x_{53} & x_{54} & x_{55} & x_{56} & 0 & \cdots & 0 \\
c_6 & 0 & 0 & 0 & x_{63} & x_{64} & x_{65} & x_{66} & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & x_{71} & x_{72} & x_{73} & x_{74} & x_{75} & x_{76} & 0 & \cdots & 0 \\
c_8 & v_8 & x_{81} & x_{82} & x_{83} & x_{84} & x_{85} & x_{86} & x_{87} & \cdots & 0 \\
\frac{1}{8} & \frac{1}{8} & x_{91} & x_{92} & x_{93} & x_{94} & x_{95} & x_{96} & x_{97} & \cdots & 0 \\
\frac{1}{4} & \frac{1}{4} & x_{10,1} & x_{10,2} & x_{10,3} & x_{10,4} & x_{10,5} & x_{10,6} & x_{10,7} & \cdots & 0 \\
\frac{3}{4} & \frac{3}{4} & x_{11,1} & x_{11,2} & x_{11,3} & x_{11,4} & x_{11,5} & x_{11,6} & x_{11,7} & \cdots & 0 \\
\frac{7}{8} & \frac{7}{8} & x_{12,1} & x_{12,2} & x_{12,3} & x_{12,4} & x_{12,5} & x_{12,6} & x_{12,7} & \cdots & 0 \\
\frac{2}{5} & \frac{2}{5} & x_{13,1} & x_{13,2} & x_{13,3} & x_{13,4} & x_{13,5} & x_{13,6} & x_{13,7} & \cdots & 0 \\
\frac{3}{5} & \frac{3}{5} & x_{14,1} & x_{14,2} & x_{14,3} & x_{14,4} & x_{14,5} & x_{14,6} & x_{14,7} & \cdots & 0 \\
\hline
 & & b_1(\theta) & b_2(\theta) & b_3(\theta) & b_4(\theta) & b_5(\theta) & b_6(\theta) & b_7(\theta) & \cdots & b_{14}(\theta)
\end{array}$$

where the coefficients for the third through sixth stages are given in equations (36)-(40), and $c_8 = v_8 = \frac{1}{2} - \frac{\sqrt{7}}{14}$. The expressions for the remaining coefficients and the weight polynomials, even with the use of specific values for the free parameters, are very complicated. Thus we provide them in a simpler form by reducing the coefficients to 16 significant digits. This gives, for the seventh stage,

$$x_{71} = -0.005208333333333333, \quad x_{72} = 0.005208333333333333, \quad x_{73} = 0.09768756687169055,$$

$$x_{74} = 0.1355546821755248, \quad x_{75} = -0.1355546821755248, \quad x_{76} = -0.09768756687169055,$$

for the eighth stage,

$$x_{81} = -0.008066287689150030, \quad x_{82} = 0.007239834759829562, \quad x_{83} = 0.1371075768778618,$$

$$x_{84} = 0.09571675661727026, \quad x_{85} = -0.05699647651059596, \quad x_{86} = -0.06921542910219575,$$

$$x_{87} = -0.1057859749530199,$$

for the ninth stage,

$$x_{91} = 0.01924728724952684, \quad x_{92} = 0.0009231770064713210, \quad x_{93} = 0.08269651783093607,$$

$$x_{94} = 0.04408472965941152, \quad x_{95} = -0.03465768595451309, \quad x_{96} = -0.02368831572171980,$$

$$x_{97} = -0.01629234729465400, \quad x_{98} = -0.07231336277545893,$$

for the tenth stage,

$$x_{10,1} = 0.04018535463530762, \quad x_{10,2} = -0.0007184435193296687, \quad x_{10,3} = -0.05149956819838338,$$

$$x_{10,4} = -0.09119800053831042, \quad x_{10,5} = -0.08421585380834320, \quad x_{10,6} = -0.04206634835217353,$$

$$x_{10,7} = 0.001968952482382824, \quad x_{10,8} = 0.03646061187208010, \quad x_{10,9} = 0.1910832954267680,$$

