# Quartic-Spline Collocation Methods for Fourth-Order Two-Point Boundary Value Problems

by

Ying Zhu

A thesis submitted in conformity with the requirements
for the degree of Master of Science
Graduate Department of Computer Science
University of Toronto

# Abstract

Quartic-Spline Collocation Methods for Fourth-Order Two-Point Boundary Value Problems

Ying Zhu

Master of Science

Graduate Department of Computer Science

University of Toronto

2001

This thesis presents numerical methods for the solution of general linear fourth-order boundary value problems in one dimension. The methods are based on quartic splines, that is, piecewise quartic polynomials with $C^3$ continuity, and the collocation discretization methodology with the midpoints of a uniform partition being the collocation points. The standard quartic-spline collocation method is shown to be second order. Two sixth-order quartic-spline collocation methods are developed and analyzed. They are both based on a high order perturbation of the differential equation and boundary conditions operators. The one-step method forces the approximation to satisfy a perturbed problem, while the three-step method proceeds in three steps and perturbs the right sides of the equations appropriately. The error analysis follows the Green's function approach and shows that both methods exhibit optimal order of convergence, that is, they are locally sixth order on the gridpoints and midpoints of the uniform partition, and fifth order globally. The properties of the matrices arising from a restricted class of problems are studied. Analytic formulae for the eigenvalues and eigenvectors are developed, and related to those arising from quadratic-spline collocation matrices. Numerical results verify the orders of convergence predicted by the analysis.

# Dedication

To my parents, Luo Ji Dai and Zhu Guo Ping, who are also my best friends.

# Acknowledgements

I would especially like to thank my supervisor, Professor Christina C. Christara. Without her diligence, her invaluable guidance, and the unwonted understanding that she has always shown me, this thesis would not have been possible.

I would also like to thank Professor Enright for his careful reading of the thesis and for providing valuable comments.

I am forever grateful for the love and sacrifices of my parents.

Lastly, I thank NSERC, University of Toronto and Professor Christara for the financial support.

# Contents

# List of Tables

# List of Figures

# Chapter 1

# Introduction

Fourth-order boundary value problems (BVPs) can be used to model several physical phenomena. For example, beam deflection under a certain load can be modelled by a fourth-order two-point BVP. In two dimensions, plate deflection can be modelled by fourth-order BVPs. The biharmonic equation, which is a fourth-order two-dimensional partial differential equation (PDE), also arises in many applications.

In this thesis, we study a collocation method to compute the numerical solution of (scalar) fourth-order two-point BVPs described by a differential equation and boundary conditions of the form

$$Lu \equiv u^{(4)}(x) + p(x)u^{(3)}(x) + q(x)u^{(2)}(x) + r(x)u^{(1)}(x) + s(x)u(x) = f(x), \quad x \in [a,b] \equiv I,$$

$$Bu(a) = g_a, Bu(b) = g_b,$$

where $u(x)$ is an unknown function, $u^{(i)} \equiv \frac{d^i u}{dx^i}$, $g_a$ and $g_b$ are given, $p(x), q(x), r(x), s(x)$ and $f(x)$ are given functions, and $B$ is a differential operator, the detailed form of which will be given later in the thesis.

In general, collocation proceeds as follows. We first choose an approximating space $X$ of dimension $n$ and a basis $\{\Phi_1(x), \Phi_2(x), \cdots, \Phi_n(x)\}$ for $X$ such that any $w_\Delta \in X$ can be written as $w_\Delta = \sum_1^n c_i\Phi_i(x)$, where the $c_i$'s are scalars referred to as *degrees of freedom*. Note that the basis functions $\Phi_i(x)$, $i = 1, \cdots, n$, are user-chosen and thus known, while the degrees of freedom are unknown. Then a set $T$ of data points, called *collocation points*, is selected, where often $T = T_L \cup T_B$, with $T_B = \{a, b\}$. In the standard formulation of collocation methods, we determine the degrees of freedom $c_i$, $i = 1, \cdots, n$, and thus the approximation $w_\Delta$ to the solution $u$ of the BVP, by forcing $w_\Delta$ to satisfy the conditions

$$\begin{aligned} Lw_\Delta(x) &= f(x), \quad \text{for } x \in T_L, \\ Bw_\Delta(x) &= g(x), \quad \text{for } x \in T_B. \end{aligned}$$

These conditions, called *collocation conditions*, give rise to a linear system with respect to the unknown degrees of freedom. The sets $T_L$ and $T_B$ are chosen so that the collocation conditions give rise to a uniquely solvable linear system.

The choice of approximating space, basis functions and collocation points plays an important role in the accuracy of the approximation and the efficiency of the calculations, especially those associated with the resulting linear system.

Spaces of piecewise polynomials of certain degree and continuity are often used as approximating spaces in various finite element methods (FEMs), such as Galerkin and collocation. The approximating space we choose is the space of quartic splines with respect to a partition $\Delta$ of $I$, that is, the space of piecewise quartic polynomials with $C^3$ continuity, defined with respect to $\Delta$. The midpoints of the subintervals of the partition and the two boundary points are picked to be collocation points. It is quite common in the literature, to pick as data points the gridpoints of the partition, when odd degree splines are used, and the midpoints when even degree splines are used. See, for example, [8], [17] and [12].

Collocation is often presented as a generalization of interpolation. More specifically, if the differential operators $L$ and $B$ reduce to the identity operator, the collocation method reduces to interpolation. Moreover, the order of convergence of collocation (or of any FEM), that is, the rate at which the collocation (or FEM) approximation error decreases as the number of data points increases, is often described related to that of the interpolant in the same approximation space. We say that we have *optimal order of convergence* or an *optimal FEM*, if the order of convergence of the approximation obtained by the FEM is the same as that of the interpolant in the same approximation space.

Several researchers have studied piecewise polynomial collocation methods. Collocation with piecewise polynomial functions for solving $m$-order two-point linear or non-linear BVPs was presented in [19] by Russell and Shampine. In the same paper, the order of convergence of certain piecewise polynomial collocation methods is analyzed. It is shown that the standard formulation of spline collocation methods as described above gives rise to sub-optimal order of convergence. In [9], piecewise polynomial collocation on the Gauss points for $m$th order two-point BVPs is introduced. The resulting orders of convergence are shown to be optimal. For certain degree piecewise polynomials the orders are superoptimal locally on the gridpoints of the partition. Chapter 5 of [2] provides a comprehensive coverage of piecewise polynomial collocation. The related software package COLNEW (formerly COLSYS) is introduced in [1] and in [2].

It is worth noticing that (smooth) spline collocation, that is, collocation based on piecewise polynomials of degree $k$ and continuity $C^{k-1}$ uses one only data point per subinterval of the partition, while piecewise polynomial collocation on the Gauss points uses more than one. Therefore, spline collocation gives rise to smaller linear systems, and therefore, in that respect, is advantageous. However, the fact that the standard formulation of spline collocation methods gives rise to sub-optimal order of convergence is a drawback to the success and wide applicability of the methods.

Relatively recently, researchers have developed and analyzed optimal spline collocation methods for certain classes of BVPs. These optimal spline collocation methods are based on an appropriate perturbation of the differential equation and boundary conditions of the BVP. Daniel and Swartz [6] apply this technique to cubic-spline collocation on gridpoints, Houstis, Christara and Rice [12] to quadratic-spline collocation on midpoints, and Irodotou-Ellina and Houstis [15] to quintic-spline collocation on gridpoints. Both [6] and [12] studied numerical solutions of second-order two-point BVPs, while [15] considered fourth-order two-point BVPs. All three papers presented methods that yielded optimal orders of convergence. These methods have been extended to two or more dimensions; see, for example, [13], [14], [4] and [5].

In this thesis, we develop optimal quartic-spline collocation methods for the numerical solution of fourth-order linear two-point BVPs. The optimal quartic-spline collocation methods are based on the same perturbation technique used in the optimal spline collocation methods of [6], [12] and [15]. In order to develop the appropriate perturbations of the differential equation and boundary conditions of the BVP, we study the properties of a quartic-spline interpolant of the solution $u$ of the BVP. The perturbations gives rise to two optimal quartic-spline collocation methods: the *one-step* or *extrapolated* method, and the *three-step* or *deferred-correction* method.

Let $P_L = P_{L_1} + P_{L_2}$ and $P_B = P_{B_1} + P_{B_2}$ be the perturbations of $L$ and $B$, respectively. The exact form of $P_{L_1}$, $P_{L_2}$, $P_{B_1}$ and $P_{B_2}$, will be given in Chapter 3 of the thesis. In the one-step or extrapolated method, the collocation approximation $z_\Delta(x)$ is determined by forcing it to satisfy the conditions

$$\begin{aligned} (L + P_{L_1} + P_{L_2}) z_\Delta(x) &= f(x), \quad \text{for } x \in T_L, \\ (B + P_{B_1} + P_{B_2}) z_\Delta(x) &= g(x), \quad \text{for } x \in T_B. \end{aligned}$$

The three-step or deferred-correction method consists of three steps. In the first step, the standard second-order quartic-spline collocation approximation $w$ is computed by the conditions

$$\begin{aligned} Lw(x) &= f(x), \quad \text{for } x \in T_L, \\ Bw(x) &= g(x), \quad \text{for } x \in T_B. \end{aligned}$$

In the second step, a fourth-order quartic-spline collocation approximation $v$ is computed by the conditions

$$\begin{aligned} Lv(x) &= f(x) - P_{L_1}w(x), \quad \text{for } x \in T_L, \\ Bv(x) &= g(x) - P_{B_1}w(x), \quad \text{for } x \in T_B. \end{aligned}$$

Finally, in the third step, a sixth-order quartic-spline collocation approximation $u_\Delta$ is computed by the conditions

$$\begin{aligned} Lu_\Delta(x) &= f(x) - P_{L_1}v(x) - P_{L_2}v(x), \quad \text{for } x \in T_L, \\ Bu_\Delta(x) &= g(x) - P_{B_1}v(x) - P_{B_2}v(x), \quad \text{for } x \in T_B. \end{aligned}$$

We study the properties of the linear systems arising in each of the two methods. For a certain class of problems and for the deferred-correction method, we give analytic formulae for the eigenvalues and eigenvectors of the matrices arising, and related those to the matrices arising from quadratic-spline collocation. We carry a convergence analysis of both the one-step and the three-step methods using Green's functions. Finally, we present numerical results that verify the results predicted by the analysis.

There are a few more papers relevant to our work. Usmani and Warsi [20], [21] presented sub-optimal numerical methods that solve special classes of second-order and fourth-order problems arising from beam deflection using quintic and sextic splines. Usmani [22] has moreover described an algorithm that uses quartic splines to numerically solve fourth-order BVPs. The order of convergence of this method is not better than that of standard quartic-spline collocation. However, a number of relations which we have derived independently and used in this thesis can also be found in [22]. Some of the relations we developed also appear in the paper by Hoskins and Meek [11] who have studied comprehensively linear dependence relations for values of splines at midpoints. Interpolation by periodic quartic splines on gridpoints has been shown by Dubeau and Savoie [10] to have $O(h^{5-k})$ error bounds for $k$th derivatives.

This thesis is organized as follows. In Chapter 2, we give the properties of a quartic-spline interpolant, that are necessary to formulate our optimal quartic-spline collocation methods. It is worth noticing that the interpolant is used only in the analysis. It is not necessary to compute it, when the optimal quartic-spline collocation solution to a BVP is computed. In Chapter 3, we formulate the one-step and three-step optimal quartic-spline collocation methods, and study the properties of the matrices arising. In Chapter 4, we present the mathematical analysis of the convergence of the methods. Results from numerical experiments are shown in Chapter 5. Finally, we conclude in Chapter 6.

# Chapter 2

# Quartic Spline Interpolation Results

## 2.1 Quartic Spline Interpolation

In this section, we present quartic spline interpolation and introduce some notation. We consider the uniform grid partition $\Delta \equiv \{a = x_0 < x_1 < \cdots < x_N = b\}$ of the interval $I = [a,b]$, and the set of points $T \equiv \{\tau_0 = x_0, \tau_i = (x_i + x_{i-1})/2; i = 1, \cdots, N, \tau_{N+1} = x_N\}$. Let $h = \frac{b-a}{N}$ be the mesh size of $\Delta$.

Let $\mathbf{S}_{4,\Delta}$ be the space of quartic splines with respect to $\Delta$, that is, the space of quartic piecewise polynomials with respect to $\Delta$ and with continuity $C^3(I)$. Note that $S_{4,\Delta}$ has dimension $N+4$. In order to represent any quartic spline $S \in \mathcal{S}_{4,\Delta}$, we choose a set of quartic spline basis functions, the quartic B-splines, $\Phi_i(x), i = -1, \cdots, N+2$. To define the $\Phi_i$'s, we first define a model quartic spline function

$$\Phi(x) = \frac{1}{24}\begin{cases} x^4 & \text{for } 0 \le x \le 1 \\ x^4 - 5(x-1)^4 & \text{for } 1 \le x \le 2 \\ x^4 - 5(x-1)^4 + 10(x-2)^4 & \text{for } 2 \le x \le 3 \\ x^4 - 5(x-1)^4 + 10(x-2)^4 - 10(x-3)^4 & \text{for } 3 \le x \le 4 \\ x^4 - 5(x-1)^4 + 10(x-2)^4 - 10(x-3)^4 + 5(x-4)^4 & \text{for } 4 \le x \le 5 \\ 0 & \text{elsewhere.} \end{cases} \tag{2.1}$$

Then the set

$$\{\Phi_i(x) = \Phi(\frac{x}{h} - i + 3)\}_{i=-1}^{N+2} \tag{2.2}$$

forms a basis for $S_{4,\Delta}$. Any quartic spline $S \in \mathcal{S}_{4,\Delta}$ can be written as $S(x) = \sum_{i=-1}^{N+2} c_i \Phi_i(x)$, where the scalars $c_i$, $i = -1, \cdots, N+2$, are called *coefficients* or *degrees of freedom.*

From the above, it is clear that, in order to uniquely determine a quartic spline, $N+4$ linearly independent conditions are required. The most common choices for the set of spline interpolation points are grid points or midpoints in a grid partition of the domain. Since the midpoints are chosen as the collocation points for the method presented in this paper and hence in the next section for interpolation, we will restrict the discussion of quartic spline interpolation to using midpoints as interpolation points. The midpoints are used for our quartic spline collocation method because they had been used in an optimal quadratic spline collocation method [3] that is similar and because it is natural to use midpoints for even-degree spline collocation/interpolation. For instance, [22] describes a numerical method that uses midpoints to solve a fourth-order boundary value problem. Grid points are also a valid choice and are, for instance, used in [10] on periodic quartic spline interpolation.

We now give an example of quartic spline interpolation. For a function $u(x)$, let $u(\tau_i)$, $i = 0, \cdots, N+1$, be given, and let also $u''(\tau_0)$ and $u''(\tau_{N+1})$ be given. The set of equations

$$S(\tau_0) = u(\tau_0); \qquad S''(\tau_0) = u''(\tau_0) \tag{2.3}$$

$$S(\tau_i) = u(\tau_i), \qquad i = 1, \cdots, N \tag{2.4}$$

$$S''(\tau_{N+1}) = u''(\tau_{N+1}); \quad S(\tau_{N+1}) = u(\tau_{N+1}) \tag{2.5}$$

gives rise to the linear system

$$\begin{bmatrix} 1 & 11 & 11 & 1 & & & & & \\ \frac{12}{h^2} & \frac{-12}{h^2} & \frac{-12}{h^2} & \frac{12}{h^2} & & & & & \\ \frac{1}{16} & \frac{76}{16} & \frac{230}{16} & \frac{76}{16} & \frac{1}{16} & & & & \\ & \frac{1}{16} & \frac{76}{16} & \frac{230}{16} & \frac{76}{16} & \frac{1}{16} & & & \\ & & & \cdot & \cdot & \cdot & & & \\ & & & & \frac{1}{16} & \frac{76}{16} & \frac{230}{16} & \frac{76}{16} & \frac{1}{16} \\ & & & & & \frac{12}{h^2} & \frac{-12}{h^2} & \frac{-12}{h^2} & \frac{12}{h^2} \\ & & & & & 1 & 11 & 11 & 1 \end{bmatrix} \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ \cdot \\ c_N \\ c_{N+1} \\ c_{N+2} \end{bmatrix} = 24 \begin{bmatrix} u(\tau_0) \\ u''(\tau_0) \\ u(\tau_1) \\ u(\tau_2) \\ \cdot \\ u(\tau_{N+1}) \\ u''(\tau_{N+2}) \\ u(\tau_{N+2}) \end{bmatrix} \quad (2.6)$$

With appropriate scaling, and after substituting the first and last row with appropriate linear combinations of the first two and last two rows, respectively, and then scaling the first and last columns with an appropriate constant, the coefficient matrix of (2.6) can be transformed to a pentadiagonal and diagonally dominant matrix (strictly except for the second and second last rows). Therefore, the linear system (2.6) has a unique solution, hence there exists a uniquely determined quartic spline interpolant $S$ of $u$, satisfying (2.3)-(2.5).

We refer to the $N$ equations (2.4) as the *interpolation conditions* and to the equations (2.3), (2.5) as the *boundary conditions* (BCs). Other BCs that lead to uniquely solvable linear systems can be defined; in fact, the BCs defined in the next section are specifically chosen in order to derive asymptotic relations that are used in convergence analysis of our collocation method. Three examples of BCs that lead to uniquely solvable linear systems are

$$S(\tau_i) = u(\tau_i), i = 0 \text{ and } N+1; \quad S'(\tau_i) = u'(\tau_i), i = 0 \text{ and } N+1 \quad (2.7)$$

$$S(\tau_i) = u(\tau_i), i = 0 \text{ and } N+1; \quad S''(\tau_i) = u''(\tau_i), i = 0 \text{ and } N+1 \quad (2.8)$$

$$S(\tau_i) = u(\tau_i), i = 0 \text{ and } N+1; \quad S'''(\tau_i) = u'''(\tau_i), i = 0 \text{ and } N+1. \quad (2.9)$$

To demonstrate the effect of BCs on the order of convergence, we present errors and respective orders of convergence for the BCs (2.7), (2.8) and (2.9) with the right-side of the interpolation and boundary conditions given so that $u(x) = x^{\frac{13}{2}}$, in $I = [0, 1]$. The order of convergence is defined as $\frac{log(E_1/E_2)}{log(N_2/N_1)}$, where $N_1, N_2$ are grid sizes with $N_2 > N_1$ and $E_1, E_2$ are errors for $N_1, N_2$, respectively. In the tables below, $|| \cdot ||_{\lambda_i}$ and $|| \cdot ||_\infty$ denote error at Gaussian points and global points, respectively. There are two Gaussian points in each of the $N$ subintervals, one of which is $\frac{3-\sqrt{3}}{6}h$ from the left grid point of the subinterval and the other is the same distance from the right grid point. To compute the global errors, we use 1030 global points where the $i$th point is defined to be $a + \frac{b-a}{1031}i$; the choice of 1030 is arbitrary aside from the fact that it ensures that none of the global points are mesh points.

It is generally known [7] that interpolation with piecewise polynomials of degree $r$ gives $O(h^{r+1})$ uniform norm errors for the interpolant and $O(h^{r+1-d})$ errors for the $d$th derivative of the interpolant. Tables 2.1, 2.2 and 2.3 verify this general result. However, we also notice some local superconvergence properties of the quartic spline interpolant with the BCs (2.8). Namely, we obtain sixth order for the interpolant on the grid points, fourth order for the second derivative of the interpolant on the midpoints and the grid points, third order for the third derivative of the interpolant on the two Gauss points of each subinterval and fourth order for the fourth derivative of the interpolant on the midpoints. This observation extends the results on quadratic splines [17], where superconvergence is obtained for the interpolant and its first and second derivatives on certain points.

It is worth noting that these superconvergence results are obtained for a function $u$ in $C^6(I)$. For $u = x^{11/2}$, whose sixth derivative is unbounded at the origin, the sixth order of convergence drops to 5.5 approximately. A similar effect of the discontinuity of the function on the order of convergence of the interpolant is obtained for quadratic splines, for which $C^4$ continuity is the minimum necessary to get local superconvergence of order 4 and for other degree splines.

