OPTION PRICING WITH REGIME SWITCHING CORRELATION: A NUMERICAL PDE APPROACH

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Abstract Modelling correlation between financial quantities is important in the accurate pricing of financial derivatives. In this paper, we introduce some stochasticity in correlation, by considering a regime-switching correlation model, in which the transition rates between regimes are given. We present a derivation of the associated Partial Differential Equation (PDE) problem. The problem involves a system of ℓ PDEs, where ℓ is the number of regimes. We formulate a finite difference method for the solution of the PDE system, and numerically demonstrate that it converges with second order. We study the effect of certain model parameters on the computed prices. We compare prices from this model, associated PDE and method with those from a stochastic correlation model, associated PDE and method in van Emmerich, 2006, Leung, 2017, Leung et al., 2016 and discuss advantages and disadvantages.

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1. Introduction

Correlation between financial quantities plays an important role in pricing financial derivatives. A lot of existing popular models assume that correlation either is constant, or exhibits some deterministic behaviour. However, market observations suggest that correlation is a more complicated process. In particular, empirical evidence suggests that there is a significant correlation risk premium (e.g. Driessen et al., 2009, Driessen et al., 2012, Buss and Vilkov, 2012). It is also observed that, during periods of financial crises, correlations between asset returns increase (e.g. Longin and Solnik, 2001, Chiang et al., 2007, Min and Hwang, 2012). These observations suggest that there is stochasticity in correlation.

There are several models that introduce stochasticity in correlation. In Teng et al., 2016, Driessen et al., 2012, van Emmerich, 2006, the dynamics of the correlation variable are modelled directly by a stochastic process. This is also the approach in Leung, 2017, Leung et al., 2016, where a related Partial Differential Equation (PDE), with the correlation as an extra variable, is derived and solved numerically.

In this paper, we consider correlation structures that are guided by regime switching, with given transition rates from one regime to another. We show a way to derive the related Partial Differential Equation (PDE) problems for pricing several types of financial derivatives, and solve them by accurate and efficient numerical methods. We also study the effect of certain model parameters to the prices. We present the PDE problems, the numerical solution, and comparison of the PDE results to Monte-Carlo simulations. We also make a comparison with results from the stochastic correlation PDE model in Leung, 2017, Leung et al., 2016.

The outline of the paper is as follows. In Section 2, we present the regime-switching correlation model and derive the associated PDE problem using an appropriate portfolio and the no-arbitrage principle. In Section 3, we formulate numerical methods for the PDE problem, based on finite differences, and discuss properties of the arising linear system. Section 4 has numerical results that demonstrate the convergence of the numerical solution and indicate the effect of certain problem parameters on it. In the same section, a comparison with the model in Leung, 2017, Leung et al., 2016 is discussed. Section 5 concludes the paper.

2. Modelling correlation

During financial crises and periods of market distress, correlation often increase, and/or exhibit a non-deterministic behaviour. A possible modelling approach for this type of variability of correlations is through regime switching. In the simplest case, there are two regimes, corresponding one to the "good" and another to "bad" times. In the general case, assume there are "states of the world" (regimes) modelled by a (continuous-time) Markov chain $X(t)$, taking values in the (finite) index set $\{1, 2, \ldots, \ell\}$, where each value corresponds to a regime. Let $q_{\alpha\beta} > 0$ be the transition rate the economy switches from regime α to β, for $\alpha \neq \beta$, and $q_{\alpha\alpha} = -\sum_{\beta=1,\beta\neq\alpha}^{\ell} q_{\alpha\beta}$. The transition rates are assumed to satisfy

$$P(X(t+h) = \beta \mid X(t) = \alpha) = q_{\alpha\beta}h + o(h), \text{ for } \alpha \neq \beta, \quad h > 0,$$

and

$$P(X(t+h) = \alpha \mid X(t) = \alpha) = 1 + q_{\alpha\alpha}h + o(h), \text{ for } h > 0.$$

For $\alpha \neq \beta$, we can view $q_{\alpha\beta}$ as the rate of a Poisson process in which a jump characterizes the transition from state α to state β. In this way, an element of randomness in the correlation structure is introduced.

Let $S_i(t)$, $i = 1, \ldots, n$, be values of respective (for simplicity non-dividend paying) risky assets evolving as

$$dS_i(t)/S_i(t) = \mu_i dt + \sigma_i dW_i(t), \ i = 1, 2, \ldots, n, \tag{2.1}$$

where

$$\mathcal{E}(dW_i(t), dW_j(t)) = \rho_{ij}(X(t))dt, i = 1, 2, \ldots, n, j = 1, 2, \ldots, n, \tag{2.2}$$

where ρ_{ij} denotes the correlation between assets i and j, μ_i and σ_i denote the drift and volatility of S_i, respectively, and $W_i(t)$ denotes a standard Wiener process. Note that, for each (i, j) pair, where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$, the correlation $\rho_{ij}(X(t))$ takes on ℓ values, one in each regime, depending on the value of $X(t)$. We wish to calculate the price $V(t, \mathbf{S}(t), X(t))$ of a (European) contingent claim on $\mathbf{S}(t) = [S_i(t), i = 1, \ldots, n]$. We assume V is Markov in $(\mathbf{S}(t), X(t))$.