for the eleventh stage,

$$x_{11,1} = -0.06747400064574176, \quad x_{11,2} = 0.01129454483158386, \quad x_{11,3} = 0.3416237837455131,$$

$$x_{11,4} = 0.3601250455762746, \quad x_{11,5} = -0.005638214280780879, \quad x_{11,6} = -0.1525401711723007,$$

$$x_{11,7} = 0.0, \quad x_{11,8} = -0.3468850433707371, \quad x_{11,9} = -0.3290534392809501,$$

$$x_{11,10} = 0.1885474945972381,$$

for the twelfth stage,

$$x_{12,1} = -0.04536130701043499, \quad x_{12,2} = -0.01881103880965994, \quad x_{12,3} = 0.2633344008411732,$$

$$x_{12,4} = 0.2972755846582715, \quad x_{12,5} = 0.04101039168870139, \quad x_{12,6} = -0.08289233968810006,$$

$$x_{12,7} = 0.0, \quad x_{12,8} = -0.4889108696816266, \quad x_{12,9} = -0.3269404305894942,$$

$$x_{12,10} = 0.3612956085915711,$$

for the thirteenth stage,

$$x_{13,1} = 0.03608279120121541, \quad x_{13,2} = 0.000131678572969903, \quad x_{13,3} = -0.06639084080129527,$$

$x_{13,4} = -0.1264523666751095, \quad x_{13,5} = -0.1290426551933866, \quad x_{13,6} = -0.06989044658201478,$

$x_{13,7} = 0.0, \quad x_{13,8} = 0.2096860531012513, \quad x_{13,9} = 0.1918513160723896,$

$$x_{13,10} = -0.04597552969601728,$$

and for the fourteenth stage,

$x_{14,1} = -0.02178576193711737, \quad x_{14,2} = 0.01258806317607745, \quad x_{14,3} = 0.1140050201965098,$

$x_{14,4} = 0.07482936616593614, \quad x_{14,5} = -0.1064001928984542, \quad x_{14,6} = -0.1308449291677193,$

$x_{14,7} = 0.0, \quad x_{14,8} = 0.2764275204217753, \quad x_{14,9} = 0.02735915640638052,$

$$x_{14,10} = -0.2461782423633799.$$

The weight polynomials are,

$$b_1(\theta) = -56.44363811998998809\theta^8 + 253.8122387156809554\theta^7 - 472.8303146326085563\theta^6 + 470.6933663638888841\theta^5 - 268.7588196996464491\theta^4 + 87.35943397959582204\theta^3 - 14.83226660692058816\theta^2 + 1.0t,$$

$$b_2(\theta) = -5.696107732450688821\theta^8 + 50.83646998900248811\theta^7 - 132.0579204347868811\theta^6 + 158.7585924649729020\theta^5 - 99.72388296679521350\theta^4 + 32.39796672751068884\theta^3 - 4.515118047453358810\theta^2,$$

$$b_3(\theta) = 139.5694629392165599\theta^8 - 619.0329051265372271\theta^7 + 1130.299160884714300\theta^6 - 1090.735595491995960\theta^5 + 590.8363598357767080\theta^4 - 173.2626835821734190\theta^3 + 22.50012796356777110\theta^2$$

$$b_4(\theta) = 261.6595694868406560\theta^8 - 1160.539562469618160\theta^7 + 2119.042271855944480\theta^6 - 2044.869990398276390\theta^5 + 1107.677741936499580\theta^4 - 324.8263498297635620\theta^3 + 42.18239199580467690\theta^2$$

$$b_5(\theta) = 261.6595694868406560\theta^8 - 1160.539562469618160\theta^7 + 2119.042271855944480\theta^6 - 2044.869990398276390\theta^5 + 1107.677741936499580\theta^4 - 324.8263498297635620\theta^3 + 42.18239199580467690\theta^2$$