Table 2.1: Errors and the corresponding orders of convergence of a quartic spline interpolant for the BCs (2.7)

| $N$ | $\|\|u - u_\Delta\|\|_{x_i,\lambda,\infty}$ | | | $\|\|u' - u'_\Delta\|\|_{\tau_i,\lambda_i,\infty}$ | | | $\|\|u'' - u''_\Delta\|\|_{\tau_i,x_i,\infty}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| 8 | 4.0-06 | 9.0-06 | 9.0-06 | 3.7-04 | 1.2-04 | 3.7-04 | 1.9-03 | 2.5-02 | 2.3-02 |
| 16 | 1.3-07 | 3.2-07 | 3.2-07 | 2.7-05 | 7.7-06 | 2.7-05 | 1.6-04 | 3.2-03 | 2.6-03 |
| 32 | 4.9-09 | 1.1-08 | 1.1-08 | 1.8-06 | 5.5-07 | 1.8-06 | 1.4-05 | 4.0-04 | 3.4-04 |
| 64 | 1.7-10 | 3.6-10 | 3.6-10 | 1.2-07 | 3.6-08 | 1.2-07 | 2.2-06 | 5.0-05 | 4.5-05 |
| 128 | 5.5-12 | 1.1-11 | 1.1-11 | 7.6-09 | 2.3-09 | 7.6-09 | 3.1-07 | 6.3-06 | 5.5-06 |
| 256 | 1.7-13 | 3.6-13 | 3.6-13 | 4.8-10 | 1.5-10 | 4.8-10 | 4.1-08 | 7.9-07 | 6.9-07 |
| | | | | orders of convergence | | | | | |
| 16 | 4.97 | 4.80 | 4.80 | 3.76 | 4.01 | 3.77 | 3.56 | 2.98 | 3.12 |
| 32 | 4.70 | 4.86 | 4.86 | 3.89 | 3.80 | 3.89 | 3.48 | 2.99 | 2.93 |
| 64 | 4.87 | 4.93 | 4.94 | 3.94 | 3.90 | 3.94 | 2.73 | 2.99 | 2.94 |
| 128 | 4.93 | 4.96 | 4.98 | 3.97 | 3.95 | 3.97 | 2.84 | 2.99 | 3.02 |
| 256 | 4.97 | 4.98 | 4.98 | 3.98 | 3.97 | 3.98 | 2.92 | 2.99 | 2.98 |

| $N$ | $\|\|u^{(3)} - u^{(3)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | | $\|\|u^{(4)} - u^{(4)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | |
|---|---|---|---|---|---|---|---|---|
| 8 | 7.0-01 | 2.3+00 | 4.5-01 | 2.2+00 | 6.3+00 | 9.0+01 | 5.3+01 | 8.8+01 |
| 16 | 2.0-01 | 5.9-01 | 1.1-01 | 5.5-01 | 3.0+00 | 4.6+01 | 2.7+01 | 4.4+01 |
| 32 | 5.5-02 | 1.5-01 | 2.8-02 | 1.2-01 | 1.4+00 | 2.3+01 | 1.3+01 | 2.1+01 |
| 64 | 1.4-02 | 3.7-02 | 7.0-03 | 2.7-02 | 7.3-01 | 1.1+01 | 7.0+00 | 1.0+01 |
| 128 | 3.6-03 | 9.4-03 | 1.7-03 | 6.5-03 | 3.6-01 | 5.8+00 | 3.5+00 | 5.3+00 |
| 256 | 9.2-04 | 2.3-03 | 4.3-04 | 1.6-03 | 1.8-01 | 2.9+00 | 1.7+00 | 2.6+00 |
| | | | | orders of convergence | | | | |
| 16 | 1.76 | 1.97 | 2.00 | 2.03 | 1.05 | 0.97 | 0.96 | 0.99 |
| 32 | 1.89 | 1.98 | 2.00 | 2.10 | 1.02 | 0.98 | 0.98 | 1.02 |
| 64 | 1.94 | 1.99 | 2.00 | 2.24 | 1.01 | 0.99 | 0.99 | 1.07 |
| 128 | 1.97 | 1.99 | 2.00 | 2.04 | 1.00 | 0.99 | 0.99 | 0.96 |
| 256 | 1.98 | 1.99 | 2.00 | 1.98 | 1.00 | 0.99 | 0.99 | 0.98 |

Table 2.2: Errors and the corresponding orders of convergence of a quartic spline interpolant for the BCs (2.8)

| $N$ | $\|\|u-u_\Delta\|\|_{x_i,\lambda,\infty}$ | | | $\|\|u'-u'_\Delta\|\|_{\tau_i,\lambda_i,\infty}$ | | | $\|\|u''-u''_\Delta\|\|_{\tau_i,x_i,\infty}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| 8 | 3.5-06 | 9.9-06 | 9.9-06 | 4.2-04 | 1.2-04 | 4.6-04 | 1.9-03 | 2.6-03 | 2.1-02 |
| 16 | 5.6-08 | 2.9-07 | 2.9-07 | 2.6-05 | 6.5-06 | 2.9-05 | 1.2-04 | 1.6-04 | 2.7-03 |
| 32 | 8.9-10 | 8.7-09 | 8.8-09 | 1.6-06 | 3.6-07 | 1.8-06 | 8.1-06 | 1.0-05 | 3.4-04 |
| 64 | 1.4-11 | 2.6-10 | 2.6-10 | 1.0-07 | 2.1-08 | 1.1-07 | 5.1-07 | 6.7-07 | 4.2-05 |
| 128 | 2.2-13 | 8.2-12 | 8.3-12 | 6.3-09 | 1.2-09 | 7.1-09 | 3.2-08 | 4.2-08 | 5.2-06 |
| 256 | 3.5-15 | 2.5-13 | 2.6-13 | 3.9-10 | 7.7-11 | 4.5-10 | 2.0-09 | 2.6-09 | 6.5-07 |
| | orders of convergence | | | | | | | | |
| 16 | 5.96 | 5.09 | 5.08 | 4.00 | 4.29 | 3.98 | 3.93 | 3.96 | 2.97 |
| 32 | 5.98 | 5.05 | 5.04 | 4.00 | 4.18 | 4.02 | 3.96 | 3.98 | 2.99 |
| 64 | 5.99 | 5.02 | 5.02 | 4.00 | 4.10 | 3.98 | 3.98 | 3.99 | 3.01 |
| 128 | 5.99 | 5.01 | 5.01 | 4.00 | 4.05 | 3.98 | 3.99 | 3.99 | 3.01 |
| 256 | 5.96 | 5.00 | 5.00 | 4.00 | 4.02 | 3.99 | 3.97 | 3.98 | 2.99 |

| $N$ | $\|\|u^{(3)}-u^{(3)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | | $\|\|u^{(4)}-u^{(4)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | |
|---|---|---|---|---|---|---|---|---|
| 8 | 8.5-01 | 1.7+00 | 7.6-02 | 1.6+00 | 1.4+00 | 8.4+01 | 4.8+01 | 8.3+01 |
| 16 | 2.2-01 | 4.4-01 | 9.8-03 | 4.0-01 | 3.8-01 | 4.3+01 | 2.4+01 | 4.1+01 |
| 32 | 5.6-02 | 1.1-01 | 1.2-03 | 1.0-01 | 9.8-02 | 2.1+01 | 1.2+01 | 2.0+01 |
| 64 | 1.4-02 | 2.8-02 | 1.5-04 | 2.6-02 | 2.4-02 | 1.0+01 | 6.3+00 | 1.0+01 |
| 128 | 3.5-03 | 7.1-03 | 1.9-05 | 6.7-03 | 6.2-03 | 5.4+00 | 3.1+00 | 5.3+00 |
| 256 | 8.9-04 | 1.7-03 | 2.5-06 | 1.7-03 | 1.5-03 | 2.7+00 | 1.5+00 | 2.6+00 |
| | orders of convergence | | | | | | | |
| 16 | 1.94 | 1.97 | 2.94 | 2.04 | 1.93 | 0.97 | 0.95 | 0.99 |
| 32 | 1.97 | 1.98 | 2.97 | 1.95 | 1.96 | 0.98 | 0.98 | 0.99 |
| 64 | 1.98 | 1.99 | 2.98 | 1.96 | 1.98 | 0.99 | 0.99 | 0.97 |
| 128 | 1.99 | 1.99 | 2.99 | 1.98 | 1.99 | 0.99 | 0.99 | 0.98 |
| 256 | 1.99 | 1.99 | 2.98 | 1.99 | 1.98 | 0.99 | 0.99 | 0.99 |

Table 2.3: Errors and the corresponding orders of convergence of a quartic spline interpolant for the BCs (2.9)

| $N$ | $\|\|u-u_\Delta\|\|_{x_i,\lambda,\infty}$ | | | $\|\|u'-u'_\Delta\|\|_{\tau_i,\lambda_i,\infty}$ | | | $\|\|u''-u''_\Delta\|\|_{\tau_i,x_i,\infty}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| 8 | 2.1-05 | 2.6-05 | 2.7-05 | 1.0-03 | 4.4-04 | 1.7-03 | 6.0-03 | 7.5-02 | 7.5-02 |
| 16 | 6.2-07 | 8.1-07 | 8.3-07 | 6.4-05 | 2.6-05 | 1.0-04 | 6.5-04 | 9.6-03 | 9.6-03 |
| 32 | 1.8-08 | 2.5-08 | 2.5-08 | 4.0-06 | 1.6-06 | 6.2-06 | 7.4-05 | 1.2-03 | 1.2-03 |
| 64 | 5.7-10 | 7.9-10 | 8.0-10 | 2.5-07 | 1.0-07 | 3.2-07 | 8.8-06 | 1.5-04 | 1.4-04 |
| 128 | 1.7-11 | 2.4-11 | 2.4-11 | 1.5-08 | 6.2-09 | 1.5-08 | 1.0-06 | 1.9-05 | 1.7-05 |
| 256 | 5.5-13 | 7.7-13 | 7.2-13 | 9.8-10 | 3.8-10 | 9.8-10 | 1.3-07 | 2.4-06 | 1.5-06 |
| | | | | orders of convergence | | | | | |
| 16 | 5.09 | 5.01 | 5.02 | 3.98 | 4.06 | 4.03 | 3.21 | 2.97 | 2.97 |
| 32 | 5.05 | 5.00 | 5.01 | 3.99 | 4.03 | 4.11 | 3.13 | 2.98 | 2.99 |
| 64 | 5.02 | 5.00 | 5.01 | 3.99 | 4.01 | 4.29 | 3.07 | 2.99 | 3.02 |
| 128 | 5.01 | 5.00 | 5.02 | 3.99 | 4.00 | 4.34 | 3.03 | 2.99 | 3.10 |
| 256 | 5.00 | 5.00 | 5.07 | 3.99 | 4.00 | 3.99 | 3.01 | 2.99 | 3.45 |

| $N$ | $\|\|u^{(3)}-u^{(3)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | | $\|\|u^{(4)}-u^{(4)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | |
|---|---|---|---|---|---|---|---|---|
| 8 | 1.5+00 | 1.8+00 | 1.2+00 | 1.8+00 | 1.6+01 | 9.2+01 | 6.1+01 | 9.1+01 |
| 16 | 3.9-01 | 5.1-01 | 3.3-01 | 4.9-01 | 8.5+00 | 4.9+01 | 3.2+01 | 4.8+01 |
| 32 | 1.0-01 | 1.3-01 | 8.4-02 | 1.2-01 | 4.3+00 | 2.5+01 | 1.6+01 | 2.5+01 |
| 64 | 2.5-02 | 3.4-02 | 2.1-02 | 3.2-02 | 2.1+00 | 1.3+01 | 8.4+00 | 1.2+01 |
| 128 | 6.3-03 | 8.5-03 | 5.3-03 | 8.2-03 | 1.1+00 | 6.5+00 | 4.2+00 | 6.4+00 |
| 256 | 1.5-03 | 2.1-03 | 1.3-03 | 2.0-03 | 5.5-01 | 3.2+00 | 2.1+00 | 3.2+00 |
| | | | orders of convergence | | | | | |
| 16 | 1.95 | 1.88 | 1.96 | 1.93 | 0.94 | 0.90 | 0.91 | 0.90 |
| 32 | 1.97 | 1.94 | 1.98 | 1.94 | 0.97 | 0.95 | 0.95 | 0.95 |
| 64 | 1.99 | 1.97 | 1.99 | 1.97 | 0.98 | 0.97 | 0.98 | 0.97 |
| 128 | 1.99 | 1.98 | 1.99 | 1.98 | 0.99 | 0.98 | 0.99 | 0.98 |
| 256 | 1.99 | 1.99 | 1.99 | 1.99 | 0.99 | 0.99 | 0.99 | 0.99 |

## 2.2 Error Formulae

In this section we define a quartic spline interpolant $S$ satisfying certain boundary conditions and prove some formulae for the error of $S$ and its derivatives on certain points. These formulae will be used later in the formulation and analysis of an optimal quartic spline collocation method.

For convenience, let $u_i = u(\tau_i)$, $i = 0, \cdots, N+1$, and also denote the $k$th derivative $D^k u$ of $u$, by $u^{(k)}$. This notation applies to all functions throughout the thesis. Now we define a quartic spline interpolant $S$ of $u$ by forcing it to satisfy the following conditions:

$$\begin{aligned} S_i &= u_i & \text{for } i = 1, \cdots, N, && (2.10)\\ S_i^{(4)} &= u_i^{(4)} - \frac{h^2}{24}u_i^{(6)} + \frac{7h^4}{5760}u_i^{(8)} & \text{for } i = 1, 2, N-1, N. && (2.11) \end{aligned}$$

The interpolation conditions (2.10)-(2.11), with appropriate ordering, result in the linear system

$$\begin{bmatrix} \frac{24}{h^4} & \frac{-96}{h^4} & \frac{144}{h^4} & \frac{-96}{h^4} & \frac{24}{h^4} & & & & & \\ & \frac{24}{h^4} & \frac{-96}{h^4} & \frac{144}{h^4} & \frac{-96}{h^4} & \frac{24}{h^4} & & & & \\ \frac{1}{16} & \frac{76}{16} & \frac{230}{16} & \frac{76}{16} & \frac{1}{16} & & & & & \\ & \frac{1}{16} & \frac{76}{16} & \frac{230}{16} & \frac{76}{16} & \frac{1}{16} & & & & \\ & & & . & . & . & & & & \\ & & & & \frac{1}{16} & \frac{76}{16} & \frac{230}{16} & \frac{76}{16} & \frac{1}{16} & \\ & & & \frac{24}{h^4} & \frac{-96}{h^4} & \frac{144}{h^4} & \frac{-96}{h^4} & \frac{24}{h^4} & & \\ & & & & \frac{24}{h^4} & \frac{-96}{h^4} & \frac{144}{h^4} & \frac{-96}{h^4} & \frac{24}{h^4} & \end{bmatrix} \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ . \\ c_N \\ c_{N+1} \\ c_{N+2} \end{bmatrix} =$$

$$24 \begin{bmatrix} u^{(4)}(\tau_1) - \frac{h^2}{24}u^{(6)}(\tau_1) + \frac{7h^4}{5760}u^{(8)}(\tau_1) \\ u^{(4)}(\tau_2) - \frac{h^2}{24}u^{(6)}(\tau_2) + \frac{7h^4}{5760}u^{(8)}(\tau_2) \\ u(\tau_1) \\ u(\tau_2) \\ \cdot \\ u(\tau_N) \\ u^{(4)}(\tau_{N-1}) - \frac{h^2}{24}u^{(6)}(\tau_{N-1}) + \frac{7h^4}{5760}u^{(8)}(\tau_{N-1}) \\ u^{(4)}(\tau_N) - \frac{h^2}{24}u^{(6)}(\tau_N) + \frac{7h^4}{5760}u^{(8)}(\tau_N) \end{bmatrix}. \tag{2.12}$$

With appropriate scaling, and after substituting each of the first two rows with appropriate linear combinations of the first two rows, and ... the coefficient matrix of (2.12) can be transformed to a diagonally dominant matrix. Therefore, the linear system (2.12) has a unique solution, hence there exists a uniquely determined quartic spline interpolant $S$ of $u$, satisfying (2.10)-(2.11).

For any function $w$ defined in $I$, let the discrete operator $\Lambda$ be defined by

$$\Lambda w_i \equiv w_{i-2} + 76w_{i-1} + 230w_i + 76w_{i+1} + w_{i+2} \text{ for } i = 3, \cdots, N-2.$$

The following relation holds for any quartic spline $S$ defined on a uniform partition:

$$\Lambda S_i^{(4)} = \frac{384}{h^4}(S_{i-2} - 4S_{i-1} + 6S_i - 4S_{i+1} + S_{i+2}) \text{ for } i = 3, \cdots, N-2. \tag{2.13}$$

If $u \in C^{10}[I]$, using Taylor's series expansion and taking into account that $S_i = u_i$, $i = 3, \cdots, N-2$, we obtain the relation

$$\Lambda S_i^{(4)} = 384u_i^{(4)} + 64h^2u_i^{(6)} + \frac{24}{5}h^4u_i^{(8)} + O(h^6), \text{ for } i = 3, \cdots, N-2. \tag{2.14}$$

**THEOREM 1** *Let $S$ be the quartic spline interpolant of $u \in C^{10}[I]$ defined by (2.10)-(2.11). The following relations hold for $i = 1, \cdots, N$:*

$$S_i^{(1)} = u_i^{(1)} - \frac{7h^4}{5760}u_i^{(5)} + O(h^6), \tag{2.15}$$

$$S_i^{(2)} = u_i^{(2)} + \frac{7h^4}{1920}u_i^{(6)} + O(h^6), \tag{2.16}$$
$$S_i^{(3)} = u_i^{(3)} + \frac{h^2}{24}u_i^{(5)} - \frac{7h^4}{1920}u_i^{(7)} + O(h^6), \tag{2.17}$$
$$S_i^{(4)} = u_i^{(4)} - \frac{h^2}{24}u_i^{(6)} + \frac{7h^4}{5760}u_i^{(8)} + O(h^6). \tag{2.18}$$

*Proof*: First we prove (2.18). Using Taylor's series expansions, it can be shown that for any function $g \in C^6[I]$, $\Lambda g_i = 384g_i + 80h^2 g_i^{(2)} + \frac{23}{3}h^4 g_i^{(4)} + O(h^6)$ for $i = 3, \cdots, N-2$. Letting $g = u^{(4)} - \frac{h^2}{24}u^{(6)} + \frac{7h^4}{5760}u^{(8)}$, we have, for $i = 3, \cdots, N-2$,

$$\Lambda(u_i^{(4)} - \frac{h^2}{24}u_i^{(6)} + \frac{7h^4}{5760}u_i^{(8)}) = 384u_i^{(4)} + 64h^2 u_i^{(6)} + \frac{24}{5}h^4 u_i^{(8)} + O(h^6). \tag{2.19}$$

Define $d_i \equiv S_i^{(4)} - u_i^{(4)} - \frac{h^2}{24}u_i^{(6)} + \frac{7h^4}{5760}u_i^{(8)}$ for $i = 1, \cdots, N$, and subtract equation (2.19) from equation (2.14). From the resulting equation and the end conditions (2.11) on the four near-boundary midpoints, we get the following equations:

$$\Lambda d_i = O(h^6) \text{ for } i = 3, \cdots, N-2 \text{ and } d_1 = d_2 = d_{N-1} = d_N = 0, \tag{2.20}$$

where the $O(h^6)$ terms involves $u^{(10)}$. Equations (2.20), if ordered in the natural ordering, result in a linear system for the $d_i$'s, $i = 1, \cdots, N$, for which the coefficient matrix is strictly diagonally dominant. This implies that $d_i = O(h^6)$ for $i = 1, \cdots, N$ and thus proves relation (2.18).

Now we prove relation (2.16). From the definition of $S$, $S^{(2)}$, and $S^{(4)}$, it can be shown [22] that the relation

$$S_i^{(2)} = \frac{1}{h^2}(S_{i-1} - 2S_i + S_{i+1}) - \frac{h^2}{384}(S_{i-1}^{(4)} + 30S_i^{(4)} + S_{i+1}^{(4)}) \tag{2.21}$$

holds for $i = 2, \cdots, N-1$. By (2.10) and (2.18), the right-side of (2.21) can be written in terms of $u$ and the derivatives of $u$

$$\begin{aligned} S_i^{(2)} = \frac{1}{h^2}(u_{i-1} - 2u_i + u_{i+1}) - \frac{h^2}{384} \quad [ & \quad (u_{i-1}^{(4)} - \frac{h^2}{24}u_{i-1}^{(6)} + \frac{7h^4}{5760}u_{i-1}^{(8)} + O(h^6)) \\ & + \quad 30(u_i^{(4)} - \frac{h^2}{24}u_i^{(6)} + \frac{7h^4}{5760}u_i^{(8)} + O(h^6)) \\ & + \quad (u_{i+1}^{(4)} - \frac{h^2}{24}u_{i-1}^{(6)} + \frac{7h^4}{5760}u_{i-1}^{(8)} + O(h^6))]. \end{aligned}$$

After applying Taylor's series expansion to $u_{i\pm1}^{(k)}$ for $k = 0, 4, 6$ and $8$, we have, for $i = 2, \cdots, N-1$,

$$S_i^{(2)} = u_i^{(2)} + \frac{7h^4}{1920} u_i^{(6)} - \frac{31h^6}{193536} u_i^{(8)} + O(h^8). \tag{2.22}$$

This proves (2.16) for $i = 2, \cdots, N-1$. To obtain the result for the two near-boundary midpoints, we use the relations [22]

$$\begin{aligned} S_i^{(2)} &= 2S_{i+1}^{(2)} - S_{i+2}^{(2)} + \frac{h^2}{8}(S_i^{(4)} + 6S_{i+1}^{(4)} + S_{i+2}^{(4)}), \text{ for } i = 1, \cdots, N-2, && (2.23) \\ S_i^{(2)} &= 2S_{i-1}^{(2)} - S_{i-2}^{(2)} + \frac{h^2}{8}(S_i^{(4)} + 6S_{i-1}^{(4)} + S_{i-2}^{(4)}) \text{ for } i = 3, \cdots, N. && (2.24) \end{aligned}$$

We consider (2.23) for $i = 1$. The terms $S_2^{(2)}, S_3^{(2)}, S_1^{(4)}, S_2^{(4)}, S_3^{(4)}$ on the right side are substituted by expressions in $u$ and the derivatives of $u$ by using the already proved equations (2.18) (for $i = 1, \cdots, N$) and (2.22) (for $i = 2, \cdots, N-1$). The $u$ terms are expanded by Taylor's series and after some grouping and cancellations, relation (2.22) is proved for $i = 1$. The case of $i = N$ in (2.16) follows similarly from (2.24). Hence (2.16) holds for $i = 1, \cdots, N$.

The following equations relate $S^{(3)}$ to $S^{(2)}$ and $S^{(4)}$, according to their definition [22]:

$$\begin{aligned} S_i^{(3)} &= \frac{1}{h}(S_{i+1}^{(2)} - S_i^{(2)}) - \frac{h}{8}(S_{i+1}^{(4)} + 3S_i^{(4)}) \text{ for } i = 1, \cdots, N-1, && (2.25) \\ S_i^{(3)} &= \frac{1}{h}(-S_{i-1}^{(2)} + S_i^{(2)}) + \frac{h}{8}(S_{i-1}^{(4)} + 3S_i^{(4)}) \text{ for } i = 2, \cdots, N. && (2.26) \end{aligned}$$

Relation (2.17) follows directly from these equations and relations (2.22), and (2.18), after expanding in Taylor's series.

Relation (2.15) can be proved in a similar way using the following equations that relate $S^{(1)}$ to $S^{(2)}$, and $S^{(4)}$:

$$\begin{aligned} S_i^{(1)} &= \frac{1}{h}(S_{i+1} - S_i) - \frac{h}{6}(S_{i+1}^{(2)} + 2S_i^{(2)}) \\ &+ \frac{h^3}{384}(7S_{i+1}^{(4)} + 9S_i^{(4)}) \text{ for } i = 1, \cdots, N-1, && (2.27) \\ S_i^{(1)} &= \frac{1}{h}(-S_{i-1} + S_i) + \frac{h}{6}(S_{i-1}^{(2)} + 2S_i^{(2)}) \\ &- \frac{h^3}{384}(7S_{i-1}^{(4)} + 9S_i^{(4)}) \text{ for } i = 2, \cdots, N. && (2.28) \end{aligned}$$

This completes the proof. $\diamondsuit$

**THEOREM 2** *Let $S$ be the quartic spline interpolant of $u \in C^{10}[I]$ defined by (2.10)-(2.11). The following relations hold on the gridpoints $x_i$ for $i = 0, \cdots, N$:*

$$\begin{aligned} S(x_i) &= u(x_i) + \frac{h^6}{3072}u^{(6)}(x_i) + O(h^8), && (2.29)\\ S^{(1)}(x_i) &= u^{(1)}(x_i) + \frac{h^4}{720}u^{(5)}(x_i) + O(h^6), && (2.30)\\ S^{(2)}(x_i) &= u^{(2)}(x_i) - \frac{h^4}{240}u^{(6)}(x_i) + O(h^6), && (2.31)\\ S^{(3)}(x_i) &= u^{(3)}(x_i) - \frac{h^2}{12}u^{(5)}(x_i) + \frac{h^4}{240}u^{(7)}(x_i) + O(h^6). && (2.32) \end{aligned}$$

*Proof*: Each of the four relations on the gridpoints can be proved similarly as (2.16) in Theorem 1 is proved, from the error bounds on midpoints stated in Theorem 1 and two corresponding equations in the list below:

$$\begin{aligned} S(x_i) &= \frac{1}{2}(S_{i+2} + S_{i+1}) - \frac{h}{8}(S_{i+2}^{(1)} + 7S_{i+1}^{(1)})\\ &- \frac{h^3}{384}(S_{i+2}^{(3)} - 17S_{i+1}^{(3)}) \ , \ i = 0, \cdots, N-2; && (2.33)\\ S(x_i) &= \frac{1}{2}(S_i + S_{i-1}) + \frac{h}{8}(7S_i^{(1)} + 7S_{i-1}^{(1)})\\ &+ \frac{h^3}{384}(17S_i^{(3)} - S_{i-1}^{(3)}) \ , \ i = 2, \cdots, N; && (2.34)\\ S^{(1)}(x_i) &= \frac{1}{h}(S_{i+2} - S_{i+1}) - \frac{h}{24}(S_{i+2}^{(2)} + 23S_{i+1}^{(2)})\\ &+ \frac{h^3}{384}(S_{i+2}^{(4)} - 17S_{i+1}^{(4)}) \ , \ i = 0, \cdots, N-2; && (2.35)\\ S^{(1)}(x_i) &= \frac{1}{h}(S_i - S_{i-1}) + \frac{h}{24}(23S_i^{(2)} + S_{i-1}^{(2)})\\ &+ \frac{h^3}{384}(17S_i^{(4)} - S_{i-1}^{(4)}) \ , \ i = 2, \cdots, N; && (2.36)\\ S^{(2)}(x_i) &= \frac{1}{h}(S_{i+2}^{(1)} - S_{i+1}^{(1)})\\ &- \frac{h}{24}(S_{i+2}^{(3)} + 23S_{i+1}^{(3)}) \ , \ i = 0, \cdots, N-2; && (2.37)\\ S^{(2)}(x_i) &= \frac{1}{h}(S_i^{(1)} - S_{i-1}^{(1)})\\ &+ \frac{h}{24}(23S_i^{(3)} + S_{i-1}^{(3)}) \ , \ i = 2, \cdots, N; && (2.38)\\ S^{(3)}(x_i) &= \frac{1}{h}(S_{i+2}^{(2)} - S_{i+1}^{(2)}) \end{aligned}$$

$$
\begin{aligned}
& - \frac{h}{8}(S^{(4)}_{i+2} + 7S^{(4)}_{i+1}) \ , \ i = 0, \cdots, N-2; \qquad (2.39) \\
S^{(3)}(x_i) &= \frac{1}{h}(S^{(2)}_i - S^{(2)}_{i-1}) \\
& + \frac{h}{8}(7S^{(4)}_i + S^{(4)}_{i-1}) \ , \ i = 2, \cdots, N. \qquad (2.40)
\end{aligned}
$$

$\diamondsuit$

**THEOREM 3** *We have the global error bounds*

$$||S^{(k)} - u^{(k)}||_\infty = O(h^{5-k}), \ \textit{for } k = 0, \cdots, 4 \qquad (2.41)$$

*Proof*: We denote the interpolation error by $e(x) = S(x) - u(x)$. By the Mean Value Theorem, there is a $\xi \in (x_i, x_{i+1})$ such that $e^{(4)}(\xi) = (e^{(3)}(x_{i+1}) - e^{(3)}(x_i))/h$. From (2.32) we have $e^{(4)}(\xi) = O(h)$. Since $S^{(4)}(x)$ is a constant piecewise polynomial by definition, $e^{(4)}(x) = S^{(4)}(\xi) - u^{(4)}(\xi) - (u^{(4)}(x) - u^{(4)}(\xi)) = e^{(4)}(\xi) - (u^{(4)}(x) - u^{(4)}(\xi))$ for any $x \in (x_i, x_{i+1}), i = 1, \cdots, N-1$. $||e^{(4)}||_\infty = O(h)$ follows from the observation that $u^{(4)}(x) - u^{(4)}(\xi) = O(h)$ by Taylor's series expansion. For $||e^{(3)}||_\infty$ we consider $\int_{x_i}^x e^{(4)}(t)dt = e^{(3)}(x) - e^{(3)}(x_i) = O(h) \cdot e^{(4)}(x_i) = O(h^2)$ which implies $||e^{(3)}||_\infty = O(h^2)$. Similar reasoning proves the remaining bounds in (2.41). $\diamondsuit$

For the rest of this thesis, we define two discrete difference operators,

$$\delta g_i \equiv g_{i-1} - 2g_i + g_{i+1}, \quad i = 2, \cdots, N-1,$$

$$\delta^2 g_i \equiv g_{i-2} - 4g_{i-1} + 6g_i - 4g_{i+1} + g_{i+2}, \quad i = 3, \cdots, N-2,$$

Note that if $g$ is sufficiently smooth we have

$$g^{(2)}_i = \frac{\delta g_i}{h^2} + O(h^2)$$

and

$$g^{(4)}_i = \frac{\delta^2 g_i}{h^4} + O(h^2).$$

**THEOREM 4** *Let $S$ be the quartic spline interpolant of $u \in C^{10}[I]$ defined by (2.10)-(2.11). Then the following relations hold for $i = 3, \cdots, N-2$:*

$$\begin{aligned}
u_i^{(5)} &= \frac{\delta^2 S_i^{(1)}}{h^4} + O(h^2) &(2.42)\\
&= \frac{8\delta S_i^{(3)} - \delta^2 S_i^{(3)}}{8h^2} + O(h^4) &(2.43)\\
u_i^{(6)} &= \frac{\delta^2 S_i^{(2)}}{h^4} + O(h^2) &(2.44)\\
&= \frac{24\delta S_i^{(4)} - \delta^2 S_i^{(4)}}{24h^2} + O(h^4) &(2.45)\\
u_i^{(7)} &= \frac{\delta^2 S_i^{(3)}}{h^4} + O(h^2) &(2.46)\\
u_i^{(8)} &= \frac{\delta^2 S_i^{(4)}}{h^4} + O(h^2) &(2.47)
\end{aligned}$$

*Proof*: We use the finite difference (FD) relation

$$u_i^{(8)} = \frac{\delta^2 u_i^{(4)}}{h^4} + O(h^2) \tag{2.48}$$

and (2.18) to prove equation (2.47). By equation (2.18), we have $u_k^{(4)} = S_k^{(4)} + \frac{h^2}{24}u_k^{(6)} - \frac{7h^4}{5760}u_k^{(8)} + O(h^6)$ for $k = i-2, i-1, i, i+1$ and $i+2$. After substituting these into the right-side (2.48) we get $u_i^{(8)} = \frac{\delta^2 S_i^{(4)}}{h^4} + \frac{\delta^2 u_i^{(6)}}{24h^2} - \frac{7\delta^2 u_i^{(8)}}{5760} + O(h^2)$. Next we expand $u_{i\pm2}^{(6)}, u_{i\pm1}^{(6)}, u_{i\pm2}^{(8)}$ and $u_{i\pm1}^{(8)}$ by Taylor's series in the expressions $\frac{\delta^2 u_i^{(6)}}{24h^2}$ and $\frac{7\delta^2 u_i^{(8)}}{5760}$ to obtain, respectively:

$$\begin{aligned}
\frac{\delta^2 u_i^{(6)}}{24h^2} &= \frac{1}{24h^2}[(u_i^{(6)} - 2hu_i^{(7)} + 2h^2u_i^{(8)} - \frac{4h^3}{3}u_i^{(9)} + O(h^4))\\
&\quad -4(u_i^{(6)} - hu_i^{(7)} + \frac{h^2}{2}u_i^{(8)} - \frac{h^3}{6}u_i^{(9)} + O(h^4))\\
&\quad +6u_i^{(6)}\\
&\quad -4(u_i^{(6)} + hu_i^{(7)} + \frac{h^2}{2}u_i^{(8)} + \frac{h^3}{6}u_i^{(9)} + O(h^4))\\
&\quad +(u_i^{(6)} + 2hu_i^{(7)} + 2h^2u_i^{(8)} + \frac{4h^3}{3}u_i^{(9)} + O(h^4))]\\
&= \frac{1}{24h^2}[O(h^4)u_i^{(10)}]\\
&= O(h^2)\\
\frac{7\delta^2 u_i^{(8)}}{5760} &= \frac{7}{5760}[(u_i^{(8)} - 2hu_i^{(9)} + \cdots) - 4(u_i^{(8)} - hu_i^{(9)} + \cdots) + 6u_i^{(8)}
\end{aligned}$$

$$
\begin{aligned}
& -4(u_i^{(8)} + hu_i^{(9)} + \cdots) + (u_i^{(8)} + 2hu_i^{(9)} + \cdots)] \\
= \; & \frac{7}{5760}[O(h^4)u_i^{(12)}] \\
= \; & O(h^4).
\end{aligned}
$$

Equation (2.47) follows immediately.

Equation (2.46) is proved similarly from the FD relation $u_i^{(7)} = \frac{\delta^2 u_i^{(3)}}{h^4} + O(h^2)$ and (2.17), equation (2.44) from FD relation $u_i^{(6)} = \frac{\delta^2 u_i^{(2)}}{h^4} + O(h^2)$ and (2.16), and equation (2.42) from FD relation $u_i^{(5)} = \frac{\delta^2 u_i^{(1)}}{h^4} + O(h^2)$ and (2.15).

To prove (2.43), we use (2.17) and the FD relation $u_i^{(5)} = \frac{\delta u_i^{(3)}}{h^2} - \frac{h^2}{12}u_i^{(7)} + O(h^4)$ to derive the equation $u_i^{(5)} = \frac{\delta S_i^{(3)}}{h^2} - \frac{h^2}{8}u_i^{(7)} + O(h^4)$. Substituting the right-side of (2.46) for $u_i^{(7)}$, we get $u_i^{(5)} = \frac{\delta S_i^{(3)}}{h^2} - \frac{\delta^2 S_i^{(3)}}{8h^2}$.

We prove (2.45) in the same manner from the FD relation $u_i^{(6)} = \frac{\delta u_i^{(4)}}{h^2} - \frac{h^2}{12}u_i^{(8)} + O(h^4)$, (2.18), and (2.47). ◇

A direct consequence of the above theorem is the following corollary, which gives approximations to the derivatives of $u$ on the interior midpoints.

**COROLLARY 1** *Under the hypotheses of Theorem 4, we have the following approximations for $i = 3, \cdots, N-2$:*

$$u_i^{(1)} = S_i^{(1)} + \frac{7\delta^2 S_i^{(1)}}{5760} + O(h^6) \tag{2.49}$$

$$u_i^{(2)} = S_i^{(2)} - \frac{7\delta^2 S_i^{(2)}}{1920} + O(h^6) \tag{2.50}$$

$$u_i^{(3)} = S_i^{(3)} - \frac{80\delta S_i^{(3)} - 17\delta^2 S_i^{(3)}}{1920} + O(h^6) \tag{2.51}$$

$$u_i^{(4)} = S_i^{(4)} + \frac{240\delta S_i^{(4)} - 17\delta^2 S_i^{(4)}}{5760} + O(h^6) \tag{2.52}$$

*Proof*: Approximations (2.49) and (2.50) follow immediately from (2.15) and (2.42), and (2.15) and (2.44), respectively. Equation (2.51) is derived from (2.17), (2.43), and (2.46), and equation (2.52) from (2.18), (2.45), and (2.47). ◇

In order to obtain approximations for the derivatives of $u$ of order $k = 5, 6, 7$ and 8 at $\{x_0, \tau_1, \tau_2, \tau_{N-1}, \tau_N, x_N\}$, we make use of some relations that are derived by finding the appropriate linear combination of Taylor's series expansions around the point $\tau_i$ or $x_i$:

$$\begin{aligned}
u^{(k)}(\tau_i) &= 2u^{(k)}(\tau_{i\pm1}) - u^{(k)}(\tau_{i\pm2}) + O(h^2), \\
u^{(k)}(\tau_i) &= 4u^{(k)}(\tau_{i\pm1}) - 6u^{(k)}(\tau_{i\pm2}) + 4u^{(k)}(\tau_{i\pm3}) - u^{(k)}(\tau_{i\pm4}) + O(h^4), \\
u^{(k)}(x_i) &= \frac{1}{2}(3u^{(k)}(\tau_{i\pm1}) - u^{(k)}(\tau_{i\pm2})) + O(h^2), \\
u^{(k)}(x_i) &= \frac{1}{16}(35u^{(k)}(\tau_{i\pm1}) - 35u^{(k)}(\tau_{i\pm2}) + 21u^{(k)}(\tau_{i\pm3}) - 5u^{(k)}(\tau_{i\pm4})) + O(h^4),
\end{aligned}$$

where $k = 5, 6, 7, 8$. The approximations to higher derivatives are thus obtained by linear combinations of approximations at interior midpoints. We let $\gamma_i \equiv i$ for $i = 0, 1, 2$ (points $x_0, \tau_1, \tau_2$, respectively), and $\gamma_i \equiv N + 1 - i$ for $i = 0, 1, 2$ (points $x_N, \tau_N, \tau_{N-1}$, respectively).

**LEMMA 1** *If $u \in C^{10}[I]$ then the following approximations to higher derivatives hold at boundary and near-boundary points*
*for $k = 5, 6, 7, 8$*

$$u_{\gamma_2}^{(k)} = \frac{1}{h^4}(2\delta^2 S_{\gamma_3}^{(k-4)} - \delta^2 S_{\gamma_4}^{(k-4)}) + O(h^2), \tag{2.53}$$

$$u_{\gamma_1}^{(k)} = \frac{1}{h^4}(3\delta^2 S_{\gamma_3}^{(k-4)} - 2\delta^2 S_{\gamma_4}^{(k-4)}) + O(h^2), \tag{2.54}$$

$$u_{\gamma_0}^{(k)} = \frac{1}{2h^4}(7\delta^2 S_{\gamma_3}^{(k-4)} - 5\delta^2 S_{\gamma_4}^{(k-4)}) + O(h^2), \tag{2.55}$$

*and for (j, l) = (5, 8) and (j, l) = (6, 24)*

$$u_{\gamma_2}^{(j)} = \frac{1}{h^2}\eta_l S_{\gamma_2}^{(j-2)} + O(h^4), \tag{2.56}$$

$$u_{\gamma_1}^{(j)} = \frac{1}{h^2}\zeta_l S_{\gamma_2}^{(j-2)} + O(h^4), \tag{2.57}$$

$$u_{\gamma_0}^{(j)} = \frac{1}{h^2}\theta_l S_{\gamma_2}^{(j-2)} + O(h^4), \tag{2.58}$$

*where $\eta_l$, $\zeta_l$, and $\theta_l$ are discrete operators defined by*

$$\begin{aligned}
\eta_l z_{\gamma_i} \equiv\ & \frac{1}{l}[4(l \cdot \delta z_{\gamma_{i+1}} - \delta^2 z_{\gamma_{i+1}}) - 6(l \cdot \delta z_{\gamma_{i+2}} - \delta^2 z_{\gamma_{i+2}}) + 4(l \cdot \delta z_{\gamma_{i+3}} - \delta^2 z_{\gamma_{i+3}}) \\
& -(l \cdot \delta z_{i+4} - \delta^2 z_{\gamma_{i+4}})],
\end{aligned} \tag{2.59}$$

$$\begin{aligned}
\zeta_l z_{\gamma_i} &\equiv \frac{1}{l}[10(l\cdot\delta z_{\gamma_{i+1}} - \delta^2 z_{\gamma_{i+1}}) - 20(l\cdot\delta z_{\gamma_{i+2}} - \delta^2 z_{\gamma_{i+2}}) + 15(l\cdot\delta z_{\gamma_{i+3}} - \delta^2 z_{\gamma_{i+3}}) \\
&\quad -4(l\cdot\delta z_{\gamma_{i+4}} - \delta^2 z_{\gamma_{i+4}})], \qquad (2.60) \\
\theta_l z_{\gamma_i} &\equiv \frac{1}{16l}[231(l\cdot\delta z_{\gamma_{i+1}} - \delta^2 z_{\gamma_{i+1}}) - 495(l\cdot\delta z_{\gamma_{i+2}} - \delta^2 z_{\gamma_{i+2}}) + 385(l\cdot\delta z_{\gamma_{i+3}} - \delta^2 z_{\gamma_{i+3}}) \\
&\quad -105(l\cdot\delta z_{\gamma_{i+4}} - \delta^2 z_{\gamma_{i+4}})]. \qquad (2.61)
\end{aligned}$$

*for any function* $z$.

The approximations to the first four derivatives of $u$ at the six boundary and near-boundary points can in turn be derived directly from these higher-derivative approximations and the equations for error bounds (2.15) to (2.18).

# Chapter 3

# Quartic Spline Collocation

## 3.1 Fourth order two-point boundary value problem

The aim of this chapter is to present and analyze an optimal quartic spline collocation method for finding a numerical solution of the fourth order two-point boundary value problem

$$Lu \equiv u^{(4)} + p(x)u^{(3)} + q(x)u^{(2)} + r(x)u^{(1)} + s(x)u = f \quad \text{on } I = [a, b] \tag{3.1}$$

subject to boundary conditions

$$\begin{aligned} Bu \equiv \{ \quad B_1 u \equiv & \sum_{i=0}^{3} \alpha_{0,i} u^{(i)}(a) = g_1, \quad B_2 u \equiv \sum_{i=0}^{3} \beta_{0,i} u^{(i)}(a) = g_2, \\ B_3 u \equiv & \sum_{i=0}^{3} \alpha_{N+1,i} u^{(i)}(b) = g_3, \quad B_4 u \equiv \sum_{i=0}^{3} \beta_{N+1,i} u^{(i)}(b) = g_4 \ \}, \end{aligned} \tag{3.2}$$

where $p(x), q(x), r(x), s(x), f(x)$ are given functions, $\alpha_{0,i}, \alpha_{N+1,i}, \beta_{0,i}, \beta_{N+1,i}$, $i = 0, \cdots, 3$, $g_1$, $g_2$, $g_3$, and $g_4$ are given scalars and $u(x)$ is the unknown function. We assume, for the remainder of the chapter, that the solution $u$ to the problem is in $C^{10}[I]$ and $S$ is the interpolant of $u$ defined by (2.10) and (2.11).

To make the presentation of the optimal method easier to understand, we first describe the straightforward quartic spline collocation method that is non-optimal. The procedure is similar to interpolation; in fact, if $L$ and $B$ are the identity operator, then collocation becomes exactly interpolation. Let $u_\Delta(x) = \sum_{i=-1}^{n+2} c_i\Phi_i(x)$ denote the quartic spline collocation approximation of $u(x)$, it follows then that $u_\Delta^{(k)}(x) = \sum_{i=-1}^{n+2} c_i\Phi_i^{(k)}(x)$, $k = 1,\cdots,4$. The collocation conditions

$$\begin{aligned} B_k u_\Delta(x_0) &= g_k, \quad k=1,2 \\ Lu_\Delta(\tau_i) &= f(\tau_i), \quad i=1,\cdots,N \\ B_k u_\Delta(x_N) &= g_k, \quad k=3,4 \end{aligned} \tag{3.3}$$

give rise to a system of $N+4$ equations with $N+4$ unknowns, the degrees of freedom, $c_i$, $i=-1,\cdots,N+2$, which determine $u_\Delta$. The existence of a solution and its uniqueness are proved later.

## 3.2 Quartic spline interpolant relations

Recall that $\gamma_i = i$ on or near the left boundary and $\gamma_i = N+1-i$ on or near the right boundary, for $i=0,1,\cdots,7$. By Theorem 1 and Theorem 2, it is clear that $S$ satisfies

$$\begin{aligned} LS_i &= f_i - \frac{h^2}{24}u_i^{(6)} + \frac{7h^4}{5760}u_i^{(8)} + p_i(\frac{h^2}{24}u_i^{(5)} - \frac{7h^4}{1920}u_i^{(7)}) + \frac{7h^4}{1920}q_i u_i^{(6)} \\ &\quad - \frac{7h^4}{5760}r_i u_i^{(5)} + O(h^6) \qquad \text{for} \quad i=1,\cdots,N \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} B_k S_{\gamma_0} &= g_k - \alpha_{\gamma_0,3}(\frac{h^2}{12}u_{\gamma_0}^{(5)} - \frac{h^4}{240}u_{\gamma_0}^{(7)}) - \frac{h^4}{240}\alpha_{\gamma_0,2}u_{\gamma_0}^{(6)} + \frac{h^4}{720}\alpha_{\gamma_0,1}u_{\gamma_0}^{(5)} \\ &\quad + O(h^6) \qquad \text{for} \quad k=1,3 \end{aligned} \tag{3.5}$$

$$\begin{aligned} B_k S_{\gamma_0} &= g_k - \beta_{\gamma_0,3}(\frac{h^2}{12}u_{\gamma_0}^{(5)} - \frac{h^4}{240}u_{\gamma_0}^{(7)}) - \frac{h^4}{240}\beta_{\gamma_0,2}u_{\gamma_0}^{(6)} + \frac{h^4}{720}\beta_{\gamma_0,1}u_{\gamma_0}^{(5)} \\ &\quad + O(h^6) \qquad \text{for} \quad k=2,4 \end{aligned} \tag{3.6}$$

Let $\eta_l$, $\zeta_l$, and $\theta_l$ be defined as by (2.59)-(2.61). Then the approximations of derivatives stated in Theorem 3, Corollary 1, and Lemma 1 allow (3.4) and (3.5)-(3.6) to be written as

$$
\begin{aligned}
LS_{\gamma_1} &+ \frac{1}{24}\zeta_{24}S^{(4)}_{\gamma_2} - \frac{7}{5760}(3\delta^2 S^{(4)}_{\gamma_3} - 2\delta^2 S^{(4)}_{\gamma_4}) - p_{\gamma_1}[\frac{1}{24}\zeta_8 S^{(3)}_{\gamma_2} - \frac{7}{1920}(3\delta^2 S^{(3)}_{\gamma_3} - 2\delta^2 S^{(3)}_{\gamma_4})] \\
&- q_{\gamma_1}\frac{7}{1920}(3\delta^2 S^{(2)}_{\gamma_3} - 2\delta^2 S^{(2)}_{\gamma_4}) + r_{\gamma_1}\frac{7}{5760}(3\delta^2 S^{(1)}_{\gamma_3} - 2\delta^2 S^{(1)}_{\gamma_4}) = f_{\gamma_1} + O(h^6) \qquad (3.7) \\
LS_{\gamma_2} &+ \frac{1}{24}\eta_{24}S^{(4)}_{\gamma_2} - \frac{7}{5760}(2\delta^2 S^{(4)}_{\gamma_3} - \delta^2 S^{(4)}_{\gamma_4}) - p_{\gamma_2}[\frac{1}{24}\eta_8 S^{(3)}_{\gamma_2} - \frac{7}{1920}(2\delta^2 S^{(3)}_{\gamma_3} - \delta^2 S^{(3)}_{\gamma_4})] \\
&- \frac{7}{1920}q_{\gamma_2}(2\delta^2 S^{(2)}_{\gamma_3} - \delta^2 S^{(2)}_{\gamma_4}) + r_{\gamma_2}\frac{7}{5760}(2\delta^2 S^{(1)}_{\gamma_3} - \delta^2 S^{(1)}_{\gamma_4}) = f_{\gamma_2} + O(h^6) \qquad (3.8) \\
LS_i &+ \frac{240\delta S^{(4)}_i - 17\delta^2 S^{(4)}_i}{5760} - p_i\frac{80\delta S^{(3)}_i - 17\delta^2 S^{(3)}_i}{1920} \\
&- q_i\frac{7}{1920}\delta^2 S^{(2)}_i + r_i\frac{7}{5760}\delta^2 S^{(1)}_i = f_i + O(h^6), \quad i = 3, \cdots, N-2 \qquad (3.9)
\end{aligned}
$$

and

$$
\begin{aligned}
B_k S_{\gamma_0} &+ \alpha_{\gamma_0,3}[\frac{1}{12}\theta_8 S^{(3)}_{\gamma_2} - \frac{1}{480}(7\delta^2 S^{(3)}_{\gamma_3} - 5\delta^2 S^{(3)}_{\gamma_4})] + \frac{1}{480}\alpha_{\gamma_0,2}(7\delta^2 S^{(2)}_{\gamma_3} - 5\delta^2 S^{(2)}_{\gamma_4}) \\
&- \frac{1}{1440}\alpha_{\gamma_0,1}(7\delta^2 S^{(1)}_{\gamma_3} - 5\delta^2 S^{(1)}_{\gamma_4}) = g_k + O(h^6), \quad k = 1, 3 \qquad (3.10) \\
B_k S_{\gamma_0} &+ \beta_{\gamma_0,3}[\frac{1}{12}\theta_8 S^{(3)}_{\gamma_2} - \frac{1}{480}(7\delta^2 S^{(3)}_{\gamma_3} - 5\delta^2 S^{(3)}_{\gamma_4})] + \frac{1}{480}\beta_{\gamma_0,2}(7\delta^2 S^{(2)}_{\gamma_3} - 5\delta^2 S^{(2)}_{\gamma_4}) \\
&- \frac{1}{1440}\beta_{\gamma_0,1}(7\delta^2 S^{(1)}_{\gamma_3} - 5\delta^2 S^{(1)}_{\gamma_4}) = g_k + O(h^6), \quad k = 2, 4
\end{aligned}
\qquad (3.11)
$$

We let $L'$ and $B' \equiv \{B'_k, k = 1,2,3,4\}$ denote the perturbations of $L$ and $B = \{B_k, k = 1,2,3,4\}$, respectively, defined by the left side of the equations (3.7)-(3.11). It is clear then that $S$ satisfies the relations

$$L'S_i = f_i + O(h^6), \qquad i = 1, \cdots, N \qquad (3.12)$$

and

$$B'_k S_{\gamma_0} = g_k + O(h^6), \qquad k = 1,2,3,4 \qquad (3.13)$$

The following lemma summarizes the observations above.