$$b_6(\theta) = 139.5694629392165599\theta^8 - 619.0329051265372271\theta^7 + 1130.299160884714300\theta^6 - 1090.735595491995960\theta^5 + 590.8363598357767080\theta^4 - 173.2626835821734190\theta^3 + 22.50012796356777110\theta^2,$$

$$b_7(\theta) \equiv 0,$$

$$b_8(\theta) = -8.311419091396759640\theta^8 + 35.83221757660262520\theta^7 - 62.94761883600197620\theta^6 +$$

$$57.52243486393713397\theta^5 - 28.74509465841990 15\theta^4 +$$

$$7.42078673440372007\theta^3 - 0.77130658912484774 2\theta^2$$

$$b_9(\theta) = 7.27347428919318217\theta^8 - 31.35742649988263380\theta^7 + 55.08660821152866631\theta^6 -$$

$$50.33893086533372225\theta^5 + 25.15535609976147 22\theta^4 -$$

$$6.49406568538233157\theta^3 + 0.674984450115374787\theta^2$$

$$b_{10}(\theta) = -386.362601167096041\theta^8 + 1665.687728683040049\theta^7 - 2926.167659615087 58\theta^6 +$$

$$2673.973880405145 92\theta^5 - 1336.237460882881 39\theta^4 +$$

$$344.960882598049104\theta^3 - 35.85477002117050 86\theta^2$$

$$b_{11}(\theta) = 259.073399405822063\theta^8 - 914.586544484416358\theta^7 + 1241.882119998521 32\theta^6 -$$

$$802.572363669290092\theta^5 + 240.317838338893073\theta^4 -$$

$$21.95473967685173 48\theta^3 - 2.15970991267827163\theta^2$$

$$b_{12}(\theta) = -290.810166756651486\theta^8 + 1160.804774052110 73\theta^7 - 1902.767040748453 15\theta^6 +$$

$$1651.047992616718 93\theta^5 - 809.114117384592055\theta^4 +$$

$$216.989550192195177\theta^3 - 26.15099197132814 57\theta^2$$

$$b_{13}(\theta) = 222.020049726361224\theta^8 - 862.278661743674837\theta^7 + 1301.057007450722 86\theta^6 -$$

$$933.709750460122755\theta^5 + 296.226753266172454\theta^4 -$$

$$16.27927459466120 59\theta^3 - 7.03612364479773211\theta^2$$

$$b_{14}(\theta) = -543.201055405906023\theta^8 + 2200.394138903847 42\theta^7 - 3599.938046875152 32\theta^6 +$$

$$3045.835950060627 54\theta^5 - 1416.148775657044 52\theta^4 +$$

$$351.777526549014724\theta^3 - 38.71973757538682 47\theta^2.$$

8 Numerical results

In this section we present numerical results for implementations of superconvergent interpolants based on several of the CRK schemes derived earlier. We consider collocation with $k = 2, 3$ and 4, and, for each value of k, we consider the corresponding superconvergent interpolant based on the CRK scheme having local error $O(h^{2k})$. (For the $k = 2$ case we consider the interpolant that has continuity to $O(h^4)$.) The double precision version of COLNEW was used to compute all collocation solutions. These tests were run on a DEC alpha system, under the DEC UNIX operating system, using the DEC UNIX FORTRAN 77 compiler.

8.1 Rates of convergence

In this subsection we present results which numerically investigate the orders of convergence for the discrete collocation solution, the collocation interpolant, and the interpolant based on an associated CRK scheme. We consider uniform meshes having, N, the number of subintervals, equal to 4, 8, 16, 32, 64, 128, and 256.