**LEMMA 2** *Let $S$ be the quartic spline interpolant of the solution $u$ to (3.1), (3.2). If $u \in C^{10}[I]$, then the following relations hold for $S$:*

$$[LS - f]_{x=\tau_i} \;=\; O(h^2), \quad i = 1, \cdots, N \qquad (3.14)$$

$$[BS - g]_{x=x_0,x_N} = O(h^2) \tag{3.15}$$

$$[L'S - f]_{x=\tau_i} = O(h^6), \quad i = 1, \cdots, N \tag{3.16}$$

$$[B'S - g]_{x=x_0,x_N} = O(h^6). \tag{3.17}$$

## 3.3 Formulation of the quartic spline collocation method

The *quartic spline collocation method* presented in this chapter consists of finding a quartic spline $z_\Delta \in \mathcal{S}_{4,\Delta}$ that satisfies

$$[L'z_\Delta - f]_{x=\tau_i} = 0, \quad \text{for } i = 1, \cdots, N \tag{3.18}$$

and

$$[B'z_\Delta - g]_{x=x_0,x_N} = 0 \tag{3.19}$$

This formulation is referred to as the *one-step quartic spline collocation* method.

An alternative formulation of the method is the *three-step quartic spline collocation* method in which $u_\Delta$ is determined by the following steps:

Step 1: Determine $w \in \mathcal{S}_{4,\Delta}$ such that it satisfies

$$[Lw - f]_{x=\tau_i} = 0, \quad \text{for } i = 1, \cdots, N \tag{3.20}$$

and

$$[Bw - g]_{x=x_0,x_N} = 0 \tag{3.21}$$

Step 2:

(i) Compute approximations of the 5th and 6th derivatives of $u$ at the points $\tau_i$, for $i = 3, \cdots, N-2$,

$$u_i^{(5)} \approx \frac{8\delta w_i^{(3)} - \delta^2 w_i^{(3)}}{8h^2}$$

and

$$u_i^{(6)} \approx \frac{24\delta w_i^{(4)} - \delta^2 w_i^{(4)}}{24h^2}$$

and for $i = \gamma_0, \gamma_1, \gamma_2$,

$$u^{(5)}_{\gamma_0} \approx \frac{1}{h^2}\theta_8 w^{(3)}\gamma_0, \quad u^{(5)}\gamma_1 \approx \frac{1}{h^2}\zeta_8 w^{(3)}\gamma_1, \quad u^{(5)}\gamma_2 \approx \frac{1}{h^2}\eta_8 w^{(3)}\gamma_2,$$

$$u^{(6)}_{\gamma_0} \approx \frac{1}{h^2}\theta_{24} w^{(4)}_{\gamma_0}, \quad u^{(6)}_{\gamma_1} \approx \frac{1}{h^2}\zeta_{24} w^{(4)}_{\gamma_1}, \quad u^{(6)}_{\gamma_2} \approx \frac{1}{h^2}\eta_{24} w^{(4)}_{\gamma_2},$$

and substitute them into only the terms involving $h^2$ in equations (3.4) and (3.5)-(3.6) to obtain new right sides $\bar{f}$ and $\bar{g}$:

$$\begin{aligned}
\bar{f}_{\gamma_1} &= f_{\gamma_1} - \frac{1}{24}\zeta_{24} w^{(4)}_{\gamma_2} + p_{\gamma_1}\frac{1}{24}\zeta_8 w^{(3)}_{\gamma_2} \\
\bar{f}_{\gamma_2} &= f_{\gamma_2} - \frac{1}{24}\eta_{24} w^{(4)}_{\gamma_2} + p_{\gamma_2}\frac{1}{24}\eta_8 w^{(3)}_{\gamma_2} \\
\bar{f}_i &= f_i - \frac{24\delta w^{(4)}_i - \delta^2 w^{(4)}_i}{576} + p_i\frac{8\delta w^{3)}_i - \delta^2 w^{(3)}_i}{192} \qquad \text{for } i = 3, \cdots, N-2 \\
\bar{g}_k &= g_k - \frac{\alpha_{\gamma_0,3}}{12}\theta_8 w^{(3)}_{\gamma_0} \qquad \text{for } k = 1, 3 \\
\bar{g}_k &= g_k - \frac{\beta_{\gamma_0,3}}{12}\theta_8 w^{(3)}_{\gamma_0} \qquad \text{for } k = 2, 4
\end{aligned}$$

(ii) Determine $v \in \mathcal{S}_{4,\Delta}$ such that it satisfies

$$[Lv - \bar{f}]_{x=\tau_i} = 0, \qquad \text{for } i = 1, \cdots, N \tag{3.22}$$

and

$$[Bv - \bar{g}]_{x=x_0,x_N} = 0. \tag{3.23}$$

Step 3:

(i) Compute approximations of the 5th and 6th derivatives of $u$ as in Step 2, except using derivatives of $v$ instead of those of $w$ in the approximations. Substitute these values into equations (3.4) and (3.5)-(3.6) exactly as in Step 2, i.e., substituting into the terms involving $h^2$ only. In addition to these approximations, estimate the four higher derivatives of $u$ at $\tau_i$,

$$\begin{aligned}
u^{(k)}_i &\approx \frac{\delta^2 v^{(k-4)}_i}{h^4}, \text{ for } i = 3, \cdots, N-2, \\
u^{(k)}_{\gamma_2} &\approx \frac{1}{h^4}(2\delta^2 v^{(k-4)}_{\gamma_3} - \delta^2 v^{(k-4)}_{\gamma_4}),
\end{aligned}$$

$$\begin{aligned}
u_{\gamma_1}^{(k)} &\approx \frac{1}{h^4}(3\delta^2 v_{\gamma_3}^{(k-4)} - 2\delta^2 v_{\gamma_4}^{(k-4)}),\\
u_{\gamma_0}^{(k)} &\approx \frac{1}{2h^4}(7\delta^2 v_{\gamma_3}^{(k-4)} - 5\delta^2 v_{\gamma_4}^{(k-4)}),
\end{aligned}$$

where $k = 5, 6, 7, 8$, and substitute these values into the terms involving $h^4$ in equations (3.4) and (3.5)-(3.6). The resulting right sides $\bar{\bar{f}}$ and $\bar{\bar{g}}$ are

$$\begin{aligned}
\bar{\bar{f}}_{\gamma_1} &= f_{\gamma_1} - \frac{1}{24}\zeta_{24} v_{\gamma_2}^{(4)} + \frac{7}{5760}(3\delta^2 v_{\gamma_3}^{(4)} - 2\delta^2 v_{\gamma_4}^{(4)})\\
&\quad + p_{\gamma_1}[\frac{1}{24}\zeta_8 v_{\gamma_2}^{(3)} - \frac{7}{1920}(3\delta^2 v_{\gamma_3}^{(3)} - 2\delta^2 v_{\gamma_4}^{(3)})]\\
&\quad + q_{\gamma_1}\frac{7}{1920}(3\delta^2 v_{\gamma_3}^{(2)} - 2\delta^2 v_{\gamma_4}^{(2)})\\
&\quad - r_{\gamma_1}\frac{7}{5760}(3\delta^2 v_{\gamma_3}^{(1)} - 2\delta^2 v_{\gamma_4}^{(1)})\\
\bar{\bar{f}}_{\gamma_2} &= f_{\gamma_2} - \frac{1}{24}\eta_{24} v_{\gamma_2}^{(4)} + \frac{7}{5760}(2\delta^2 v_{\gamma_3}^{(4)} - \delta^2 v_{\gamma_4}^{(4)})\\
&\quad + p_{\gamma_2}[\frac{1}{24}\eta_8 v_{\gamma_2}^{(3)} - \frac{7}{1920}(2\delta^2 v_{\gamma_3}^{(3)} - \delta^2 v_{\gamma_4}^{(3)})]\\
&\quad + q_{\gamma_2}\frac{7}{1920}(2\delta^2 v_{\gamma_3}^{(2)} - \delta^2 v_{\gamma_4}^{(2)})\\
&\quad - r_{\gamma_2}\frac{7}{5760}(2\delta^2 v_{\gamma_3}^{(1)} - \delta^2 v_{\gamma_4}^{(1)})\\
\bar{\bar{f}}_i &= f_i - \frac{240\delta v_i^{(4)} - 17\delta^2 v_i^{(4)}}{5760} + p_i\frac{80\delta v_i^{(3)} - 17\delta^2 v_i^{(3)}}{1920}\\
&\quad + q_i\frac{7}{1920}\delta^2 v_i^{(2)} - r_i\frac{7}{5760}\delta^2 v_i^{(1)} \text{ for } i = 3, \cdots, N-2\\
\bar{\bar{g}}_k &= g_k - \alpha_{\gamma_0,3}[\frac{1}{12}\theta_8 v_{\gamma_2}^{(3)} - \frac{1}{480}(7\delta^2 v_{\gamma_3}^{(3)} - 5\delta^2 v_{\gamma_4}^{(3)}] - \frac{\alpha_{\gamma_0,2}}{480}(7\delta^2 v_{\gamma_3}^{(2)} - 5\delta^2 v_{\gamma_4}^{(2)})\\
&\quad + \frac{\alpha_{\gamma_0,1}}{1440}(7\delta^2 v_{\gamma_3}^{(1)} - 5\delta^2 v_{\gamma_4}^{(1)}), \text{ for } k = 1, 3\\
\bar{\bar{g}}_k &= g_k - \beta_{\gamma_0,3}[\frac{1}{12}\theta_8 v_{\gamma_2}^{(3)} - \frac{1}{480}(7\delta^2 v_{\gamma_3}^{(3)} - 5\delta^2 v_{\gamma_4}^{(3)}] - \frac{\beta_{\gamma_0,2}}{480}(7\delta^2 v_{\gamma_3}^{(2)} - 5\delta^2 v_{\gamma_4}^{(2)})\\
&\quad + \frac{\beta_{\gamma_0,1}}{1440}(7\delta^2 v_{\gamma_3}^{(1)} - 5\delta^2 v_{\gamma_4}^{(1)}), \text{ for } k = 2, 4
\end{aligned}$$

(ii) Find $u_\Delta \in \mathcal{S}_{4,\Delta}$ such that it satisfies

$$[Lu_\Delta - \bar{\bar{f}}]_{x=\tau_i} = 0, \quad \text{for } i = 1, \cdots, N \tag{3.24}$$

and

$$[Bu_\Delta - \bar{\bar{g}}]_{x=x_0,x_N} = 0. \tag{3.25}$$

## 3.4 Matrix analysis

In this section, we present the properties of the quartic spline collocation matrices arising from some special cases of (3.1)-(3.2). We also related the quartic spline collocation matrices to the quadratic spline collocation matrices arising from second-order boundary value problems.

Consider the special case of (3.2) where $\alpha_{\gamma_0,0} = \beta_{\gamma_0,2} = 1$, $\alpha_{\gamma_0,i} = \beta_{\gamma_0,j} = 0$, for $i = 1, 2, 3$, and $j = 0, 1, 3$, resulting in boundary conditions

$$\begin{aligned} u(a) &= u''(a) = 0, & (3.26) \\ u(b) &= u''(b) = 0. & (3.27) \end{aligned}$$

The space of quartic splines that satisfy (3.26)-(3.27) has dimension $N$, since 4 degrees of freedom are absorbed by the boundary conditions (3.26)-(3.27). To form a set of quartic spline basis functions $\tilde{\Phi}_i$, $i = 1, \cdots, N$, that satisfy (3.26)-(3.27) by construction, we adjust the quartic spline basis functions $\Phi_i$, $i = -1, \cdots, N+2$ of (2.2) as follows:

$$\begin{aligned} \tilde{\Phi}_1 = \Phi_1 - \Phi_0, &\qquad \tilde{\Phi}_2 = \Phi_2 - \Phi_{-1}, \\ \tilde{\Phi}_i = \Phi_i, &\qquad i = 3, \cdots, N-2, \qquad (3.28) \\ \tilde{\Phi}_{N-1} = \Phi_{N-1} - \Phi_{N+2}, &\qquad \tilde{\Phi}_N = \Phi_N - \Phi_{N+1}. \end{aligned}$$

Now any $\tilde{u}_\Delta(x) = \sum_{i=1}^{N} \tilde{c}_i \tilde{\Phi}_i(x)$ satisfies the boundary conditions (3.26)-(3.27) by construction of the $\tilde{\Phi}_i$'s.

Consider the special case of (3.1) where $p(x) = q(x) = r(x) = s(x) = 1$, resulting in the operator equation

$$u^{(4)} + u^{(3)} + u^{(2)} + u^{(1)} + u = f. \qquad (3.29)$$

The straightforward quartic spline collocation method, as described in Section 2.1, applied to the boundary value problem (3.29), (3.26)-(3.27) determines $\tilde{u}_\Delta(x) = \sum_{i=1}^{N} \tilde{c}_i \tilde{\Phi}_i(x)$ by the collocation conditions

$$\tilde{u}_\Delta^{(4)}(\tau_i) + \tilde{u}_\Delta^{(3)}(\tau_i) + \tilde{u}_\Delta^{(2)}(\tau_i) + \tilde{u}_\Delta^{(1)}(\tau_i) + \tilde{u}_\Delta(\tau_i) = f(\tau_i), \ i = 1, \cdots, N. \qquad (3.30)$$

Equations (3.30) lead to the $N \times N$ linear system

$$(Q_4 + Q_3 + Q_2 + Q_1 + Q_0)\bar{\tilde{c}} = \overline{f} \tag{3.31}$$

where the matrices $Q_i$, $i = 0, \cdots, 4$, arise from the terms $u^{(i)}$, $i = 0, \cdots, 4$, respectively, of (3.29), $\bar{\tilde{c}} = (\tilde{c}_1, \tilde{c}_2, \cdots, \tilde{c}_N)^T$ and $\overline{f} = (f(\tau_1), f(\tau_2), \cdots, f(\tau_N))^T$. Note that the matrices in (3.31) also arise when the three-step quartic spline collocation method is applied to (3.29), (3.26)-(3.27). Therefore, any properties of $Q_i$, $i = 0, \cdots, 4$, apply to both the straightforward and the three-step quartic spline collocation methods.

In the following, we give the form of $Q_i$, $i = 0, \cdots, 4$:

$$Q_4 = \frac{1}{h^4} \begin{bmatrix} 10 & -5 & 1 & & & & & & \\ -5 & 6 & -4 & 1 & & & & & \\ 1 & -4 & 6 & -4 & 1 & & & & \\ & 1 & -4 & 6 & -4 & 1 & & & \\ & & & \cdot & \cdot & \cdot & & & \\ & & & & 1 & -4 & 6 & -4 & 1 \\ & & & & & 1 & -4 & 6 & -5 \\ & & & & & & 1 & -5 & 10 \end{bmatrix} \tag{3.32}$$

$$Q_3 = \frac{1}{2h^3} \begin{bmatrix} -2 & -1 & 1 & & & & & & \\ 3 & 0 & -2 & 1 & & & & & \\ -1 & 2 & 0 & -2 & 1 & & & & \\ & -1 & 2 & 0 & -2 & 1 & & & \\ & & & \cdot & \cdot & \cdot & & & \\ & & & & -1 & 2 & 0 & -2 & 1 \\ & & & & & -1 & 2 & 0 & -3 \\ & & & & & & -1 & 1 & 2 \end{bmatrix} \tag{3.33}$$

$$Q_2 = \frac{1}{8h^2}\begin{bmatrix} -14 & 3 & 1 & & & & & \\ 3 & -10 & 4 & 1 & & & & \\ 1 & 4 & -10 & 4 & 1 & & & \\ & 1 & 4 & -10 & 4 & 1 & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & 1 & 4 & -10 & 4 & 1 \\ & & & & 1 & 4 & -10 & 3 \\ & & & & & 1 & 3 & -14 \end{bmatrix} \tag{3.34}$$

$$Q_1 = \frac{1}{48h}\begin{bmatrix} 22 & 23 & 1 & & & & & \\ -21 & 0 & 22 & 1 & & & & \\ -1 & -22 & 0 & 22 & 1 & & & \\ & -1 & -22 & 0 & 22 & 1 & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & -1 & -22 & 0 & 22 & 1 \\ & & & & -1 & -22 & 0 & 21 \\ & & & & & -1 & -23 & -22 \end{bmatrix} \tag{3.35}$$

$$Q_0 = \frac{1}{24}\begin{bmatrix} 154 & 75 & 1 & & & & & \\ 75 & 230 & 76 & 1 & & & & \\ 1 & 76 & 230 & 76 & 1 & & & \\ & 1 & 76 & 230 & 76 & 1 & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & 1 & 76 & 230 & 76 & 1 \\ & & & & 1 & 76 & 230 & 75 \\ & & & & & 1 & 75 & 154 \end{bmatrix} \tag{3.36}$$

To study the properties of the matrices $Q_i$, $i = 1, \cdots, 4$, we recall the quadratic spline collocation matrices arising from second-order boundary value problems [12], [4]. More specifically, consider the boundary value problem

$$\begin{aligned} u^{(2)} + u^{(1)} + u &= f \text{ in } (a,b) \\ u(a) = u(b) &= 0 \end{aligned} \tag{3.37}$$

and its discretization by quadratic spline collocation, where the quadratic spline basis functions are adjusted so that they satisfy homogeneous Dirichlet boundary conditions by construction. The matrix of the linear system arising from (3.37) is of the form $T_2 + T_1 + T_0$, where the matrices $T_i$, $i = 0, \cdots, 2$, arise from the terms $u^{(i)}$, $i = 0, \cdots, 2$, respectively, of (3.37). Recall the form of $T_i$, $i = 0, \cdots, 2$:

$$T_2 = \frac{1}{h^2}\begin{bmatrix} -3 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \cdot & \cdot & \cdot & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -3 \end{bmatrix}, \quad T_1 = \frac{1}{2h}\begin{bmatrix} 1 & 1 & & & & \\ -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & \cdot & \cdot & \cdot & \\ & & & -1 & 0 & 1 \\ & & & & -1 & -1 \end{bmatrix},$$

$$T_0 = \frac{1}{8}\begin{bmatrix} 5 & 1 & & & & \\ 1 & 6 & 1 & & & \\ & 1 & 6 & 1 & & \\ & & \cdot & \cdot & \cdot & \\ & & & 1 & 6 & 1 \\ & & & & 1 & 5 \end{bmatrix}$$

Let $I_N$ be the identity matrix of size $N$. It is easy to show that

$$Q_4 = T_2^2 \tag{3.38}$$

$$Q_3 = T_1 \cdot T_2 \tag{3.39}$$

$$Q_2 = T_2 \cdot T_0 \tag{3.40}$$

$$Q_1 = 16T_1 + T_1 \cdot T_0 \tag{3.41}$$

and

$$Q_0 = T_0^2 + 64(T_0 - 3I_N). \tag{3.42}$$

The matrix $T_2$ is non-singular [4], hence $Q_4$ is also non-singular and a unique solution to $u^{(4)} = f$ subject to (3.26)-(3.27) exists. Since the matrices $T_2$ and $T_0$ are non-singular and, furthermore, their eigenvalues and eigenvectors are known [4], the eigenvalues and eigenvectors of the matrices $Q_4$, $Q_2$ and $Q_0$ can be easily derived. Moreover, the unique solvability of $Q_4$ also proves the unique solvability of the (straightforward and three-step) quartic spline collocation equations arising from boundary value problems with general differential operator equation (3.1) subject to (3.26)-(3.27), for sufficiently small $h$, since the entries of $Q_4$ dominate those of $Q_i$, $i = 0, \cdots, 3$.

We now consider the boundary value problem consisting of the operator equation $u^{(4)} = f$ subject to boundary conditions (3.26)- (3.27), discretized by the one-step quartic spline collocation method. We will show that $z_\Delta$ of (3.18)-(3.19) can be determined uniquely. First we consider the system of equations with respect to $\{z_\Delta^{(4)}(\tau_i)\}_1^N$ arising from (3.18). Let $Q$ be the $N \times N$ coefficient matrix of $\{z_\Delta^{(4)}(\tau_i)\}_1^N$. We can show that $Q = \frac{1}{5760} Q'$, where the non-zero entries of $Q'$ are given by

$$\begin{array}{lllll}
Q'_{11} = 5639, & Q'_{12} = 3098, & Q'_{13} = -11332, & Q'_{14} = 18008, & Q'_{15} - 14997, \\
Q'_{16} = 6574, & Q'_{17} = -1270, & Q'_{18} = 40, & & \\
Q'_{21} = -54, & Q'_{22} = 7003, & Q'_{23} = -3992, & Q'_{24} = 5588, & Q'_{25} = -4202, \\
Q'_{26} = 1727, & Q'_{27} = -320, & Q'_{28} = 10, & & \\
Q'_{i,i-2} = -17, & Q'_{i,i-1} = 308, & Q'_{ii} = 5178, & Q'_{i,i+1} = 308, & Q'_{i,i+2} = -17, \\
i = 3, \cdots, N-2 & & & &
\end{array}$$

and $Q'_{N,N-j+1} = Q'_{1,j}$, $Q'_{N-1,N-j+1} = Q'_{2,j}$, $j = 1, \cdots, N$. We can show that the matrix $Q$ is non-singular. More specifically, by substituting the second and second last columns by appropriate linear combinations of the first two and last two columns, respectively, and then scaling the first column appropriately, the matrix $Q$ can be transformed into a diagonally dominant matrix that is pentadiagonal except for the first two and last two rows which have 7 or 6 nonzero entries. Hence, $\{z_\Delta^{(4)}(\tau_i)\}_1^N$ in (3.18) are uniquely determined. Then the degrees of freedom of $z_\Delta = \sum_{i=-1}^{N+2} c_i \Phi_i$ can be determined by the following system of equations where the first two and last two rows arise from the boundary conditions (3.19):

$$
\begin{bmatrix}
1 & 11 & 11 & 1 & & & & & & & \\
1927 & -1925 & -2060 & 2468 & -950 & 898 & -460 & 100 & 7 & -5 & \\
1 & -4 & 6 & -4 & 1 & & & & & & \\
 & & & & & . & . & . & & & \\
 & & & & & & 1 & -4 & 6 & -4 & 1 \\
 & -5 & 7 & 100 & -460 & 898 & -950 & 2468 & -2060 & -1925 & 1927 \\
 & & & & & & & 1 & 11 & 11 & 1
\end{bmatrix}
\begin{bmatrix}
c_{-1} \\ c_0 \\ c_1 \\ . \\ c_N \\ c_{N+1} \\ c_{N+2}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
24g_1 \\ 3840h^2 g_2 \\ h^4 z_\Delta^{(4)}(\tau_1) \\ . \\ h^4 z_\Delta^{(4)}(\tau_N) \\ 3840h^2 g_3 \\ 24g_4
\end{bmatrix} \tag{3.43}
$$

Due to the complexity of certain rows, we do not show the nonsingularity of the coefficient matrix in equation (3.43). However, the existence and uniqueness of the quartic-spline solution to (3.43) is proved via the Green's function approach in Section 4.2.

# Chapter 4

# Convergence Analysis and Error Bounds

The convergence and error bound analysis in this section is based on the analysis for quadratic-spline collocation methods in [12]. We consider the boundary value problem (3.1)-(3.2) with homogeneous boundary conditions, i.e. we assume $g_k = 0$, $k = 1, 2, 3, 4$. We also assume that $u$, $u_\Delta$, $w$, $v$ and $z_\Delta$ satisfy the homogeneous boundary conditions.

We introduce the notion of the *Green's function* associated with a differential operator in order to proceed with the analysis of the two quartic-spline collocation methods. Suppose that the boundary value problem

$$U^{(4)} = 0 \text{ in } I, BU = 0 \text{ on } \partial I \equiv \text{boundary of } I, \tag{4.1}$$

where $B$ is defined in (3.2), is uniquely solvable. Then it is known [19] that there exists a Green's function $G(x,t)$ associated with (4.1). The Green's function allows us to obtain the numerical solution $u_\Delta$ from $u_\Delta^{(4)}$. In general, letting $y \equiv u^{(4)}, y_\Delta \equiv u_\Delta^{(4)}, m \equiv w^{(4)}, l \equiv v^{(4)}$, and $\zeta_\Delta \equiv z_\Delta^{(4)}$,

$$\begin{aligned} u^{(k)}(x) &= \int_a^b \frac{\partial^k G(x,t)}{\partial x^k} y(t)\mathrm{d}t, \\ u_\Delta^{(k)}(x) &= \int_a^b \frac{\partial^k G(x,t)}{\partial x^k} y_\Delta(t)\mathrm{d}t, \end{aligned}$$

$$\begin{aligned}
w^{(k)}(x) &= \int_a^b \frac{\partial^k G(x,t)}{\partial x^k} m(t)\mathrm{d}t, \\
v^{(k)}(x) &= \int_a^b \frac{\partial^k G(x,t)}{\partial x^k} l(t)\mathrm{d}t, \\
z_\Delta^{(k)}(x) &= \int_a^b \frac{\partial^k G(x,t)}{\partial x^k} \zeta_\Delta(t)\mathrm{d}t,
\end{aligned}$$

hold for $k = 0, \cdots, 3$. For brevity, we denote $\frac{\partial G(x,t)}{\partial x}$ by $G_x(x,t)$, $\frac{\partial^2 G(x,t)}{\partial x^2}$ by $G_{xx}(x,t)$, and $\frac{\partial^3 G(x,t)}{\partial x^3}$ by $G_{xxx}(x,t)$.

Now we present two more operators. The integral operator $K$ is defined by

$$\begin{aligned}
Ky(x) &\equiv p(x)\int_a^b G_{xxx}(x,t)y(t)\mathrm{d}t + q(x)\int_a^b G_{xx}(x,t)y(t)\mathrm{d}t \\
&+ r(x)\int_a^b G_x(x,t)y(t)\mathrm{d}t + s(x)\int_a^b G(x,t)y(t)\mathrm{d}t
\end{aligned} \tag{4.2}$$

and maps $L_2(I)$ to $C(I)$. The linear projection $P_\Delta$ maps $C(I)$ to $\mathcal{S}_{0,\Delta}$ by piecewise constant polynomial interpolation at the middle points $\{\tau_i\}_1^N$.

## 4.1 Convergence analysis of the three-step method

Making use of the given notation, equation (3.1) can be rewritten as

$$y + Ky = f \tag{4.3}$$

Equations (3.20), (3.22) and (3.24) can now be written as

$$\begin{aligned}
P_\Delta(m + Km) &= P_\Delta f \\
P_\Delta(l + Kl) &= P_\Delta \bar{f} \\
P_\Delta(y_\Delta + Ky_\Delta) &= P_\Delta \bar{\bar{f}}
\end{aligned}$$

respectively, which, because $P_\Delta m = m$, $P_\Delta l = l$ and $P_\Delta y_\Delta = y_\Delta$, simplify to

$$m + P_\Delta K m = P_\Delta f \tag{4.4}$$

$$l + P_\Delta K l = P_\Delta \bar{f} \tag{4.5}$$

$$y_\Delta + P_\Delta K y_\Delta = P_\Delta \bar{\bar{f}} \tag{4.6}$$

respectively. By definition of $P_\Delta$, $P_\Delta g(x) = g(\tau_i)$ for $x \in (x_{i-1}, x_i)$, and it can be assumed that $P_\Delta g(x_i) = g(\tau_i)$ for $i = 1, \cdots, N$ and $P_\Delta g(x_0) = g(\tau_1)$. As $h \to 0$, $|\tau_i - x| \to 0$ for any $x \in [x_i, x_{i+1}]$, and this implies that $|g(\tau_i) - g(x)| \to 0$ by the continuity of $g$. I.e., $||P_\Delta g - g||_\infty \to 0$ as $h \to 0$. (For the proof of a much more general result, see [19].) This fact and the complete continuity of $K$ [19] imply $||P_\Delta K - K||_\infty \to 0$. We proceed to prove the convergence of each of the three collocation approximations in the three-step method.

**THEOREM 5** *Suppose*

*(a1) the coefficients* $p(x), q(x), r(x), s(x)$, *and* $f(x)$ *are continuous on* $[a, b]$,

*(a2) the boundary value problem* $Lu = f, Bu = 0$ *has a unique solution,*

*and*

*(a3) the problem* $u^{(4)} = 0, Bu = 0$ *is uniquely solvable,*

*then*

*(i) the approximation* $w \in \mathcal{S}_{4,\Delta}$ *that satisfies equations (3.20)-(3.21) in step 1 exists,*

*(ii) the global error bounds*

$$||(u - w)^{(k)}||_\infty = O(h^2),\ \textit{for } k = 0, 1, 2, 3, \quad ||(u - w)^{(4)}||_\infty = O(h), \tag{4.7}$$

*(iii) and the local error bounds*

$$|(u - w)^{(k)}(\tau_i)| = O(h^2),\ \textit{for } k = 0, 1, \cdots, 4,\ \textit{and } i = 1, \cdots, N \tag{4.8}$$

*hold.*

*Proof:* Assumption (a3) implies that (4.3) is uniquely solvable for any $f$, therefore the bounded operator $(I + K)^{-1}$ exists. Since $||P_\Delta K - K||$ converges to 0 as $h$ approaches 0, it may be concluded by Neumann's Theorem [18] that the operators $(I + P_\Delta K)^{-1}$ exist for sufficiently small $h$ and are uniformly bounded. The unique existence of $w$ (and in fact, of $v$ and $u_\Delta$) follows.

Now we prove the error bounds (4.7) and (4.8). Recall the interpolant $S$ of $u$ defined by (2.10), (2.11). We have shown that $BS = O(h^2)$. Also note that a cubic function $\xi$ exists that satisfies $B\xi = BS = O(h^2)$ because of (a3) and the fact that any problem with non-homogeneous BCs can be converted to one with homogeneous BCs. It is easy to see that $||\xi^{(k)}||_\infty = O(h^2)$ for $k = 0, 1, 2, 3$. It is clear that the problem $(S-\xi)^{(4)} = S^{(4)}, B(S-\xi) = 0$ is solvable. Then writing equation (3.4) in the operator notations introduced and substituting $(S-\xi)^{(4)}$ for $S^{(4)}$, we get

$$(I + P_\Delta K)(S^{(4)} - \xi^{(4)}) = P_\Delta f + O(h^2),$$

from which (4.4) is subtracted to yield

$$(I + P_\Delta K)(S^{(4)} - \xi^{(4)} - w^{(4)}) = O(h^2)$$

Since $(I + P_\Delta K)^{-1}$ is uniformly bounded, we have

$$||S^{(4)} - \xi^{(4)} - w^{(4)}||_\infty = O(h^2) \tag{4.9}$$

Assumption (a3) ensures the unique solvability of $(S - \xi - w)^{(4)} = e, B(S - \xi - w) = 0$, thus we can use the Green's function to obtain

$$(S - \xi - w)^{(k)}(x) = \int_a^b \frac{\partial^k G(x,t)}{\partial x^k}(S^{(4)} - \xi^{(4)} - w^{(4)})(t)\mathrm{d}t$$

which implies that

$$||S^{(k)} - \xi^{(k)} - w^{(k)}||_\infty = O(h^2) \text{ for } k = 0, 1, 2, 3 \tag{4.10}$$

Using Theorems 1 and 3, relations (4.9) and (4.10), the definition of $\xi$, and the triangle inequality, the error bounds (4.7) and (4.8) can now be shown easily. $\Diamond$

**THEOREM 6** *Under the hypotheses of Theorem 5,*

*(i) the approximation $v \in \mathcal{S}_{4,\Delta}$ that satisfies equations (3.22)-(3.23) in step 2 exists,*

*(ii) the global error bounds*

$$||u - v||_\infty = O(h^4), \quad ||(u - v)^{(k)}||_\infty = O(h^{5-k}), k = 1, \cdots, 4 \tag{4.11}$$

*(iii) and the local error bounds*

$$\begin{aligned} |(u-v)(x_i)| &= O(h^4), && (4.12)\\ |(u-v)^{(k)}(\tau_i)| &= O(h^4), \quad k=0,1,2, && (4.13)\\ |(u-v)^{(k)}(\tau_i)| &= O(h^2), \quad k=3,4 && (4.14)\end{aligned}$$

*hold.*

By triangle inequality, the bounds (4.9) and (4.10), and the definition of $\xi$, we have $||S^{(k)}-w^{(k)}||_\infty \le ||S^{(k)}-\xi^{(k)}-w^{(k)}||_\infty + ||\xi^{(k)}||_\infty = O(h^2)$ for $k=0,\cdots,4$, which leads to the relations $\delta S_i^{(k)} = \delta w_i^{(k)} + O(h^2)$ and $\delta^2 S_i^{(k)} = \delta^2 w_i^{(k)} + O(h^2)$, for $k=0,\cdots,4$ and $i=0,\cdots,N+1$. Then we can rewrite equations (3.7)-(3.11) as

$$\begin{aligned} LS_i &= \bar{f}_i + O(h^4), \quad i=1,\cdots,N && (4.15)\\ BS_{\gamma_0} &= \bar{g}_j + O(h^4), \quad j=1,2,3,4 && (4.16)\end{aligned}$$

Thus we have

$$\begin{aligned} L(S-v)(\tau_i) &= O(h^4), \quad i=1,\cdots,N && (4.17)\\ B(S-v) &= O(h^4) && (4.18)\end{aligned}$$

Once again, note that there exists a cubic function $\bar{\xi}$ that satisfies $B\bar{\xi} = B(S-v) = O(h^4)$. It is clear that $||\bar{\xi}^{(k)}||_\infty = O(h^4)$ for $k=0,1,2,3$ and $B(S-\bar{\xi}-v)=0$. From the solvability of $(S-\bar{\xi}-v)^{(4)} = (S-v)^{(4)}, B(S-\bar{\xi}-v)=0$ due to (a3), we conclude that equations (4.17) can be written as

$$(I+P_\Delta K)(S^{(4)}-\bar{\xi}^{(4)}-v^{(4)}) = O(h^4)$$

By the same reasoning used in the proof of Theorem 5, we derive the bounds

$$||S^{(k)}-\bar{\xi}^{(k)}-v^{(k)}||_\infty = O(h^4), \quad k=0,\cdots,4 \qquad (4.19)$$

We know that $||S^{(k)} - v^{(k)}||_\infty \leq ||S^{(k)} - \bar{\xi}^{(k)} - v^{(k)}||_\infty + ||\bar{\xi}^{(k)}||_\infty$, and since both norms in the right side of the inequality are of order $O(h^4)$, it follows that

$$||S^{(k)} - v^{(k)}||_\infty = O(h^4), \quad k = 0, \cdots, 4. \tag{4.20}$$

The error bounds (4.11) and (4.12)-(4.14) follow immediately from relations (4.20), the definition of $\bar{\xi}$, Theorems 1, 2 and 3, relation (2.29) and the triangle inequality. $\diamondsuit$

The convergence analysis of step 3 is done using similar arguments as for the first two steps and yields optimal results.

**THEOREM 7** *Under the hypotheses of Theorem 5,*

*(i) the approximation $u_\Delta \in \mathcal{S}_{4,\Delta}$ that satisfies equations (3.24)-(3.25) in step 3 exists,*

*(ii) the global error bounds*

$||(u - u_\Delta)^{(k)}||_\infty = O(h^{5-k}), k = 0, \cdots, 4,$

*and the local error bounds*

*(iii)* $|(u - u_\Delta)(x)| = O(h^6)$ *for* $x = x_i$ *and* $\tau_i$,

*(iv)* $|(u - u_\Delta)^{(k)}(\tau_i)| = O(h^4), k = 1, 2,$

*(v)* $|(u - u_\Delta)^{(k)}(\tau_i)| = O(h^2), k = 3, 4$

*hold.*

## 4.2 Convergence analysis of the one-step method

We denote $z_\Delta^{(4)}$ by $\zeta_\Delta$ and the coefficient matrices of $z_\Delta^{(k)}, k = 1, 2, 3, 4$ by $\Omega, \Upsilon, \Phi, \Psi$, respectively. The matrix $\Psi \equiv Q$ was already presented in the matrix analysis section. We give the nonzero entries of the scaled matrices $\Omega' = \frac{1}{5760}\Omega$, $\Upsilon' = \frac{1}{1920}\Upsilon$, and $\Phi' = \frac{1}{1920}\Phi$. Note that $M_{1,j} = M_{N,N-j+1}$ and $M_{2,j} = M_{N-1,N-j+1}$ for $j = 1, \cdots, 8$, where $M$ is any of the coefficient matrices $\Omega, \Upsilon, \Phi$ and $\Psi$.

$$\begin{array}{llllll}
\Omega'_{1,1} = 5781, & \Omega'_{1,2} = -98, & \Omega'_{1,3} = 182, & \Omega'_{1,4} = -168, & \Omega'_{1,5} = 77, & \Omega'_{1,6} = -14, \\
\Omega'_{2,1} = 14, & \Omega'_{2,2} = 5697, & \Omega'_{2,3} = 112, & \Omega'_{2,4} = -98, & \Omega'_{2,5} = 42, & \Omega'_{2,6} = -7, \\
\Omega'_{i,i-2} = 7, & \Omega'_{i,i-1} = -28, & \Omega'_{i,i} = 5802, & \Omega'_{i,i+1} = -28, & \Omega'_{i,i+2} = 7, &
\end{array}$$

$$\Upsilon'_{1,1} = 1899,\quad \Upsilon'_{1,2} = 98,\quad \Upsilon'_{1,3} = -182,\quad \Upsilon'_{1,4} = 168,\quad \Upsilon'_{1,5} = -77,\quad \Upsilon'_{1,6} = 14,$$
$$\Upsilon'_{2,1} = -14,\quad \Upsilon'_{2,2} = 1983,\quad \Upsilon'_{2,3} = -112,\quad \Upsilon'_{2,4} = 98,\quad \Upsilon'_{2,5} = -42,\quad \Upsilon'_{2,6} = 7,$$
$$\Upsilon'_{i,i-2} = -7,\quad \Upsilon'_{i,i-1} = 28,\quad \Upsilon'_{i,i} = 1878,\quad \Upsilon'_{i,i+1} = 28,\quad \Upsilon'_{i,i+2} = -7,$$

$$\Phi'_{1,1} = 2041,\quad \Phi'_{1,2} = -1498,\quad \Phi'_{1,3} = 4932,\quad \Phi'_{1,4} = -7608,\quad \Phi'_{1,5} = 6357,$$
$$\Phi'_{1,6} = -2894,\quad \Phi'_{1,7} = 630,\quad \Phi'_{1,8} = -40,$$
$$\Phi'_{2,1} = 54,\quad \Phi'_{2,2} = 1317,\quad \Phi'_{2,3} = 1752,\quad \Phi'_{2,4} = -2388,\quad \Phi'_{2,5} = 1802,$$
$$\Phi'_{2,6} = -767,\quad \Phi'_{2,7} = 160,\quad \Phi'_{2,8} = -10,$$
$$\Phi'_{i,i-2} = 17,\quad \Phi'_{i,i-1} = -148,\quad \Phi'_{i,i} = 2182,\quad \Phi'_{i,i+1} = -148,\quad \Phi'_{i,i+2} = 17$$

where $i = 3, \cdots, N-2$. Recall that from the solvability of $U^{(4)} = 0, BU = 0$, and the assumption that $z_\Delta$ satisfies the homogeneous boundary conditions, we can conclude that there exists a Green's function $G(x,t)$ such that for $k = 0, 1, 2, 3$

$$z_\Delta^{(k)}(x) = \int_a^b \frac{\partial^k G(x,t)}{\partial x^k} \zeta_\Delta(t) \mathrm{d}t \quad .$$

We define an operator $D_\Delta : C[I] \to R^N$ by $[D_\Delta g]_i = g(\tau_i), i = 1, \cdots, N$ for any $g \in C[I]$, i.e., $D_\Delta g$ is a vector of size $N$ with components the values of $g$ at the midpoints. Next we define $P_\Delta : R^N \to \mathcal{S}_{0,\Delta}$ to be the operator that maps vectors $\vec{g} = (g_1, g_2, \cdots, g_N)^T$ to the associated constant polynomial interpolant $g_\Delta$ so that $g_\Delta(\tau_i) = g_i, i = 1, \cdots, N$. Note that, in this section, the definition of $P_\Delta$ is slightly different than that in Section 4.1. Also let $E_p, E_q, E_r$ and $E_s$ be $N \times N$ diagonal matrices with $\{p(\tau_i)\}_1^N, \{q(\tau_i)\}_1^N, \{r(\tau_i)\}_1^N$ and $\{s(\tau_i)\}_1^N$, on the diagonals, respectively. Then equation (3.18) can be written as

$$\Psi D_\Delta z_\Delta^{(4)} + E_p \Phi D_\Delta z_\Delta^{(3)} + E_q \Upsilon D_\Delta z_\Delta^{(2)} + E_r \Omega D_\Delta z_\Delta^{(1)} + E_s D_\Delta z_\Delta = D_\Delta f$$

and since it has been shown that $\Psi$ is invertible, we multiply both sides of the equation by $\Psi^{-1}$ and then apply the operator $P_\Delta$ to get

$$\zeta_\Delta + R_\Delta K \zeta_\Delta = P_\Delta \Psi^{-1} D_\Delta f \tag{4.21}$$

where

$$R_\Delta K g \;\equiv\; P_\Delta \Psi^{-1} E_p \Phi D_\Delta \int_a^b G_{xxx}(x,t) \zeta_\Delta(t) \mathrm{d}t$$

$$
\begin{aligned}
&+ \quad P_\Delta \Psi^{-1} E_q \Upsilon D_\Delta \int_a^b G_{xx}(x,t)\zeta_\Delta(t)\mathrm{d}t \\
&+ \quad P_\Delta \Psi^{-1} E_r \Omega D_\Delta \int_a^b G_{x}(x,t)\zeta_\Delta(t)\mathrm{d}t \\
&+ \quad P_\Delta \Psi^{-1} E_s D_\Delta \int_a^b G(x,t)\zeta_\Delta(t)\mathrm{d}t
\end{aligned}
$$

Our aim is to show that $R_\Delta K$ converges to $K$ and to do that, we first prove that $||R_\Delta Kg - P_\Delta D_\Delta Kg||_\infty$ converges to zero. Let $\widehat{G}^k(x)$ denote $\int_a^b \frac{\partial^k G(x,t)}{\partial x^k}\zeta_\Delta(t)\mathrm{d}t$, then by triangle inequality, we can write

$$
\begin{aligned}
||R_\Delta Kg \quad &- \quad P_\Delta D_\Delta Kg||_\infty \\
&\leq \quad ||P_\Delta \Psi^{-1} E_p \Phi D_\Delta \widehat{G}^3(x) - P_\Delta D_\Delta p(x)\widehat{G}^3(x)||_\infty \\
&+ \quad ||P_\Delta \Psi^{-1} E_q \Upsilon D_\Delta \widehat{G}^2(x) - P_\Delta D_\Delta q(x)\widehat{G}^2(x)||_\infty \\
&+ \quad ||P_\Delta \Psi^{-1} E_r \Omega D_\Delta \widehat{G}^1(x) - P_\Delta D_\Delta r(x)\widehat{G}^1(x)||_\infty \\
&+ \quad ||P_\Delta \Psi^{-1} E_s D_\Delta \widehat{G}(x) - P_\Delta D_\Delta s(x)\widehat{G}(x)||_\infty \,. \qquad (4.22)
\end{aligned}
$$