The test problems are

(a)

$$\epsilon^2 y''(t) = y(t) + y^2(t) - e^{-2t/\epsilon}, \quad y(0) = 1, y(1) = e^{-1/\epsilon},$$

$$\text{(with } \epsilon = 0.05\text{)},$$
$$\text{true solution: } y(t) = e^{-t/\epsilon},$$

(b) Swirling Flow III [3],

$$\epsilon f''''(t) + f(t)f'''(t) + g(t)g'(t) = 0,$$
$$\epsilon g''(t) + f(t)g'(t) - f'(t)g(t) = 0,$$
$$f(0) = f(1) = f'(0) = f'(1) = 0,$$
$$g(0) = 1, \quad g(1) = -1,$$
$$\text{(with } \epsilon = 0.075.\text{)}$$

(Since no closed form solution is available for problem (b), the COLNEW code was run in quadruple precision with a stringent tolerance to obtain a high accuracy reference solution.) For each problem and solution component, the initial solution approximation is a straight line through the boundary conditions for components with boundary conditions and zero otherwise.

To obtain discrete solutions on specific uniform meshes, COLNEW was run with the communication parameters IPAR(10)=2 and IPAR(10)=1. When the corresponding Newton iterations converged this meant that COLNEW would (i) compute a solution on the given initial mesh, then (ii) generate a new mesh by doubling the initial mesh, then (iii) compute a second solution on the mesh from step (ii), and then (iv) terminate. This helped us to control the meshes used by COLNEW so that comparisons of solutions obtained from meshes of N and $2N$ subintervals, as is necessary for experimental analysis of convergence rates, would be possible.

The discrete solution values at the mesh points were obtained by subsequent calls to the APPSLN routine, which is part of the COLNEW package. The global error for the corresponding collocation polynomial was based on evaluating the approximate solution, through calls to the APPSLN routine, at a large number of nonmesh sample points on each subinterval. Once the COLNEW code has computed a numerical solution, it is straightforward to employ the CRK schemes derived in the previous sections of this paper to construct a high order CRK interpolant. The algorithm will be discussed in detail in the next subsection. Subsequent to the execution of this algorithm, the CRK interpolant can be evaluated through calls to the INTERPEVAL routine, from the MIRKDC package. The global error estimate for the CRK interpolant was based on calls to the INTERPEVAL routine at a large number of sample points on each subinterval.

In Tables 2, 3, 4, for problem (a), and in Tables 5, 6, 7, for problem (b), we give the ratio, for each mesh, of the maximum value of the global error of the approximate solution computed on the previous mesh to that for the current mesh, for the discrete collocation solution, the collocation polynomial, and the CRK interpolant. For each ratio, we also give, in brackets, the maximum value of the global error. For a given k, the rates of convergence predicted by theory for these three solution approximations are, respectively, $O(h^{2k})$, $O(h^{k+1})$, and $O(h^{2k})$.

N	Discrete Solution	Collocation Polynomial	CRK Interpolant
4	3.3(2.0x10^{0})	2.3(2.9x10^{0})	3.8(2.4x10^{0})
8	4.1(5.0x10^{-1})	2.9(1.0x10^{0})	3.3(7.4x10^{-1})
16	6.0(8.3x10^{-2})	4.7(2.2x10^{-1})	4.2(1.8x10^{-1})
32	11.1(7.5x10^{-3})	6.4(3.4x10^{-2})	8.5(2.1x10^{-2})
64	14.3(5.2x10^{-4})	7.3(4.6x10^{-3})	11.7(1.8x10^{-3})
128	15.5(3.4x10^{-5})	7.8(5.9x10^{-4})	13.6(1.3x10^{-4})
256	15.9(2.1x10^{-6})	7.8(7.5x10^{-5})	14.8(8.9x10^{-6})
Theoretical	16	8	16

Table 2: Error ratios (maximum errors): $k = 2$: Problem(a)

Note that we have not included results (for some of the smaller meshes) when COLNEW was not able to obtain convergence of the Newton iteration on the nonlinear discrete system corresponding to the given mesh. Thus, for example, there is no entry in Table 3 for $N = 4$. Note also that as the accuracy of the approximate solutions begins to approach that of the reference solution, the estimate of the global error becomes unreliable, and thus the estimates of the convergence rates become irrelevant.