We study the first norm on the right side of the inequality (4.22). Note that $D_\Delta p(x)\widehat{G}^3(x)$ is equivalent to $E_p D_\Delta \widehat{G}^3(x)$. Also, we know that $||\Psi^{-1}||_\infty$ and $||P_\Delta||_\infty$ are bounded independently of $h$. Using Taylor's series, it can be easily seen that $\Psi E_p = E_p\Psi + O(\max\{p(\tau_{i+j}) - p(\tau_i) : j = 1, \cdots, 8, \quad i = 1, \cdots, N-j\}) = E_p\Psi + O(h)$. Furthermore, it is clear that $||D_\Delta \widehat{G}^3||_\infty \leq ||\widehat{G}^3||_\infty$ which is bounded since $\widehat{G}^3 \in C[I]$. Thus, we have

$$
\begin{aligned}
||P_\Delta \Psi^{-1} E_p \Phi D_\Delta \widehat{G}^3(x) \quad &- \quad P_\Delta D_\Delta p(x)\widehat{G}^3(x)||_\infty \\
&= \quad ||P_\Delta \Psi^{-1} E_p \Phi D_\Delta \widehat{G}^3(x) - P_\Delta E_p D_\Delta \widehat{G}^3(x)||_\infty \\
&\leq \quad ||P_\Delta \Psi^{-1}||_\infty ||E_p \Phi D_\Delta \widehat{G}^3(x) - \Psi E_p D_\Delta \widehat{G}^3(x)]||_\infty \\
&\leq \quad C||E_p||_\infty ||(\Phi - \Psi) D_\Delta \widehat{G}^3(x)||_\infty + O(h)
\end{aligned}
$$

where $C$ is a constant independent of $h$. Similar derivations can be done for the other norms on the right side of (4.22) to obtain

$$
||R_\Delta Kg \quad - \quad P_\Delta D_\Delta Kg||_\infty \leq
$$

$$
\begin{aligned}
& C||E_p||_\infty||(\Phi-\Psi)D_\Delta\widehat{G}^3(x)||_\infty \\
+\;& C||E_q||_\infty||(\Upsilon-\Psi)D_\Delta\widehat{G}^2(x)||_\infty \\
+\;& C||E_r||_\infty||(\Omega-\Psi)D_\Delta\widehat{G}^1(x)||_\infty \\
+\;& C||E_s||_\infty||(I-\Psi)D_\Delta\widehat{G}(x)||_\infty
\end{aligned}
\tag{4.23}
$$

where $I$ is the identity matrix. It is obvious that $E_p, E_q, E_r$ and $E_s$ are bounded independently of $h$. Furthermore, after calculating the entries of the matrices $(\Phi-\Psi), (\Upsilon-\Psi), (\Omega-\Psi)$ and $(I-\Psi)$, we have ($[V]_i$ denotes the $i$th component of the vector $V$)

$$
\begin{aligned}
||(\Phi \;&-\; \Psi)D_\Delta\widehat{G}^3(x)||_\infty \\
&= \max\{[(\Phi-\Psi)D_\Delta\widehat{G}^3(x)]_i : i=1,\cdots,N\} \\
&= \frac{1}{5760}\max\{ \\
&\quad 484\widehat{G}^3_{\gamma_1} - 7592\widehat{G}^3_{\gamma_2} + 26128\widehat{G}^3_{\gamma_3} - 40832\widehat{G}^3_{\gamma_4} + 34068\widehat{G}^3_{\gamma_5} - 15256\widehat{G}^3_{\gamma_6} \\
&\quad +3160\widehat{G}^3_{\gamma_7} - 160\widehat{G}^3_{\gamma_8}, \\
&\quad 216\widehat{G}^3_{\gamma_1} - 3052\widehat{G}^3_{\gamma_2} + 9248\widehat{G}^3_{\gamma_3} - 12752\widehat{G}^3_{\gamma_4} + 9608\widehat{G}^3_{\gamma_5} - 4028\widehat{G}^3_{\gamma_6} + 800\widehat{G}^3_{\gamma_7} \\
&\quad -40\widehat{G}^3_{\gamma_8}, \\
&\quad 68\widehat{G}^3_{i-2} - 752\widehat{G}^3_{i-1} + 1368\widehat{G}^3_i - 752\widehat{G}^3_{i+1} + 68\widehat{G}^3_{i+2} : i=3,\cdots,N-2\ \} \\
&= O(\omega(\widehat{G}^3, 7h))
\end{aligned}
$$

$$
\begin{aligned}
||(\Upsilon \;&-\; \Psi)D_\Delta\widehat{G}^2(x)||_\infty \\
&= \max\{[(\Upsilon-\Psi)D_\Delta\widehat{G}^2(x)]_i : i=1,\cdots,N\} \\
&= \frac{1}{5760}\max\{ \\
&\quad 58\widehat{G}^2_{\gamma_1} - 2804\widehat{G}^2_{\gamma_2} + 10786\widehat{G}^2_{\gamma_3} - 17504\widehat{G}^2_{\gamma_4} + 14766\widehat{G}^2_{\gamma_5} - 6532\widehat{G}^2_{\gamma_6} + 1270\widehat{G}^2_{\gamma_7} \\
&\quad -40\widehat{G}^2_{\gamma_8}, \\
&\quad 12\widehat{G}^2_{\gamma_1} - 1054\widehat{G}^2_{\gamma_2} + 3656\widehat{G}^2_{\gamma_3} - 5294\widehat{G}^2_{\gamma_4} + 4076\widehat{G}^2_{\gamma_5} - 1706\widehat{G}^2_{\gamma_6} + 320\widehat{G}^2_{\gamma_7} \\
&\quad -10\widehat{G}^2_{\gamma_8},
\end{aligned}
$$

$$
\begin{aligned}
& -38\widehat{G}^2_{i-2} + 392\widehat{G}^2_{i-1} - 708\widehat{G}^2_i + 392\widehat{G}^2_{i+1} - 38\widehat{G}^2_{i+2} : i = 3, \ldots, N-2 \ \} \\
= \ & O(\omega(\widehat{G}^2, 7h))
\end{aligned}
$$

$$
\begin{aligned}
||(\Omega \ - \ & \Psi)D_\Delta \widehat{G}^1(x)||_\infty \\
= \ & \max\{[(\Omega - \Psi)D_\Delta \widehat{G}^1(x)]_i : i = 1, \cdots, N\} \\
= \ & \frac{1}{5760} \max\{ \\
& 142\widehat{G}^1_{\gamma_1} - 3196\widehat{G}^1_{\gamma_2} + 11514\widehat{G}^1_{\gamma_3} - 18176\widehat{G}^1_{\gamma_4} + 15074\widehat{G}^1_{\gamma_5} - 6588\widehat{G}^1_{\gamma_6} + 1270\widehat{G}^1_{\gamma_7} \\
& -40\widehat{G}^1_{\gamma_8}, \\
& 68\widehat{G}^1_{\gamma_1} - 1306\widehat{G}^1_{\gamma_2} + 4104\widehat{G}^1_{\gamma_3} - 5686\widehat{G}^1_{\gamma_4} + 4244\widehat{G}^1_{\gamma_5} - 1734\widehat{G}^1_{\gamma_6} + 320\widehat{G}^1_{\gamma_7} \\
& -10\widehat{G}^1_{\gamma_8}, \\
& 24\widehat{G}^1_{i-2} - 336\widehat{G}^1_{i-1} + 624\widehat{G}^1_i - 336\widehat{G}^1_{i+1} + 24\widehat{G}^1_{i+2} : i = 3, \cdots, N-2 \ \} \\
= \ & O(\omega(\widehat{G}^1, 7h))
\end{aligned}
$$

$$
\begin{aligned}
||(I \ - \ & \Psi)D_\Delta \widehat{G}(x)||_\infty \\
= \ & \max\{[(I - \Psi)D_\Delta \widehat{G}(x)]_i : i = 1, \cdots, N\} \\
= \ & \frac{1}{5760} \max\{ \\
& 5881\widehat{G}_{\gamma_1} - 3098\widehat{G}_{\gamma_2} + 11332\widehat{G}_{\gamma_3} - 18008\widehat{G}_{\gamma_4} + 14997\widehat{G}_{\gamma_5} - 6574\widehat{G}_{\gamma_6} \\
& +1270\widehat{G}_{\gamma_7} - 40\widehat{G}_{\gamma_8}, \\
& 54\widehat{G}_{\gamma_1} + 4517\widehat{G}_{\gamma_2} + 3992\widehat{G}_{\gamma_3} - 5588\widehat{G}_{\gamma_4} + 4202\widehat{G}_{\gamma_5} - 1727\widehat{G}_{\gamma_6} + 320\widehat{G}_{\gamma_7} \\
& -10\widehat{G}_{\gamma_8}, \\
& 17\widehat{G}_{i-2} - 308\widehat{G}_{i-1} + 6342\widehat{G}_i - 308\widehat{G}_{i+1} + 17\widehat{G}_{i+2} : i = 3, \cdots, N-2 \ \} \\
= \ & O(\omega(\widehat{G}, 7h))
\end{aligned}
$$

where $\omega(g, j) \equiv \sup\{|g(x) - g(y)| : |x - y| \leq j\}$ is the modulus of continuity of $g$. It is easy to see that $\omega(\widehat{G}^k, 7h)$ approaches 0 as $h$ converges to 0. Hence,

$$||R_\Delta Kg - P_\Delta D_\Delta Kg||_\infty \to 0 \text{ as } h \to 0. \tag{4.24}$$

We are ready to state the following lemma.

**LEMMA 3** *The sequence of operators $R_\Delta K$ converges to $K$ as $h$ converges to 0.*

*Proof:* By the triangular inequality and the equivalence of $P_\Delta D_\Delta K$ and $P_\Delta K$, $||R_\Delta K - K||_\infty \leq ||R_\Delta K - P_\Delta D_\Delta K||_\infty + ||P_\Delta K - K||_\infty$. Since both norms on the right side have been shown to converge to zero as $h$ goes to 0, the assertion is proved. $\diamondsuit$

Now we can prove the following for the one-step method.

**THEOREM 8** *Under the assumptions of Theorem 5, we conclude that*

*(i) the approximation $z_\Delta \in \mathcal{S}_{4,\Delta}$ that satisfies equations (3.18)-(3.19) in the one-step method exists,*

*and the global error bounds*

*(ii) $||(u - z_\Delta)^{(k)}||_\infty = O(h^{5-k}), k = 0, \cdots, 4$*

*and the local error bounds*

*(iii) $|(u - z_\Delta)(x)| = O(h^6), x = x_i$ and $x = \tau_i$,*

*(iv) $|(u - z_\Delta)^{(k)}(\tau_i)| = O(h^4), k = 1, 2, \quad |(u - z_\Delta)^{(k)}(\tau_i)| = O(h^2), k = 3, 4$*

*hold.*

*Proof:* The proof uses the same arguments as the proofs of Theorems 5, 6 and 7 for the three-step method. The existence and boundedness of $(I + K)^{-1}$ follow from the assumption that the problem $Lu = 0, Bu = 0$ is uniquely solvable. By Lemma 3 and Neumann's Theorem, $(I + R_\Delta K)^{-1}$ exists and is uniformly bounded for sufficiently small $h$.

There exists a cubic function $\xi$ such that $BS = B\xi = O(h^6)$, $||\xi^{(k)}||_\infty = O(h^6), k = 0, \cdots, 3$, and $(S - \xi)^{(4)} = S^{(k)}, B(S - \xi) = 0$ is solvable. We have, from equations (3.18) and (3.19), that

$$(I + R_\Delta K)(S^{(4)} - \xi^{(4)}) = P_\Delta f + O(h^6)$$

from which we obtain

$$(I + R_\Delta K)(S^{(4)} - \xi^{(4)} - u^{(4)}) = O(h^6)$$

The error bounds are then derived by applying the same reasoning as in the proof of Theorem 6.

# Chapter 5

# Numerical Results

In this chapter, we present results from numerical experiments to demonstrate the performance of the methods presented and verify the results of the analysis.

We implemented our methods in MATLAB (in double precision), and ran all experiments on a Ultra 4 sparc machine. The methods are tested on several BVPs, many of which are found in the literature. Most problems considered involve one of the following types of boundary conditions:

$$u(a) = g_1, u^{(1)}(a) = g_2, u(b) = g_3, u^{(1)}(b) = g_4, \tag{5.1}$$

$$u(a) = g_1, u^{(2)}(a) = g_2, u(b) = g_3, u^{(2)}(b) = g_4. \tag{5.2}$$

Whenever a problem involves boundary conditions other than (5.1) or (5.2) these are presented explicitly.

The first problem considered is used in [15], [21] and [22].

Problem 1: Consider the differential equation

$$Lu \equiv u^{(4)} + xu = f(x), \quad x \in [0,1]$$

with boundary conditions (5.1) or (5.2). The functions $f$ and $g_i$, $i = 1, \cdots, 4$, are determined so that the exact solution is $u(x) = x(1-x)e^x$. We apply the three-step quartic-spline collocation method to Problem 1. The maximum absolute errors and the respective orders of convergences are presented in the tables below. The notation used in the tables is the same as that used in Chapter 2.

The effect of the boundary conditions on the order of convergence is not as significant as in the case of interpolation (see tables in Chapter 2). The order of convergence of our quartic-spline collocation method with BCs either (5.1) or (5.2) is close to 6 locally on the midpoints and gridpoints. The global order of convergence is almost 0.5 higher for BCs (5.2) than for BCs (5.1), which is 5 as predicted by our analysis.

It can be observed that round-off effects occur when errors approach $10^{-11}$ which is still significantly larger than the machine epsilon which is approximately $10^{-15}$. This may be caused by large condition numbers of the linear systems for the relatively large $N$'s. For Problem 1 with BCs (5.1), the condition number of the linear system for $N = 8$ is in the order of $10^5$ while those for $N = 64$ and $N = 128$ are about $10^9$ and $10^{10}$, respectively.

Table 5.1: Errors and corresponding orders of convergence for Problem 1, BCs (5.1).

| $N$ | $\|\|u-u_\Delta\|\|_{\tau_i,x_i,\infty}$ | | | $\|\|u'-u'_\Delta\|\|_{\tau_i,\lambda_i,\infty}$ | | | $\|\|u''-u''_\Delta\|\|_{\tau_i,x_i,\infty}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| 8 | 1.4-07 | 2.2-07 | 4.8-07 | 1.9-05 | 4.2-06 | 2.0-05 | 7.1-05 | 1.0-04 | 9.9-04 |
| 16 | 3.1-09 | 4.3-09 | 1.3-08 | 1.2-06 | 2.7-07 | 1.3-06 | 5.0-06 | 6.3-06 | 1.2-04 |
| 32 | 5.6-11 | 7.6-11 | 4.1-10 | 7.7-08 | 1.6-08 | 8.7-08 | 3.3-07 | 3.9-07 | 1.6-05 |
| 64 | 1.9-11 | 2.0-11 | 2.4-11 | 4.8-09 | 9.9-10 | 5.4-09 | 2.1-08 | 2.4-08 | 2.0-06 |
| 128 | 2.8-11 | 2.9-11 | 2.8-11 | 4.4-10 | 3.6-10 | 4.4-10 | 1.9-09 | 1.8-09 | 2.5-07 |
| | | | | orders of convergence | | | | | |
| 16 | 5.51 | 5.69 | 5.17 | 3.99 | 3.95 | 3.90 | 3.82 | 4.07 | 2.94 |
| 32 | 5.78 | 5.83 | 5.00 | 3.99 | 4.04 | 4.00 | 3.93 | 4.02 | 2.97 |
| 64 | 1.49 | 1.91 | 4.07 | 3.99 | 4.05 | 3.99 | 3.96 | 4.02 | 3.00 |
| 128 | -0.49 | -0.53 | -0.22 | 3.45 | 1.43 | 3.62 | 3.46 | 3.72 | 3.01 |

| $N$ | $\|\|u^{(3)}-u^{(3)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | | $\|\|u^{(4)}-u^{(4)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | |
|---|---|---|---|---|---|---|---|---|
| 8 | 4.0-02 | 8.8-02 | 2.9-03 | 8.4-02 | 5.8-02 | 4.1+00 | 2.3+00 | 4.0+00 |
| 16 | 1.0-02 | 2.2-02 | 3.7-04 | 2.0-02 | 1.5-02 | 2.0+00 | 1.2+00 | 2.0+00 |
| 32 | 2.7-03 | 5.5-03 | 4.7-05 | 5.0-03 | 3.8-03 | 1.0+00 | 6.0-01 | 1.0+00 |
| 64 | 6.8-04 | 1.3-03 | 5.9-06 | 1.2-03 | 9.8-04 | 5.2-01 | 3.0-01 | 5.1-01 |
| 128 | 1.7-04 | 3.4-04 | 7.4-07 | 3.2-04 | 2.4-04 | 2.6-01 | 1.5-01 | 2.5-01 |
| | | | | orders of convergence | | | | |
| 16 | 1.93 | 1.99 | 2.96 | 2.06 | 1.93 | 0.97 | 0.97 | 1.00 |
| 32 | 1.96 | 1.99 | 2.98 | 1.98 | 1.96 | 0.98 | 0.98 | 1.00 |
| 64 | 1.98 | 1.99 | 2.99 | 1.96 | 1.98 | 0.99 | 0.99 | 0.97 |
| 128 | 1.99 | 2.00 | 2.99 | 1.98 | 1.99 | 0.99 | 0.99 | 0.98 |

Table 5.2: Errors and corresponding orders of convergence for Problem 1, BCs (5.2).

| $N$ | $\|\|u - u_\Delta\|\|_{\tau_i,x_i,\infty}$ | | | $\|\|u' - u'_\Delta\|\|_{\tau_i,\lambda_i,\infty}$ | | | $\|\|u'' - u''_\Delta\|\|_{\tau_i,x_i,\infty}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| 8 | 6.9-07 | 6.3-07 | 8.9-07 | 6.9-07 | 6.3-07 | 8.9-07 | 8.6-05 | 9.1-05 | 1.0-03 |
| 16 | 1.3-08 | 1.2-08 | 2.0-08 | 1.1-06 | 3.6-07 | 1.4-06 | 5.3-06 | 6.0-06 | 1.2-04 |
| 32 | 2.2-10 | 2.0-10 | 4.6-10 | 7.6-08 | 1.8-08 | 8.8-08 | 3.3-07 | 3.8-07 | 1.6-05 |
| 64 | 2.0-11 | 2.0-11 | 2.4-11 | 4.8-09 | 9.7-10 | 5.4-09 | 2.1-08 | 2.4-08 | 2.0-06 |
| 128 | 2.2-10 | 2.2-10 | 2.2-10 | 8.9-10 | 9.4-10 | 9.9-10 | 2.3-09 | 3.0-09 | 2.5-07 |
| | | | | orders of convergence | | | | | |
| 16 | 5.69 | 5.66 | 5.45 | 3.79 | 4.56 | 4.11 | 4.01 | 3.90 | 2.96 |
| 32 | 5.90 | 5.89 | 5.44 | 3.91 | 4.32 | 4.06 | 3.99 | 3.97 | 2.98 |
| 64 | 3.49 | 3.37 | 4.24 | 3.95 | 4.21 | 4.02 | 3.99 | 3.99 | 3.00 |
| 128 | -3.47 | -3.46 | -3.18 | 2.44 | 0.03 | 2.44 | 3.19 | 3.01 | 3.00 |

| $N$ | $\|\|u^{(3)} - u^{(3)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | | $\|\|u^{(4)} - u^{(4)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | |
|---|---|---|---|---|---|---|---|---|
| 8 | 4.0-02 | 8.8-02 | 2.9-03 | 8.4-02 | 5.8-02 | 4.1+00 | 2.3+00 | 4.0+00 |
| 16 | 1.0-02 | 2.2-02 | 3.7-04 | 2.0-02 | 1.5-02 | 2.0+00 | 1.2+00 | 2.0+00 |
| 32 | 2.7-03 | 5.5-03 | 4.7-05 | 5.0-03 | 3.8-03 | 1.0+00 | 6.0-01 | 1.0+00 |
| 64 | 6.8-04 | 1.3-03 | 5.9-06 | 1.2-03 | 9.8-04 | 5.2-01 | 3.0-01 | 5.1-01 |
| 128 | 1.7-04 | 3.4-04 | 7.4-07 | 3.2-04 | 2.4-04 | 2.6-01 | 1.5-01 | 2.5-01 |
| | | | | orders of convergence | | | | |
| 16 | 1.93 | 1.99 | 2.95 | 2.06 | 1.93 | 0.97 | 0.97 | 1.00 |
| 32 | 1.96 | 1.99 | 2.98 | 1.98 | 1.96 | 0.98 | 0.98 | 1.00 |
| 64 | 1.98 | 1.99 | 2.99 | 1.96 | 1.98 | 0.99 | 0.99 | 0.97 |
| 128 | 1.99 | 2.00 | 2.99 | 1.98 | 1.99 | 0.99 | 0.99 | 0.98 |

For comparison, in the two tables below we give numerical results from [21], for Problem 1 with boundary conditions (5.2). Errors for both the quintic-spline solution and the sextic-spline solution of the methods in [21] are shown. The former is a second-order method, while the latter a fourth-order one. By comparing the results in Tables 5.2, 5.3 and 5.4, we note that our quartic-spline method gives smaller errors even for the smallest grids considered, and, of course, exhibits sixth order of convergence, until it reaches errors close to the round-off level. The sixth-order method of [15] gives slightly smaller errors than our quartic-spline method.

Table 5.3: Errors for quintic-spline solution of [19].

| $N$ | $\|\|u - u_\Delta\|\|_{x_i}$ | $\|\|u^{(1)} - u_\Delta^{(1)}\|\|_{x_i}$ | $\|\|u^{(2)} - u_\Delta^{(2)}\|\|_{x_i}$ | $\|\|u^{(3)} - u_\Delta^{(3)}\|\|_{x_i}$ | $\|\|u^{(4)} - u_\Delta^{(4)}\|\|_{x_i}$ |
|---|---|---|---|---|---|
| 8 | 0.87-3 | 0.29-2 | 0.83-2 | 0.42-1 | 0.51-3 |
| 16 | 0.22-3 | 0.73-3 | 0.21-2 | 0.11-1 | 0.13-3 |
| 32 | 0.54-4 | 0.18-3 | 0.52-3 | 0.26-2 | 0.32-4 |
| 64 | 0.14-4 | 0.45-4 | 0.13-3 | 0.66-3 | 0.8-5 |

Table 5.4: Errors for sextic-spline solution of [19].

| $N$ | $\|\|u - u_\Delta\|\|_{x_i}$ | $\|\|u^{(1)} - u_\Delta^{(1)}\|\|_{x_i}$ | $\|\|u^{(2)} - u_\Delta^{(2)}\|\|_{x_i}$ | $\|\|u^{(3)} - u_\Delta^{(3)}\|\|_{x_i}$ | $\|\|u^{(4)} - u_\Delta^{(4)}\|\|_{x_i}$ |
|---|---|---|---|---|---|
| 8 | 0.13-5 | 0.42-5 | 0.12-4 | 0.10-3 | 0.74-6 |
| 16 | 0.79-7 | 0.26-6 | 0.76-6 | 0.64-5 | 0.46-7 |
| 32 | 0.49-8 | 0.16-7 | 0.48-7 | 0.41-6 | 0.29-8 |
| 64 | 0.29-9 | 0.97-9 | 0.28-8 | 0.26-7 | 0.17-9 |
| 128 | 0.26-9 | 0.83-9 | 0.25-8 | 0.11-7 | 0.15-9 |

The second problem considered is extracted from [16].

Problem 2: Consider the differential equation

$$Lu \equiv u^{(4)} + u^{(3)} + \frac{1}{x}u^{(2)} - \frac{1}{x^2}u^{(1)} = f, \quad x \in [1,2]$$

subject to boundary conditions (5.1).

The functions $f$ and $g_i$, $i = 1, \cdots, 4$, are determined so that the exact solution is

$$u(x) = \frac{x^2}{4}[\ln\frac{x}{2} - 1] - \frac{x^2}{8}[\frac{0.7}{1.3} + \frac{2}{3}\ln 2] - \frac{2.6}{2.1}\ln 2 \ln\frac{x}{2} + \frac{3.3}{2.6} + \frac{\ln 2}{3}\ .$$

We apply three quartic-spline collocation methods to Problem 2: the three-step, the one-step and the standard quartic-spline collocation methods. The three tables below show the maximum absolute errors and the respective orders of convergence for the three-step, the one-step and the standard quartic-spline collocation methods, respectively. Both the three-step and the one-step quartic-spline collocation methods are sixth-order (up to the point that the errors reach the round-off level), while the standard quartic-spline collocation method is second-order, as predicted by the analysis. It should be noted again that while the three-step and one-step methods are predicted theoretically to be fifth-order globally, experimentally the observed global order is six for this problem.

Table 5.5: Errors and corresponding orders of convergence for Problem 2 using the three-step quartic-spline collocation method.

| $N$ | $\|\|u - u_\Delta\|\|_{x_i,\lambda_i,\infty}$ | | | $\|\|u' - u'_\Delta\|\|_{\tau_i,\lambda_i,\infty}$ | | | $\|\|u'' - u''_\Delta\|\|_{\tau_i,x_i,\infty}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| 8 | 1.2-06 | 1.2-06 | 1.2-06 | 9.3-06 | 6.2-06 | 9.5-06 | 7.2-05 | 6.3-05 | 2.9-04 |
| 16 | 1.8-08 | 1.8-08 | 1.9-08 | 4.0-07 | 1.1-07 | 4.1-07 | 4.7-06 | 4.9-06 | 3.5-05 |
| 32 | 3.1-10 | 3.1-10 | 3.4-10 | 2.2-08 | 4.4-09 | 2.3-08 | 3.1-07 | 3.8-07 | 4.5-06 |
| 64 | 2.2-11 | 2.3-11 | 2.6-11 | 1.2-09 | 4.2-10 | 1.5-09 | 2.1-08 | 2.4-08 | 5.7-07 |
| 128 | 2.4-10 | 2.4-10 | 2.4-10 | 5.5-10 | 6.6-10 | 7.2-10 | 4.9-09 | 2.1-09 | 7.6-08 |
| | | | | orders of convergence | | | | | |
| 16 | 6.03 | 6.05 | 6.01 | 4.51 | 5.83 | 4.54 | 3.93 | 3.69 | 3.04 |
| 32 | 5.88 | 5.88 | 5.82 | 4.15 | 4.64 | 4.13 | 3.92 | 3.66 | 2.94 |
| 64 | 3.78 | 3.73 | 3.72 | 4.15 | 3.38 | 3.90 | 3.89 | 3.98 | 2.98 |
| 128 | -3.42 | -3.35 | -3.22 | 1.21 | -0.65 | 1.12 | 2.09 | 3.47 | 2.91 |

| $N$ | $\|\|u^{(3)} - u^{(3)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | | $\|\|u^{(4)} - u^{(4)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | |
|---|---|---|---|---|---|---|---|---|
| 8 | 9.8-03 | 2.3-02 | 2.2-03 | 2.2-02 | 3.3-02 | 1.0+00 | 6.0-01 | 1.0+00 |
| 16 | 2.7-03 | 6.2-03 | 3.1-04 | 5.7-03 | 1.2-02 | 5.7-01 | 3.2-01 | 5.6-01 |
| 32 | 7.3-04 | 1.5-03 | 4.5-05 | 1.3-03 | 3.6-03 | 2.9-01 | 1.7-01 | 2.7-01 |
| 64 | 1.9-04 | 3.9-04 | 5.9-06 | 3.5-04 | 9.6-04 | 1.5-01 | 8.6-02 | 1.4-01 |
| 128 | 4.8-05 | 9.9-05 | 7.4-07 | 9.1-05 | 2.4-04 | 7.6-02 | 4.3-02 | 7.3-02 |
| | | | | orders of convergence | | | | |
| 16 | 1.85 | 1.89 | 2.81 | 1.96 | 1.38 | 0.90 | 0.86 | 0.92 |
| 32 | 1.89 | 1.98 | 2.77 | 2.12 | 1.79 | 0.95 | 0.94 | 1.00 |
| 64 | 1.94 | 1.99 | 2.94 | 1.89 | 1.92 | 0.98 | 0.97 | 0.98 |
| 128 | 1.97 | 1.99 | 2.99 | 1.94 | 1.97 | 0.99 | 0.98 | 0.95 |

Table 5.6: Errors and corresponding orders of convergence for Problem 2 using the one-step quartic-spline collocation method.

| $N$ | $\|\|u - u_\Delta\|\|_{\tau_i,x_i,\infty}$ | | | $\|\|u' - u'_\Delta\|\|_{\tau_i,\lambda_i,\infty}$ | | | $\|\|u'' - u''_\Delta\|\|_{\tau_i,x_i,\infty}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| 8 | 1.2-06 | 1.2-06 | 1.2-06 | 9.3-06 | 6.2-06 | 9.5-06 | 7.2-05 | 6.3-05 | 2.9-04 |
| 16 | 1.8-08 | 1.8-08 | 1.9-08 | 4.0-07 | 1.0-07 | 4.1-07 | 4.7-06 | 4.9-06 | 3.5-05 |
| 32 | 3.0-10 | 3.0-10 | 3.3-10 | 2.2-08 | 4.4-09 | 2.3-08 | 3.1-07 | 3.8-07 | 4.5-06 |
| 64 | 2.6-11 | 2.7-11 | 3.0-11 | 1.3-09 | 3.8-10 | 1.5-09 | 1.9-08 | 2.5-08 | 5.7-07 |
| 128 | 3.7-10 | 3.7-10 | 3.7-10 | 2.4-09 | 2.5-09 | 2.6-09 | 7.7-09 | 7.3-09 | 7.3-08 |
| | | | | orders of convergence | | | | | |
| 16 | 6.04 | 6.06 | 6.02 | 4.51 | 5.83 | 4.54 | 3.93 | 3.70 | 3.04 |
| 32 | 5.90 | 5.90 | 5.83 | 4.15 | 4.62 | 4.13 | 3.92 | 3.66 | 2.94 |
| 64 | 3.51 | 3.49 | 3.47 | 4.11 | 3.53 | 3.92 | 3.98 | 3.91 | 2.98 |
| 128 | -3.80 | -3.76 | -3.62 | -0.92 | -2.76 | -0.78 | 1.36 | 1.79 | 2.97 |

| $N$ | $\|\|u^{(3)} - u^{(3)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | | $\|\|u^{(4)} - u^{(4)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | |
|---|---|---|---|---|---|---|---|---|
| 8 | 9.8-03 | 2.3-02 | 2.2-03 | 2.2-02 | 3.3-02 | 1.0+00 | 6.0-01 | 1.0+00 |
| 16 | 2.7-03 | 6.2-03 | 3.1-04 | 5.7-03 | 1.2-02 | 5.7-01 | 3.2-01 | 5.6-01 |
| 32 | 7.3-04 | 1.5-03 | 4.5-05 | 1.3-03 | 3.6-03 | 2.9-01 | 1.7-01 | 2.7-01 |
| 64 | 1.9-04 | 3.9-04 | 5.9-06 | 3.5-04 | 9.6-04 | 1.5-01 | 8.6-02 | 1.4-01 |
| 128 | 4.8-05 | 9.9-05 | 7.6-07 | 9.1-05 | 2.4-04 | 7.6-02 | 4.3-02 | 7.3-02 |
| | | | | orders of convergence | | | | |
| 16 | 1.85 | 1.89 | 2.81 | 1.96 | 1.38 | 0.90 | 0.86 | 0.92 |
| 32 | 1.89 | 1.98 | 2.77 | 2.12 | 1.79 | 0.95 | 0.94 | 1.00 |
| 64 | 1.94 | 1.99 | 2.94 | 1.89 | 1.92 | 0.98 | 0.97 | 0.98 |
| 128 | 1.97 | 1.99 | 2.96 | 1.94 | 2.00 | 0.99 | 0.98 | 0.95 |

Table 5.7: Errors and corresponding orders of convergence for Problem 2 using the standard quartic-spline collocation method.

| $N$ | $\|\|u - u_\Delta\|\|_{\tau_i,x_i,\infty}$ | | | $\|\|u' - u'_\Delta\|\|_{\tau_i,\lambda_i,\infty}$ | | | $\|\|u'' - u''_\Delta\|\|_{\tau_i,x_i,\infty}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| 8 | 2.6-05 | 2.5-05 | 2.6-05 | 9.5-05 | 9.4-05 | 9.6-05 | 7.1-04 | 1.3-03 | 1.3-03 |
| 16 | 6.4-06 | 6.4-06 | 6.4-06 | 2.3-05 | 2.3-05 | 2.3-05 | 2.5-04 | 3.4-04 | 3.5-04 |
| 32 | 1.6-06 | 1.6-06 | 1.6-06 | 5.7-06 | 5.7-06 | 5.7-06 | 7.5-05 | 8.8-05 | 8.8-05 |
| 64 | 4.0-07 | 4.0-07 | 4.0-07 | 1.4-06 | 1.4-06 | 1.4-06 | 2.0-05 | 2.2-05 | 2.2-05 |
| 128 | 9.9-08 | 9.9-08 | 9.9-08 | 3.6-07 | 3.6-07 | 3.6-07 | 5.3-06 | 5.5-06 | 5.5-06 |
| | | | | orders of convergence | | | | | |
| 16 | 2.04 | 1.99 | 2.03 | 2.02 | 2.02 | 2.03 | 1.48 | 1.90 | 1.94 |
| 32 | 2.00 | 2.01 | 2.01 | 2.02 | 2.00 | 2.02 | 1.76 | 1.97 | 1.99 |
| 64 | 2.00 | 2.00 | 2.00 | 1.99 | 2.00 | 2.00 | 1.88 | 1.99 | 2.00 |
| 128 | 2.00 | 2.00 | 2.00 | 2.00 | 2.00 | 2.00 | 1.94 | 1.99 | 2.00 |

| $N$ | $\|\|u^{(3)} - u^{(3)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | | $\|\|u^{(4)} - u^{(4)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | |
|---|---|---|---|---|---|---|---|---|
| 8 | 1.8-02 | 1.1-02 | 9.6-03 | 1.8-02 | 1.8-02 | 1.0+00 | 5.4-01 | 1.0+00 |
| 16 | 5.6-03 | 2.7-03 | 2.9-03 | 5.6-03 | 5.4-03 | 5.6-01 | 3.1-01 | 5.4-01 |
| 32 | 1.5-03 | 6.9-04 | 8.1-04 | 1.5-03 | 1.4-03 | 2.9-01 | 1.6-01 | 2.7-01 |
| 64 | 4.0-04 | 1.7-04 | 2.1-04 | 4.0-04 | 3.8-04 | 1.4-01 | 8.5-02 | 1.4-01 |
| 128 | 1.0-04 | 4.3-05 | 5.4-05 | 1.0-04 | 9.8-05 | 7.5-02 | 4.3-02 | 7.3-02 |
| | | | | orders of convergence | | | | |
| 16 | 1.73 | 2.00 | 1.70 | 1.73 | 1.75 | 0.88 | 0.82 | 0.90 |
| 32 | 1.86 | 2.00 | 1.85 | 1.86 | 1.87 | 0.93 | 0.90 | 0.98 |
| 64 | 1.93 | 2.00 | 1.92 | 1.93 | 1.93 | 0.96 | 0.95 | 0.94 |
| 128 | 1.96 | 2.00 | 1.96 | 1.96 | 1.96 | 0.98 | 0.97 | 0.95 |

The third problem is extracted from [15].

Problem 3: Consider the differential equation

$$Lu \equiv u^{(4)} + \frac{24}{(1+x)^3}u^{(1)} - \frac{72}{(1+x)^4}u = 0, \quad x \in [0,1]$$

subject to boundary conditions (5.2). The true solution is $u(x) = \frac{4}{(1+x)^2}$. The following three tables show errors and orders of convergence for the three-step, the one-step and the standard quartic-spline collocation methods, respectively. Again, the three-step and one-step methods are sixth-order globally in experimental results for this problem as opposed to fifth-order predicted by the analysis.

Table 5.8: Errors and corresponding orders of convergence for Problem 3 using the three-step quartic-spline collocation method.

| $N$ | $\|\|u-u_\Delta\|\|_{\tau_i,x_i,\infty}$ | | | $\|\|u'-u'_\Delta\|\|_{\tau_i,\lambda_i,\infty}$ | | | $\|\|u''-u''_\Delta\|\|_{\tau_i,x_i,\infty}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| 8 | 2.8-04 | 2.8-04 | 2.8-04 | 1.6-03 | 1.3-03 | 1.7-03 | 1.4-02 | 1.0-02 | 4.3-02 |
| 16 | 5.0-06 | 5.1-06 | 5.1-06 | 6.6-05 | 3.0-05 | 6.8-05 | 9.6-04 | 8.2-04 | 5.1-03 |
| 32 | 8.6-08 | 8.7-08 | 8.9-08 | 3.4-06 | 5.6-07 | 3.4-06 | 6.2-05 | 7.5-05 | 6.5-04 |
| 64 | 1.5-09 | 1.5-09 | 1.6-09 | 2.0-07 | 4.3-08 | 2.0-07 | 4.0-06 | 4.9-06 | 8.3-05 |
| 128 | 3.1-10 | 3.1-10 | 3.1-10 | 1.4-08 | 1.0-09 | 1.4-08 | 2.6-07 | 3.1-07 | 1.0-05 |
| | | | | orders of convergence | | | | | |
| 16 | 5.79 | 5.79 | 5.77 | 4.61 | 5.50 | 4.65 | 3.89 | 3.66 | 3.09 |
| 32 | 5.87 | 5.87 | 5.85 | 4.27 | 5.74 | 4.29 | 3.95 | 3.46 | 2.95 |
| 64 | 5.81 | 5.81 | 5.78 | 4.06 | 3.69 | 4.05 | 3.92 | 3.91 | 2.97 |
| 128 | 2.30 | 2.30 | 2.35 | 3.83 | 5.35 | 3.84 | 3.97 | 3.97 | 2.98 |

| $N$ | $\|\|u^{(3)}-u^{(3)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | | $\|\|u^{(4)}-u^{(4)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | |
|---|---|---|---|---|---|---|---|---|
| 8 | 1.3+00 | 3.2+00 | 3.9-01 | 3.0+00 | 5.2+00 | 1.5+02 | 8.2+01 | 1.4+02 |
| 16 | 3.7-01 | 9.1-01 | 5.8-02 | 8.3-01 | 2.3+00 | 8.3+01 | 4.6+01 | 8.0+01 |
| 32 | 1.0-01 | 2.3-01 | 8.9-03 | 1.9-01 | 7.1-01 | 4.3+01 | 2.4+01 | 4.0+01 |
| 64 | 2.7-02 | 5.8-02 | 1.1-03 | 5.0-02 | 1.9-01 | 2.2+01 | 1.2+01 | 2.0+01 |
| 128 | 7.1-03 | 1.4-02 | 1.5-04 | 1.3-02 | 4.9-02 | 1.1+01 | 6.4+00 | 1.0+01 |
| | | | | orders of convergence | | | | |
| 16 | 1.82 | 1.82 | 2.75 | 1.89 | 1.18 | 0.86 | 0.80 | 0.88 |
| 32 | 1.85 | 1.96 | 2.71 | 2.11 | 1.70 | 0.94 | 0.92 | 0.98 |
| 64 | 1.91 | 1.99 | 2.91 | 1.93 | 1.89 | 0.97 | 0.96 | 1.00 |
| 128 | 1.95 | 1.99 | 2.97 | 1.92 | 1.95 | 0.98 | 0.98 | 0.93 |

Table 5.9: Errors and corresponding orders of convergence for Problem 3 using the one-step quartic-spline collocation method.

| $N$ | $\|\|u - u_\Delta\|\|_{\tau_i,x_i,\infty}$ | | | $\|\|u' - u'_\Delta\|\|_{\tau_i,\lambda_i,\infty}$ | | | $\|\|u'' - u''_\Delta\|\|_{\tau_i,x_i,\infty}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| 8 | 2.8-04 | 2.8-04 | 2.8-04 | 1.6-03 | 1.3-03 | 1.7-03 | 1.4-02 | 1.0-02 | 4.3-02 |
| 16 | 5.0-06 | 5.0-06 | 5.1-06 | 6.6-05 | 3.0-05 | 6.8-05 | 9.6-04 | 8.2-04 | 5.1-03 |
| 32 | 8.5-08 | 8.6-08 | 8.8-08 | 3.4-06 | 5.6-07 | 3.4-06 | 6.2-05 | 7.5-05 | 6.5-04 |
| 64 | 1.4-09 | 1.4-09 | 1.5-09 | 2.0-07 | 4.3-08 | 2.0-07 | 4.0-06 | 4.9-06 | 8.3-05 |
| 128 | 1.3-09 | 1.3-09 | 1.3-09 | 7.7-09 | 7.8-09 | 1.9-08 | 2.5-07 | 3.2-07 | 1.0-05 |
| | | | | orders of convergence | | | | | |
| 16 | 5.80 | 5.80 | 5.78 | 4.62 | 5.50 | 4.66 | 3.89 | 3.66 | 3.09 |
| 32 | 5.88 | 5.88 | 5.86 | 4.27 | 5.74 | 4.29 | 3.95 | 3.46 | 2.95 |
| 64 | 5.89 | 5.90 | 5.86 | 4.06 | 3.70 | 4.05 | 3.92 | 3.91 | 2.97 |
| 128 | 0.13 | 0.14 | 0.20 | 4.73 | 2.46 | 3.45 | 4.00 | 3.95 | 2.98 |

| $N$ | $\|\|u^{(3)} - u^{(3)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | | $\|\|u^{(4)} - u^{(4)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | |
|---|---|---|---|---|---|---|---|---|
| 8 | 1.3+00 | 3.2+00 | 3.9-01 | 3.0+00 | 5.2+00 | 1.5+02 | 8.2+01 | 1.4+02 |
| 16 | 3.7-01 | 9.1-01 | 5.8-02 | 8.3-01 | 2.3+00 | 8.3+01 | 4.6+01 | 8.0+01 |
| 32 | 1.0-01 | 2.3-01 | 8.9-03 | 1.9-01 | 7.1-01 | 4.3+01 | 2.4+01 | 4.0+01 |
| 64 | 2.7-02 | 5.8-02 | 1.1-03 | 5.0-02 | 1.9-01 | 2.2+01 | 1.2+01 | 2.0+01 |
| 128 | 7.1-03 | 1.4-02 | 1.5-04 | 1.3-02 | 4.9-02 | 1.1+01 | 6.4+00 | 1.0+01 |
| | | | | orders of convergence | | | | |
| 16 | 1.82 | 1.82 | 2.75 | 1.89 | 1.17 | 0.86 | 0.80 | 0.88 |
| 32 | 1.85 | 1.96 | 2.71 | 2.11 | 1.70 | 0.94 | 0.92 | 0.98 |
| 64 | 1.91 | 1.99 | 2.91 | 1.93 | 1.89 | 0.97 | 0.96 | 1.00 |
| 128 | 1.95 | 1.99 | 2.97 | 1.92 | 1.95 | 0.98 | 0.98 | 0.93 |

Table 5.10: Errors and corresponding orders of convergence for Problem 3 using the standard quartic-spline collocation method.

| $N$ | $\|\|u - u_\Delta\|\|_{\tau_i,x_i,\infty}$ | | | $\|\|u' - u'_\Delta\|\|_{\tau_i,\lambda_i,\infty}$ | | | $\|\|u'' - u''_\Delta\|\|_{\tau_i,x_i,\infty}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| 8 | 2.6-03 | 2.5-03 | 2.6-03 | 9.3-03 | 9.2-03 | 9.4-03 | 7.3-02 | 1.2-01 | 1.4-01 |
| 16 | 6.2-04 | 6.3-04 | 6.3-04 | 2.3-03 | 2.2-03 | 2.3-03 | 2.5-02 | 3.5-02 | 3.7-02 |
| 32 | 1.5-04 | 1.5-04 | 1.5-04 | 5.6-04 | 5.6-04 | 5.6-04 | 7.6-03 | 9.0-03 | 9.2-03 |
| 64 | 3.8-05 | 3.8-05 | 3.8-05 | 1.4-04 | 1.4-04 | 1.4-04 | 2.0-03 | 2.2-03 | 2.3-03 |
| 128 | 9.7-06 | 9.7-06 | 9.7-06 | 3.5-05 | 3.5-05 | 3.5-05 | 5.4-04 | 5.7-04 | 5.7-04 |
| | | | | orders of convergence | | | | | |
| 16 | 2.06 | 2.00 | 2.05 | 2.02 | 2.03 | 2.03 | 1.50 | 1.82 | 1.92 |
| 32 | 2.00 | 2.01 | 2.01 | 2.02 | 2.00 | 2.02 | 1.75 | 1.95 | 1.99 |
| 64 | 2.00 | 2.00 | 2.00 | 2.00 | 2.00 | 2.00 | 1.87 | 1.98 | 2.00 |
| 128 | 2.00 | 2.00 | 2.00 | 2.00 | 2.00 | 2.00 | 1.93 | 1.99 | 2.00 |

| $N$ | $\|\|u^{(3)} - u^{(3)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | | $\|\|u^{(4)} - u^{(4)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | |
|---|---|---|---|---|---|---|---|---|
| 8 | 2.1+00 | 2.0+00 | 1.0+00 | 2.1+00 | 1.3-01 | 1.4+02 | 7.6+01 | 1.4+02 |
| 16 | 6.9-01 | 5.2-01 | 3.3-01 | 6.9-01 | 3.1-02 | 8.0+01 | 4.4+01 | 7.8+01 |
| 32 | 1.9-01 | 1.3-01 | 9.3-02 | 1.9-01 | 7.7-03 | 4.2+01 | 2.4+01 | 3.9+01 |
| 64 | 5.1-02 | 3.2-02 | 2.4-02 | 5.1-02 | 1.9-03 | 2.1+01 | 1.2+01 | 2.0+01 |
| 128 | 1.3-02 | 8.2-03 | 6.2-03 | 1.3-02 | 4.8-04 | 1.1+01 | 6.3+00 | 1.0+01 |
| | | | | orders of convergence | | | | |
| 16 | 1.65 | 1.98 | 1.67 | 1.65 | 2.09 | 0.85 | 0.78 | 0.87 |
| 32 | 1.83 | 1.99 | 1.84 | 1.83 | 2.05 | 0.92 | 0.89 | 0.97 |
| 64 | 1.91 | 1.99 | 1.92 | 1.91 | 1.99 | 0.96 | 0.94 | 0.96 |
| 128 | 1.95 | 1.99 | 1.96 | 1.95 | 1.99 | 0.98 | 0.97 | 0.94 |

The fourth problem is designed so that the coefficient functions in the differential operator are all nonzero with some being variable, and so that the derivatives of order 0 to 3 are used in the boundary conditions.

Problem 4: Consider the differential equation

$$Lu \equiv u^{(4)} + \frac{1}{1+x^2}u^{(3)} - e^{x/2}u^{(2)} + x^{5/2}u^{(1)} + x^3 u = f, \quad x \in [0,5]$$

subject to boundary conditions

$$u(0) = g_1, u^{(1)}(0) = g_2, u^{(2)}(5) = g_3, u^{(3)}(5) = g_4.$$

The functions $f$ and $g_i$, $i = 1, \cdots, 4$, are determined so that the exact solution is $u(x) = e^x$. We apply the three-step quartic-spline collocation method to Problem 4. The maximum absolute errors and the respective orders of convergences are presented in the table below. The presence of all (acceptable) order terms in the differential equation and boundary conditions operators does not affect the order of convergence of the method. As for Problem 2 and 3, the global order of convergence for the function is 6, the same as the order of convergence on the superconvergence points (midpoints and grid points).

Table 5.11: Errors and corresponding orders of convergence for Problem 4 using the three-step quartic-spline collocation method.

| $N$ | $\|\|u-u_\Delta\|\|_{\tau_i,x_i,\infty}$ | | | $\|\|u'-u'_\Delta\|\|_{\tau_i,\lambda_i,\infty}$ | | | $\|\|u''-u''_\Delta\|\|_{\tau_i,x_i,\infty}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| 8 | 1.9-02 | 8.5-02 | 8.5-02 | 2.0-01 | 2.4-01 | 2.4-01 | 4.3-01 | 4.3-01 | 6.9-01 |
| 16 | 4.9-04 | 8.7-04 | 8.7-04 | 3.9-03 | 2.1-03 | 3.9-03 | 1.7-03 | 9.9-03 | 3.4-02 |
| 32 | 2.7-05 | 3.4-05 | 3.4-05 | 1.9-04 | 8.7-05 | 1.9-04 | 2.0-04 | 4.8-04 | 4.2-03 |
| 64 | 5.1-07 | 5.6-07 | 6.3-07 | 8.0-06 | 7.2-07 | 8.0-06 | 1.8-05 | 2.2-05 | 5.4-04 |
| 128 | 9.4-09 | 9.7-09 | 1.2-08 | 4.3-07 | 6.2-08 | 4.5-07 | 1.2-06 | 1.4-06 | 6.8-05 |
| | | | | orders of convergence | | | | | |
| 16 | 5.28 | 6.60 | 6.60 | 5.71 | 6.84 | 5.97 | 7.96 | 5.46 | 4.31 |
| 32 | 4.17 | 4.65 | 4.64 | 4.29 | 4.61 | 4.29 | 3.13 | 4.36 | 3.01 |
| 64 | 5.72 | 5.93 | 5.78 | 4.62 | 6.91 | 4.62 | 3.41 | 4.40 | 2.97 |
| 128 | 5.77 | 5.86 | 5.67 | 4.20 | 3.52 | 4.15 | 3.92 | 3.98 | 2.98 |

| $N$ | $\|\|u^{(3)}-u^{(3)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | | $\|\|u^{(4)}-u^{(4)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | |
|---|---|---|---|---|---|---|---|---|
| 8 | 2.3+00 | 3.9+00 | 1.4+00 | 3.9+00 | 4.9+00 | 3.4+01 | 2.2+01 | 3.4+01 |
| 16 | 5.3-01 | 1.1+00 | 6.2-02 | 1.1+00 | 3.6-01 | 2.1+01 | 1.2+01 | 2.1+01 |
| 32 | 1.4-01 | 2.9-01 | 8.8-03 | 2.9-01 | 1.2-01 | 1.1+01 | 6.4+00 | 1.1+01 |
| 64 | 3.6-02 | 7.5-02 | 1.0-03 | 7.5-02 | 3.6-02 | 5.7+00 | 3.2+00 | 5.7+00 |
| 128 | 9.2-03 | 1.8-02 | 1.3-04 | 1.8-02 | 9.2-03 | 2.8+00 | 1.6+00 | 2.8+00 |
| | | | | orders of convergence | | | | |
| 16 | 2.16 | 1.77 | 4.49 | 1.77 | 3.75 | 0.67 | 0.89 | 0.67 |
| 32 | 1.92 | 1.93 | 2.81 | 1.93 | 1.49 | 0.94 | 0.92 | 0.94 |
| 64 | 1.95 | 1.98 | 3.02 | 1.98 | 1.84 | 0.97 | 0.97 | 0.97 |
| 128 | 1.97 | 1.99 | 2.96 | 1.99 | 1.96 | 0.99 | 0.98 | 0.99 |

The fifth problem is found in [19] and in [16]. The differential equation of this problem subject to boundary conditions (5.2) with a zero right side, models the bending of a thin beam clamped at both ends.