From Tables 2-7, we can observe that the expected rates of convergence are being approached by the numerical approximations. From Tables 2-4, we can

N	Discrete Solution	Collocation Polynomial	CRK Interpolant
8	9.2(1.3x10^{-2})	3.8(4.5x10^{-2})	13.3(2.1x10^{-2})
16	23.8(5.3x10^{-4})	6.5(6.9x10^{-3})	23.0(9.1x10^{-4})
32	49.8(1.1x10^{-5})	9.6(7.2x10^{-4})	36.7(2.5x10^{-5})
64	67.2(1.6x10^{-7})	12.3(5.8x10^{-5})	48.1(5.2x10^{-7})
128	64.9(2.5x10^{-9})	14.0(4.2x10^{-6})	55.2(9.4x10^{-9})
256	64.2(3.8x10^{-11})	15.0(2.8x10^{-7})	59.9(1.6x10^{-10})
Theoretical	64	16	64

Table 3: Error ratios (maximum errors): $k = 3$: Problem(a)

N	Discrete Solution	Collocation Polynomial	CRK Interpolant
16	31.0(1.5x10^{-4})	7.1(2.1x10^{-3})	46.9(3.1x10^{-4})
32	94.4(1.6x10^{-6})	13.3(1.6x10^{-4})	86.7(3.7x10^{-6})
64	196.4(8.3x10^{-9})	20.1(7.7x10^{-6})	136.3(2.7x10^{-8})
128	259.9(3.2x10^{-11})	25.2(3.1x10^{-7})	182.2(1.5x10^{-10})
256	258.5(1.2x10^{-13})	28.5(1.1x10^{-8})	215.3(6.8x10^{-13})
Theoretical	256	32	256

Table 4: Error ratios (maximum errors): $k = 4$: Problem(a)

observe that the maximum error for the superconvergent interpolant is usually only a 3 or 4 times larger than that of the discrete collocation solution. From Tables 5-7, we see that the maximum error for the superconvergent interpolant is usually about an order of magnitude larger than that of the discrete collocation solution.

8.2 Execution times

In this subsection we consider execution time results for the calculation and evaluation of the collocation solution and the setup and evaluation of the associated superconvergent interpolant. The initial mesh is a uniform mesh of 5 subintervals and a tolerance of 10^{-6} is applied to each solution component. We use test problem (b) from the previous subsection. The COLNEW code ran to completion, using a sequence of meshes and Newton iterations, until a converged solution within the tolerance was obtained.

The construction of the superconvergent CRK interpolant is based on the following algorithm which assumes that COLNEW has already been used to

N	Discrete Solution	Collocation Polynomial	CRK Interpolant
4	14.4(7.2x10^{-3})	6.7(3.5x10^{-2})	5.9(5.5x10^{-2})
8	19.7(3.7x10^{-4})	5.5(6.3x10^{-3})	9.9(5.6x10^{-3})
16	16.8(2.2x10^{-5})	5.9(1.1x10^{-3})	12.7(4.4x10^{-4})
32	16.2(1.3x10^{-6})	6.9(1.5x10^{-4})	14.0(3.1x10^{-5})
64	16.1(8.3x10^{-8})	7.5(2.1x10^{-5})	14.4(2.2x10^{-6})
128	16.0(5.2x10^{-9})	7.7(2.7x10^{-6})	15.2(1.4x10^{-7})
256	16.0(3.3x10^{-10})	7.9(3.4x10^{-7})	15.6(9.1x10^{-9})
Theoretical	16	8	16

Table 5: Error ratios (maximum errors): $k = 2$: Problem (b)

N	Discrete Solution	Collocation Polynomial	CRK Interpolant
4	24.5(3.8x10^{-4})	4.7(7.7x10^{-3})	11.7(1.3x10^{-3})
8	41.7(9.1x10^{-6})	7.9(9.8x10^{-4})	31.3(4.2x10^{-5})
16	54.9(1.7x10^{-7})	11.2(8.7x10^{-5})	40.7(1.0x10^{-6})
32	61.3(2.7x10^{-9})	13.4(6.5x10^{-6})	50.5(2.1x10^{-8})
64	63.3(4.3x10^{-11})	14.7(4.4x10^{-7})	56.7(3.6x10^{-10})
128	63.1(6.8x10^{-13})	15.3(2.9x10^{-8})	60.1(6.1x10^{-12})
256	34.7(2.0x10^{-14})	15.7(1.8x10^{-9})	41.6(1.5x10^{-13})
Theoretical	64	16	64

Table 6: Error ratios (maximum errors): $k = 3$: Problem (b)

compute an approximate solution.