Problem 5: Consider the differential equation

$$Lu \equiv u^{(4)} = (x^4 + 14x^3 + 49x^2 + 32x - 12)e^x, \quad x \in [0, 1]$$

subject to boundary conditions (5.1) or (5.2). The exact solution is $u(x) = x^2(1-x)^2e^x$. The maximum absolute errors and the estimated orders of convergences for the three-step method are presented in the tables below. In [19], two quintic-spline collocation methods are considered for the solution of the differential equation of Problem 5 with boundary conditions (5.1), and they are both sub-optimal and give rise to larger errors than our quartic-spline collocation method. However, while the quintic-spline collocation method of [16] is also a sixth-order method as is our quartic-spline collocation method, the former gives rise to slightly smaller errors for this problem.

For BCs (5.1), the global order of convergence of our quartic-spline collocation method is 0.5 to 1 lower than the local order on the midpoints and grid points, which is close to 6; for BCs (5.2), the global order is approximately 0.5 higher than for BCs(5.1).

Table 5.12: Errors and corresponding orders of convergence for Problem 5, BCs (5.1).

| $N$ | $\|\|u-u_\Delta\|\|_{\tau_i,x_i,\infty}$ | | | $\|\|u'-u'_\Delta\|\|_{\tau_i,\lambda_i,\infty}$ | | | $\|\|u''-u''_\Delta\|\|_{\tau_i,x_i,\infty}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| 8 | 4.8-06 | 5.8-06 | 7.5-06 | 2.1-04 | 3.9-05 | 2.1-04 | 1.1-03 | 1.8-03 | 9.8-03 |
| 16 | 9.9-08 | 1.1-07 | 1.7-07 | 1.2-05 | 2.8-06 | 1.4-05 | 8.5-05 | 1.1-04 | 1.3-03 |
| 32 | 1.8-09 | 2.1-09 | 4.3-09 | 8.0-07 | 1.7-07 | 9.0-07 | 5.6-06 | 6.8-06 | 1.6-04 |
| 64 | 3.1-11 | 3.5-11 | 1.3-10 | 5.0-08 | 1.0-08 | 5.5-08 | 3.6-07 | 4.2-07 | 2.1-05 |
| 128 | 1.0-11 | 1.0-11 | 1.1-11 | 3.1-09 | 6.4-10 | 3.5-09 | 2.3-08 | 2.6-08 | 2.6-06 |
| | | | | orders of convergence | | | | | |
| 16 | 5.59 | 5.63 | 5.39 | 4.03 | 3.82 | 3.92 | 3.76 | 4.05 | 2.90 |
| 32 | 5.74 | 5.78 | 5.34 | 4.00 | 3.98 | 3.96 | 3.90 | 4.03 | 2.96 |
| 64 | 5.86 | 5.88 | 5.01 | 3.99 | 4.06 | 4.00 | 3.96 | 4.00 | 3.00 |
| 128 | 1.54 | 1.72 | 3.61 | 3.99 | 4.05 | 3.96 | 3.98 | 4.00 | 3.00 |

| $N$ | $\|\|u^{(3)}-u^{(3)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | | $\|\|u^{(4)}-u^{(4)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | |
|---|---|---|---|---|---|---|---|---|
| 8 | 3.9-01 | 9.1-01 | 4.9-02 | 8.7-01 | 9.7-01 | 4.1+01 | 2.3+01 | 4.1+01 |
| 16 | 1.0-01 | 2.2-01 | 6.4-03 | 2.0-01 | 2.6-01 | 2.1+01 | 1.2+01 | 2.0+01 |
| 32 | 2.7-02 | 5.7-02 | 8.2-04 | 5.1-02 | 6.7-02 | 1.0+01 | 6.2+00 | 1.0+01 |
| 64 | 7.0-03 | 1.4-02 | 1.0-04 | 1.3-02 | 1.7-02 | 5.4+00 | 3.1+00 | 5.2+00 |
| 128 | 1.7-03 | 3.5-03 | 1.3-05 | 3.3-03 | 4.3-03 | 2.7+00 | 1.5+00 | 2.6+00 |
| | | | | orders of convergence | | | | |
| 16 | 1.88 | 1.99 | 2.93 | 2.06 | 1.89 | 0.96 | 0.95 | 0.98 |
| 32 | 1.94 | 1.99 | 2.97 | 2.03 | 1.95 | 0.98 | 0.97 | 1.02 |
| 64 | 1.97 | 1.99 | 2.98 | 1.94 | 1.97 | 0.99 | 0.98 | 0.95 |
| 128 | 1.98 | 1.99 | 2.99 | 1.97 | 1.98 | 0.99 | 0.99 | 0.97 |

Table 5.13: Errors and corresponding orders of convergence for Problem 5, BCs (5.2).

| $N$ | $\|\|u-u_\Delta\|\|_{\tau_i,x_i,\infty}$ | | | $\|\|u'-u'_\Delta\|\|_{\tau_i,\lambda_i,\infty}$ | | | $\|\|u''-u''_\Delta\|\|_{\tau_i,x_i,\infty}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| 8 | 1.6-05 | 1.5-05 | 1.7-05 | 1.1-04 | 1.4-04 | 3.1-04 | 1.5-03 | 1.4-03 | 1.0-02 |
| 16 | 4.0-07 | 3.9-07 | 4.4-07 | 1.0-05 | 5.4-06 | 1.6-05 | 9.4-05 | 1.0-04 | 1.3-03 |
| 32 | 7.1-09 | 6.9-09 | 8.5-09 | 7.5-07 | 2.2-07 | 9.4-07 | 5.8-06 | 6.7-06 | 1.6-04 |
| 64 | 1.1-10 | 1.1-10 | 1.6-10 | 4.9-08 | 1.1-08 | 5.6-08 | 3.6-07 | 4.2-07 | 2.1-05 |
| 128 | 8.1-12 | 8.5-12 | 8.5-12 | 3.1-09 | 6.6-10 | 3.6-09 | 2.3-08 | 2.6-08 | 2.6-06 |
| | orders of convergence | | | | | | | | |
| 16 | 5.36 | 5.32 | 5.31 | 3.46 | 4.74 | 4.22 | 4.03 | 3.81 | 2.95 |
| 32 | 5.80 | 5.80 | 5.68 | 3.78 | 4.58 | 4.14 | 4.00 | 3.94 | 2.97 |
| 64 | 5.92 | 5.92 | 5.66 | 3.92 | 4.30 | 4.06 | 3.99 | 3.98 | 3.00 |
| 128 | 3.84 | 3.75 | 4.30 | 3.97 | 4.10 | 3.97 | 3.99 | 3.99 | 3.00 |

| $N$ | $\|\|u^{(3)}-u^{(3)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | | $\|\|u^{(4)}-u^{(4)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | |
|---|---|---|---|---|---|---|---|---|
| 8 | 3.9-01 | 9.1-01 | 4.9-02 | 8.7-01 | 9.7-01 | 4.1+01 | 2.3+01 | 4.1+01 |
| 16 | 1.0-01 | 2.2-01 | 6.4-03 | 2.0-01 | 2.6-01 | 2.1+01 | 1.2+01 | 2.0+01 |
| 32 | 2.7-02 | 5.7-02 | 8.2-04 | 5.1-02 | 6.7-02 | 1.0+01 | 6.2+00 | 1.0+01 |
| 64 | 7.0-03 | 1.4-02 | 1.0-04 | 1.3-02 | 1.7-02 | 5.4+00 | 3.1+00 | 5.2+00 |
| 128 | 1.7-03 | 3.5-03 | 1.3-05 | 3.3-03 | 4.3-03 | 2.7+00 | 1.5+00 | 2.6+00 |
| | orders of convergence | | | | | | | |
| 16 | 1.88 | 1.99 | 2.92 | 2.06 | 1.89 | 0.96 | 0.95 | 0.98 |
| 32 | 1.94 | 1.99 | 2.97 | 2.03 | 1.95 | 0.98 | 0.97 | 1.02 |
| 64 | 1.97 | 1.99 | 2.98 | 1.94 | 1.97 | 0.99 | 0.98 | 0.95 |
| 128 | 1.98 | 1.99 | 2.99 | 1.97 | 1.98 | 0.99 | 0.99 | 0.97 |

Finally, we design a problem to test numerically the minimum continuity required by the solution in order for our quartic-spline collocation method to be sixth order.

Problem 6: Consider the differential equation

$$u^{(4)} = f \text{ in } [0, 8]$$

subject to boundary conditions (5.1). The functions $f$ and $g_i, i = 1, \cdots, 4$, are determined so that the exact solution is

$$u = x^{(13/2)} \tag{5.3}$$

or

$$u = x^{(15/2)}. \tag{5.4}$$

We apply the one-step quartic-spline collocation method to Problem 6. The maximum absolute errors and the respective orders of convergences are presented in the tables below. The numerical results of Problem 6 below show that superconvergence holds for functions in $C^7(I)$ and for a function $u = x^{13/2}$, with discontinuous $u^{(7)}$ in $I$, the order of convergence is approximately 5.5 instead of 6. This means that the continuity condition $u \in C^7(I)$ set in the analysis is only sufficient and not necessary.

Table 5.14: Errors and corresponding orders of convergence for Problem 6 with exact solution (5.3).

| $N$ | $\|\|u - u_\Delta\|\|_{\tau_i,x_i,\infty}$ | | | $\|\|u' - u'_\Delta\|\|_{\tau_i,\lambda_i,\infty}$ | | | $\|\|u'' - u''_\Delta\|\|_{\tau_i,x_i,\infty}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| 8 | 7.3+00 | 6.0+00 | 8.1+00 | 3.5+01 | 1.0+01 | 4.1+01 | 1.8+01 | 2.8+01 | 2.3+02 |
| 16 | 1.6-01 | 1.5-01 | 2.0-01 | 2.3+00 | 5.7-01 | 2.7+00 | 1.2+00 | 1.6+00 | 3.0+01 |
| 32 | 3.9-03 | 3.6-03 | 6.4-03 | 1.4-01 | 3.2-02 | 1.6-01 | 8.3-02 | 9.8-02 | 3.9+00 |
| 64 | 8.9-05 | 8.5-05 | 2.0-04 | 9.3-03 | 1.9-03 | 1.0-02 | 5.2-03 | 6.0-03 | 4.8-01 |
| 128 | 4.9-06 | 4.9-06 | 8.2-06 | 5.8-04 | 1.1-04 | 6.6-04 | 3.2-04 | 3.8-04 | 6.0-02 |
| | | | | orders of convergence | | | | | |
| 16 | 5.44 | 5.31 | 5.27 | 3.94 | 4.25 | 3.94 | 3.83 | 4.16 | 2.94 |
| 32 | 5.42 | 5.37 | 5.01 | 3.97 | 4.12 | 4.01 | 3.95 | 4.03 | 2.97 |
| 64 | 5.45 | 5.41 | 5.01 | 3.98 | 4.08 | 3.99 | 3.98 | 4.01 | 3.00 |
| 128 | 4.16 | 4.12 | 4.60 | 3.99 | 4.04 | 3.98 | 4.00 | 3.99 | 3.00 |

| $N$ | $\|\|u^{(3)} - u^{(3)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | | $\|\|u^{(4)} - u^{(4)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | |
|---|---|---|---|---|---|---|---|---|
| 8 | 1.2+03 | 2.6+03 | 9.7+01 | 2.5+03 | 2.4+02 | 1.5+04 | 8.8+03 | 1.5+04 |
| 16 | 3.1+02 | 6.6+02 | 1.1+01 | 6.0+02 | 6.1+01 | 7.8+03 | 4.5+03 | 7.5+03 |
| 32 | 8.1+01 | 1.6+02 | 1.4+00 | 1.5+02 | 1.5+01 | 3.9+03 | 2.2+03 | 3.7+03 |
| 64 | 2.0+01 | 4.1+01 | 1.8-01 | 3.8+01 | 3.8+00 | 1.9+03 | 1.1+03 | 1.9+03 |
| 128 | 5.1+00 | 1.0+01 | 2.3-02 | 9.8+00 | 9.6-01 | 9.9+02 | 5.7+02 | 9.7+02 |
| | | | | orders of convergence | | | | |
| 16 | 1.92 | 2.00 | 3.02 | 2.07 | 1.98 | 0.97 | 0.97 | 1.00 |
| 32 | 1.96 | 1.99 | 2.99 | 1.99 | 1.98 | 0.98 | 0.98 | 1.00 |
| 64 | 1.98 | 2.00 | 2.99 | 1.96 | 1.99 | 0.99 | 0.99 | 0.97 |
| 128 | 1.99 | 2.00 | 3.00 | 1.98 | 1.99 | 0.99 | 0.99 | 0.98 |

Table 5.15: Errors and corresponding orders of convergence for Problem 6 with exact solution (5.4).

| $N$ | $\|\|u-u_\Delta\|\|_{\tau_i,x_i,\infty}$ | | | $\|\|u'-u'_\Delta\|\|_{\tau_i,\lambda_i,\infty}$ | | | $\|\|u''-u''_\Delta\|\|_{\tau_i,x_i,\infty}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| 8 | 1.4+02 | 1.9+02 | 2.5+02 | 9.3+02 | 1.9+02 | 9.3+02 | 6.8+02 | 1.1+03 | 5.4+03 |
| 16 | 2.7+00 | 3.5+00 | 5.6+00 | 5.6+01 | 1.3+01 | 6.2+01 | 4.9+01 | 6.4+01 | 7.2+02 |
| 32 | 4.9-02 | 6.0-02 | 1.5-01 | 3.5+00 | 8.0-01 | 4.0+00 | 3.2+00 | 3.9+00 | 9.3+01 |
| 64 | 8.2-04 | 9.9-04 | 4.8-03 | 2.2-01 | 4.7-02 | 2.4-01 | 2.0-01 | 2.4-01 | 1.1+01 |
| 128 | 1.6-05 | 1.4-05 | 1.4-04 | 1.4-02 | 2.8-03 | 1.5-02 | 1.3-02 | 1.5-02 | 1.4+00 |
| | | | | orders of convergence | | | | | |
| 16 | 5.72 | 5.81 | 5.47 | 4.04 | 3.88 | 3.90 | 3.76 | 4.10 | 2.90 |
| 32 | 5.81 | 5.86 | 5.19 | 3.99 | 4.02 | 3.96 | 3.92 | 4.03 | 2.96 |
| 64 | 5.90 | 5.92 | 5.00 | 3.98 | 4.08 | 4.01 | 3.97 | 4.00 | 3.00 |
| 128 | 5.63 | 6.09 | 5.00 | 3.98 | 4.06 | 3.96 | 3.99 | 4.00 | 3.00 |

| $N$ | $\|\|u^{(3)}-u^{(3)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | | $\|\|u^{(4)}-u^{(4)}_\Delta\|\|_{\tau_i,x_i,\lambda_i,\infty}$ | | | |
|---|---|---|---|---|---|---|---|---|
| 8 | 2.6+04 | 6.3+04 | 3.6+03 | 6.0+04 | 9.0+03 | 3.6+05 | 2.0+05 | 3.5+05 |
| 16 | 7.3+03 | 1.5+04 | 4.6+02 | 1.4+04 | 2.3+03 | 1.8+05 | 1.0+05 | 1.8+05 |
| 32 | 1.9+03 | 3.9+03 | 5.9+01 | 3.5+03 | 6.0+02 | 9.4+04 | 5.4+04 | 8.8+04 |
| 64 | 4.8+02 | 9.9+02 | 7.4+00 | 9.1+02 | 1.5+02 | 4.7+04 | 2.7+04 | 4.5+04 |
| 128 | 1.2+02 | 2.4+02 | 9.3-01 | 2.3+02 | 3.8+01 | 2.3+04 | 1.3+04 | 2.3+04 |
| | | | | orders of convergence | | | | |
| 16 | 1.87 | 1.99 | 2.96 | 2.06 | 1.92 | 0.96 | 0.94 | 0.98 |
| 32 | 1.94 | 1.99 | 2.98 | 2.03 | 1.96 | 0.98 | 0.97 | 1.02 |
| 64 | 1.97 | 1.99 | 2.99 | 1.94 | 1.98 | 0.99 | 0.98 | 0.95 |
| 128 | 1.98 | 2.00 | 2.99 | 1.97 | 1.99 | 0.99 | 0.99 | 0.97 |

# Chapter 6

# Conclusions

An optimal quartic-spline collocation method for solving one-dimensional fourth-order BVPs is presented in this thesis. In order to develop the method, we first define an appropriate quartic-spline interpolant and derive a number of asymptotic relations for the error of the interpolant and its derivatives on the midpoints and gridpoints of a uniform partition. Although these results are not used in the actual computation of the optimal quartic-spline collocation method, they are needed to explain the derivation of the method and in the analysis of error bounds and orders of convergence of the collocation approximation. Two formulations of the optimal quartic-spline collocation method are described: the one-step or extrapolated method, and the three-step or deferred-correction method. We study the properties of the matrices that arise from the two formulations. For a restricted class of BVPs, we derive analytic formulae for the eigenvalues and eigenvectors of the matrices arising from the deferred-correction method. Furthermore, a convergence analysis based on Green's functions shows that for the $j$th derivative of the quartic-spline collocation approximation, $j = 0, 1, 2, 3$, the orders of convergence are $O(h^6), O(h^4), O(h^4), O(h^2)$, respectively, on the midpoints and gridpoints of the uniform partition. Numerical results verify the convergence rates expected from theory, and show that the errors of our optimal quartic-spline collocation approximation are smaller than those from existing sub-optimal collocation methods based on quartic, quintic and sextic splines. While $C^{10}$ is shown in theory to be a sufficient continuity condition for our methods to yield the superconvergence results, it can be seen from the numerical results that superconvergence can hold for functions in $C^7$.

Even though only one-dimensional problems are considered in this thesis, the methods we have developed can be extended to higher dimensions, using a tensor product formulation. There is a lot of interest in solving fourth-order PDEs in applications from science and engineering. In particular, the biharmonic equation $\nabla^4 u \equiv u_{xxxx} + 2u_{xxyy} + u_{yyyy} = f(x,y)$ arises in plane elasticity. Equations of the form $u_{xxxx} + u_{yy} = 0$ must be solved when studying vibrations of a thin beam.

Once we move beyond one dimension, the systems that need to be solved become so large that fast solvers must be developed for them. For possible future research, FFT (Fast Fourier Transform) solvers can be developed for various types of boundary conditions and certain fourth-order PDE operators, since the eigenvalues and eigenvectors of some of the coefficient matrices arising from the discretization are given in Chapter 3. From the matrix analysis in Chapter 3 we can predict that an FFT solver for the biharmonic equation discretized by our quartic-spline collocation method will take no more floating-point operations than the respective solver for the Laplace operator discretized by the optimal quadratic-spline collocation method. Furthermore, iterative methods like multigrid solvers and preconditioned iterative solvers can be studied for the linear system arising from quartic-spline collocation applied to general fourth-order PDEs; FFT solvers can be used to solve the preconditioners. Another future problem could be to extend our quartic-spline collocation methods to solve systems of linear partial differential equations and to solve nonlinear scalar problems.

# Bibliography


[1] U. Ascher, J. Christiansen, and R. D. Russell. A collocation solver for mixed order systems of boundary value problems. *Mathematics of Computation*, 33(146):659–679, 1979.

[2] U. M. Ascher, R. M. Mattheij, and R. D. Russell. *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations.* SIAM, 1995.

[3] C. C. Christara. *Spline Collocation Methods, Software and Architectures for Linear Elliptic Partial Differential Equations.* PhD thesis, Department of Computer Science, Purdue University, IN, U.S.A., 1988.

[4] C. C. Christara. Quadratic spline collocation methods for elliptic partial differential equations. *BIT*, 34:33–61, 1994.

[5] A. Constas. Fast Fourier transform solvers for quadratic spline collocation. M.Sc. Thesis, Department of Computer Science, University of Toronto, Toronto, Ontario, Canada, July 1996.

[6] J. W. Daniel and B. K. Swartz. Extrapolated collocation for two-point boundary-value problems using cubic splines. *J. Inst. Maths Applics*, 16:161–174, 1975.

[7] C. de Boor. On uniform approximation by splines. *Journal of Approximation Theory*, 1:219–235, 1968.

[8] C. de Boor. *A Practical Guide to Splines.* Springer-Verlag, New York, 1978.

[9] C. de Boor and B. Swartz. Collocation at Gaussian points. *SIAM J. Numer. Anal.*, 10(4):582–606, 1973.

[10] F. Dubeau and J. Savoie. Periodic quartic splines with equi-spaced knots. *IMA Journal of Numerical Analysis*, 5:183–189, 1985.

[11] W. D. Hoskins and D. S. Meek. Linear dependence relations for polynomial splines at midknots. *BIT*, 15:272–276, 1975.

[12] E. N. Houstis, C. C. Christara, and J. R. Rice. Quadratic-spline collocation methods for two-point boundary value problems. *International Journal for Numerical Methods in Engineering*, 26:935–952, 1988.

[13] E. N. Houstis, E. A. Vavalis, and J. R. Rice. Convergence of an $O(h^4)$ cubic spline collocation method for elliptic partial differential equations. *SIAM J. Numer. Anal.*, 25(1):54–74, 1988.

[14] M. Irodotou-Ellina. *Spline collocation methods for high order elliptic boundary value problems*. PhD thesis, Department of Mathematics, Aristotle University of Thessaloniki, Greece, 1987.

[15] M. Irodotou-Ellina and E. N. Houstis. An $O(h^6)$ quintic spline collocation method for fourth order two-point boundary value problems. *BIT*, 28:288–301, 1988.

[16] M. Irodotou-Ellina and E. N. Houstis. An $O(h^6)$ quintic spline collocation method for fourth order two-point boundary value problems. Technical Report 618, Department of Computer Science, Purdue University, 1988.

[17] W. J. Kammerer, G. W. Reddien, and R. S. Varga. Quadratic interpolatory splines. *Numer. Math.*, 22:241–259, 1974.

[18] P. M. Prenter. *Splines and Variational Methods*. John Wiley & Sons, 1975.

[19] R. D. Russell and L. F. Shampine. A collocation method for boundary value problems. *Numer. Math.*, 19:1–28, 1971.

[20] R. A. Usmani and S. A. Warsi. Quintic spline solutions of boundary value problems. *Comp. & Maths. with Appls.*, 6:197–203, 1980.

[21] R. A. Usmani and S. A. Warsi. Smooth spline solutions for boundary value problems in plate deflection theory. *Comp. & Maths. with Appls.*, 6:205–211, 1980.

[22] Riaz A. Usmani. The use of quartic splines in the numerical solution of a fourth-order boundary value problem. *Journal of Computational and Applied Mathematics*, 44:187–199, 1992.