Algorithm for the Construction of a CRK Interpolant

1. The APPSLN routine, from the COLNEW package, is used to evaluate the solution at the meshpoints and at the k collocation points within each subinterval.

2. The user-defined FSUB function is then called to compute the endpoint and collocation stages from the solution values computed in the previous step.

3. The INTERPSETUP routine, from the MIRKDC package, is called, once for each subinterval, to setup the extra MIRK stages, as specified by the CRK scheme corresponding to the value of k, identified in the previous sections of this paper.

N	Discrete Solution	Collocation Polynomial	CRK Interpolant
4	170.4(1.1x10^{-5})	10.9(1.5x10^{-3})	73.9(6.5x10^{-5})
8	138.1(8.1x10^{-8})	18.4(8.1x10^{-5})	71.6(9.1x10^{-7})
16	203.9(4.0x10^{-10})	24.8(3.3x10^{-6})	136.6(6.7x10^{-9})
32	241.8(1.6x10^{-12})	28.6(1.1x10^{-7})	188.7(3.5x10^{-11})
64	205.0(8.0x10^{-15})	30.4(3.8x10^{-9})	64.1(5.5x10^{-13})
128	0.9(9.3x10^{-15})	31.7(1.2x10^{-10})	2.6(2.1x10^{-13})
256	1.0(9.8x10^{-15})	32.2(3.7x10^{-12})	2.6(9.0x10^{-14})
Theoretical	256	32	256

Table 7: Error ratios (maximum errors): $k = 4$: Problem (b)

In step 1. we obtain the arguments for the collocation stages to be computed in step 2. by calling the APPSLN routine. A somewhat more efficient implementation would involve recovery of the collocation stages by accessing them from the COLNEW code. This would avoid k calls to the APPSLN and the FSUB routine for each subinterval. However, the approach implemented here is more direct and thus somewhat simpler, and, as we will show shortly, introduces negligible extra cost.

Upon completion of the above algorithm, the CRK interpolant is then available for evaluation. It can be evaluated using a call to the INTERPEVAL routine, from the MIRKDC package.

The important execution costs here are (i) the original call to COLNEW, which yields the discrete solution on the final mesh of N subintervals, and (ii) the construction of the CRK interpolant, using the above algorithm, in which the costs are attributable largely to calls to the APPSLN routine (step 1) and the INTERPSETUP routine (step 3). (For each call to the APPSLN routine there is a subsequent call to the user-defined FSUB function (step 2) but the latter cost is usually negligible.)

Another concern is the cost associated with the subsequent evaluation of the superconvergent interpolant. Since the superconvergent interpolant typically employs about twice as many stages as the collocation polynomial, we expect its evaluation cost to be roughly twice that of the collocation polynomial, i.e. a call to the INTERPEVAL routine will likely cost about twice as much as a call to the APPSLN routine.

In Table 8, for $k = 2, 3$, and 4, we list execution times (in seconds) for the initial calculation of the collocation solution, in the column labelled COLNEW, and for each of the two main steps of the algorithm for the construction of the CRK interpolant. The times for step 1, which consists of $N + 1 + kN$ calls to the APPSLN routine, are given in the column labelled APPSLN, and those for step 3, which consists of N calls to the INTERPSETUP routine, are in the column labelled INTERPSETUP. The column labelled TOTAL SETUP gives the sum of the previous two columns and represents the total setup time for the construction of the CRK interpolant. We also give the number of subintervals in the final mesh required by COLNEW to compute its solution in the column labelled N. Note that for $k = 2$, no extra stages are needed for the CRK interpolant and thus no calls to the INTERPSETUP routine are required.

k	N	COLNEW	APPSLN	INTERPSETUP	TOTAL SETUP
2	160	0.35	8.3x10^{-3}	-	8.3x10^{-3}
3	40	0.14	2.1x10^{-3}	7.0x10^{-4}	2.8x10^{-3}
4	20	0.08	1.4x10^{-3}	1.1x10^{-3}	2.5x10^{-3}

Table 8: Size of Final Mesh and Total Execution times

From Table 8, one can observe that the costs associated with setting up the superconvergent interpolant are negligible with respect to those associated with the original computation of the discrete solution. For $k = 2, 3$, and 4, the total setup costs represent approximately 2.4%, 2.0%, and 3.1%, respectively, of the cost associated with the calculation of the original collocation solution by COLNEW.

In Table 9, we provide maximum errors for the discrete collocation solution, the collocation polynomial, and the CRK interpolant for the cases considered in Table 8. These results show that COLNEW yields a continuous solution that approximately meets the tolerance, 10^{-6}, while the discrete solution and the CRK interpolant are substantially more accurate.

k	N	Discrete Solution	Collocation Polynomial	CRK Interpolant
2	160	2.1x10^{-9}	1.4x10^{-6}	5.9x10^{-8}
3	40	7.1x10^{-10}	2.8x10^{-6}	5.7x10^{-9}
4	20	6.8x10^{-11}	1.1x10^{-6}	1.3x10^{-9}

Table 9: Size of Final Mesh and Maximum Error

In Table 10 we give the execution time for a single call to APPSLN and a single call to INTERPEVAL; we observe that the cost associated with an evaluation of the superconvergent interpolant is within a factor of at most 2.6 times that of the collocation polynomial.

Since, the execution costs for COLNEW are $O(Nn^3k^3)$ while the costs for computing the interpolant (e.g. with the boot strapping approach) are

k	APPSLN	INTERPEVAL
2	1.7x10^{-5}	1.7x10^{-5}
3	1.3x10^{-5}	3.0x10^{-5}
4	1.4x10^{-5}	3.7x10^{-5}

Table 10: Execution time for a single call

$O(Nk^2n)$, for larger values of k one would expect to see that the cost of constructing a superconvergent interpolant would continue to be a negligible fraction of the cost of computing the discrete solution.

9 Summary, Conclusions, and Future Work

In this paper, the question of developing an efficient method for augmenting the high order discrete solution yielded by a collocation method is explored. It is observed that it is possible to make use of the solution values at the mesh points and derivative values at the collocation points (information that is naturally produced during the collocation calculation) in the construction a C^1 continuous interpolant to the high order mesh point solution. The approach is based on the use of continuous Runge-Kutta schemes, which incorporate the available information from the collocation computation and augment it with additional derivative information obtained using efficiently computable mono-implicit Runge-Kutta stages. C^1 superconvergent interpolants having global errors of $O(h^{2k})$ based on two continuous Runge-Kutta schemes with local errors of $O(h^{2k})$ and $O(h^{2k+1})$, respectively, are derived for k, the number of collocation points, taking on the values 1,2,3, and 4. An alternative approach for deriving continuous Runge-Kutta schemes, based on the use of a boot strapping algorithm, is also discussed.

Although the derivation process is more tedious, the direct derivation of continuous Runge-Kutta schemes, in which the coefficients which define the scheme are determined by requiring them to satisfy continuous order conditions, leads to schemes which use fewer stages, and thus are more efficient than those obtained from the boot strapping algorithm, at least for the values of k considered in this paper. The implementation or construction costs for these interpolants, when compared with the cost of computing the original discrete solution using a collocation method, are demonstrated through some numerical tests, to be negligible. The cost of evaluating these interpolants is low; it is about twice that of evaluating the lower order collocation polynomial produced by the collocation process. Since the algorithm can be implemented through calls to existing routines from the COLNEW and MIRKDC codes, the software development efforts associated with preparing the software that performs the construction and evaluation of the superconvergent interpolants are also quite reasonable.

There are a variety of directions for future work. One is to continue the development of these interpolants for higher values of k, at least as far as $k = 7$ (the maximum value supported by the COLNEW package). Another is to investigate the development of continuous Runge-Kutta schemes for higher order differential equations, since the collocation method is able to handle such equations quite naturally. An example of work in this direction for initial value problems is the paper [12], where continuous Runge-Kutta extensions of Nyström methods for second order initial value ODEs are considered. A third area of investigation involves modifying the COLNEW code so that it computes a superconvergent interpolant which it then uses for global error estimation and mesh selection.

10 Acknowledgements

The authors would like to thank Graeme Day for his assistance with the numerical testing aspects of this work.

References

[1] R. Alexander, Diagonally implicit Runge-Kutta methods for stiff ODEs, SIAM J. Numer. Anal., 14, 1977, 1006-1021.

[2] U.M. Ascher, J. Christiansen, and R.D. Russell, Collocation software for boundary value ODEs, ACM Trans. Math. Softw., 7, 1981, 209-222.

[3] U.M. Ascher, R.M.M. Mattheij, R.D. Russell, *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations*, Classics in Applied Mathematics Series, Society for Industrial and Applied Mathematics, Philadelphia, 1995.

[4] G. Bader and U.M. Ascher, A new basis implementation for a mixed order boundary value ode solver, SIAM J. Sci. Stat. Comp., 8, 1987, 483-500.

[5] K. Burrage, F.H. Chipman, P.H. Muir, Order results for mono-implicit Runge-Kutta methods, SIAM J. Numer. Anal., 31, 1994, 876-891.

[6] W. M. G. van Bokhoven, Efficient higher order implicit one-step methods for integration of stiff differential equations, BIT, 20, 1980, 34-43.

[7] J.C. Butcher, *The Numerical Analysis of Ordinary Differential Equations*, Wiley, Chichester, 1987.

[8] J.C. Butcher and J. Cash, Towards efficient Runge-Kutta methods for stiff systems, SIAM J. Numer. Anal., 27, 1990, 753-761.

[9] W.H. Enright, K.R. Jackson, S.P. Nørsett, and P.G. Thomsen, Interpolants for Runge-Kutta formulas, ACM Trans. Math. Softw., 12, 1986, 193-218.

[10] W.H. Enright and P.H. Muir, Efficient classes of Runge-Kutta methods for two-point boundary value problems, Computing, 37, 1986, 315-334.

[11] W.H. Enright and P.H. Muir, Runge-Kutta software with defect control for boundary value ODEs, SIAM J. Sci. Comput., 17, 1996, 479-497.

[12] A. Marthinsen, Continuous extensions to Nyström methods for second order initial value problems, BIT 36, 1996, 309-332.

[13] P.H. Muir, Discrete and continuous mono-implicit Runge-Kutta schemes for BVODEs, submitted to Adv. in Comp. Math., 1997.

[14] P.H. Muir and W.H. Enright, Relationships among some classes of implicit Runge-Kutta methods and their stability functions, BIT 27, 1987, 403-423.

[15] P. Muir and B. Owren, Order barriers and characterizations for continuous mono-implicit Runge-Kutta schemes, Math. Comp., 61, 1993, 675-699.

[16] S. Pruess, Interpolation schemes for collocation solutions of two point boundary value problems, SIAM J. Sci. Stat. Comput., 7, 1986, 322-333.

[17] S. Pruess and H. Jin, A stable high-order interpolation scheme for superconvergent data, SIAM J. Sci. Comput., 17, 1996, 714-724.

[18] H. Stetter, *The Analysis of Discretization Methods for Ordinary Differential Equations*, Springer-Verlag, Berlin, 1982.

[19] J. Verner, Differentiable interpolants, SIAM J. Numer. Anal., 1993.

[20] R. Weiss, The application of implicit Runge-Kutta and collocation methods to boundary value problems, Math. Comp., 28, 1974, 449-464.