Since $X(t)$ takes on ℓ discrete values, there are ℓ unknown price functions $V(t, \mathbf{S}, \alpha)$, $\alpha = 1, \ldots, \ell$, one for each regime. Let $\mathbf{V}(t, \mathbf{S})$ be the vector $[V(t, \mathbf{S}, \alpha), \alpha = 1, \ldots, \ell]$. We will derive a system of PDEs that $\mathbf{V}$ satisfies. To do this, we consider an appropriate portfolio of financial instruments that is riskless and, abiding with the no-arbitrage principle, equate its return to the return from an equivalent amount invested at the risk-free interest rate r.

Consider a portfolio of $\hat{\ell}$ financial instruments, and denote its value at time t by $\Pi(t)$. Let $V^{(l)}(t, \mathbf{S}(t), X(t))$ be the value of the l-th instrument, where $l = 1, 2, \ldots, \hat{\ell}$. We take $V^{(1)} = V$. The other $\hat{\ell} - 1$ instruments are auxiliary and help in the derivation of the system of PDEs. Let our holding of the l-th instrument be $a^{(l)}$, which may vary with time, and $a^{(1)} \equiv 1$. Assume also the existence of a money-market account $B(t)$ that pays an instantaneous short rate r. Mathematically, we have

$$\Pi(t, \mathbf{S}(t), X(t)) = \sum_{l=1}^{\hat{\ell}} a^{(l)} V^{(l)}(t, \mathbf{S}(t), X(t)) \tag{2.3}$$

$$\begin{aligned} dV^{(l)} = \Big(&\frac{\partial V^{(l)}}{\partial t} + \sum_{i,j=1}^{n} \sigma_i \sigma_j \rho_{ij}(X(t)) S_i(t) S_j(t) \frac{\partial^2 V^{(l)}}{\partial S_i S_j} \\ &+ \sum_{i=1}^{n} \mu_i S_i(t) \frac{\partial V^{(l)}}{\partial S_i}\Big) dt \\ &+ \sum_{i=1}^{n} \sigma_i S_i(t) \frac{\partial V^{(l)}}{\partial S_i} dW_i + \sum_{\beta=1}^{\ell} (V^{(l)}(t, \mathbf{S}(t), \beta) - V^{(l)}(t, \mathbf{S}(t), \alpha)) dX_{\alpha\beta}(t), \\ l = 1, \ldots, \hat{\ell}, \end{aligned} \tag{2.4}$$

$$dB(t) = rB(t)dt, \tag{2.5}$$

where $X(t) = \alpha$, and $X_{\alpha\beta}(t)$ is the Poisson process that corresponds to the switch from the α to the β regime. For notational convenience, we suppress the dependence of $V^{(l)}$ on $(t, \mathbf{S}(t), X(t))$. Equation (2.3) is the value equation of the portfolio, Equation (2.4) follows from Ito's lemma for jump processes and Equation (2.5) is the value equation for the money market account. We do not exclude the money market account B from the set of instruments to trade. Assume that we are always able to trade in such a way that (a) the portfolio is self-financing, and (b) instantaneous risks corresponding to $dX(t)$ and $dW_i(t)$ are eliminated. Denote

$$\Delta_{\alpha\beta} V^{(l)}(t, \mathbf{S}, \alpha) \equiv V^{(l)}(t, \mathbf{S}, \beta) - V^{(l)}(t, \mathbf{S}, \alpha),$$

and

$$\hat{\mathcal{L}} V^{(l)}(t, \mathbf{S}, \alpha) \equiv \sum_{i,j=1}^{n} \sigma_i \sigma_j \rho_{ij}(\alpha) S_i S_j \frac{\partial^2 V^{(l)}}{\partial S_i S_j} + \sum_{i=1}^{n} \mu_i S_i \frac{\partial V^{(l)}}{\partial S_i}.$$

The first condition (self-financing portfolio) requires that

$$d\Pi = \sum_{l=1}^{\hat{\ell}} a^{(l)} dV^{(l)},$$

and the second condition (elimination of risks) requires that

$$\sum_{l=1}^{\hat{\ell}} a^{(l)} \frac{\partial V^{(l)}}{\partial S_i} = 0, \text{ for each } i \text{ and} \tag{2.6}$$

$$\sum_{l=1}^{\hat{\ell}} a^{(l)} \Delta_{\alpha\beta} V^{(l)} = 0, \text{ for each } \alpha, \beta. \tag{2.7}$$

The no-arbitrage principle implies that Π should grow at risk-free rate r: $d\Pi = r\Pi dt$. In other words, for each α,

$$\sum_{l=1}^{\hat{\ell}} a^{(l)} (\frac{\partial V^{(l)}}{\partial t} + \hat{\mathcal{L}} V^{(l)}) = r(\sum_{l=1}^{\hat{\ell}} a^{(l)} V^{(l)}) \tag{2.8}$$

$$\iff \sum_{l=1}^{\hat{\ell}} a^{(l)} (\frac{\partial V^{(l)}}{\partial t} * \hat{\mathcal{L}} V^{(l)} - rV^{(l)}) = 0.$$

Notice that (2.6), (2.7) and (2.8) form a linear system in the portfolio weights $a^{(l)}$, with $a^{(1)} = 1$. We assume the market is such that there exist $\hat{\ell}$ instruments that these simultaneous equations are solvable. To have a non-trivial solution to the homogeneous system, the rows must be linearly dependent. Therefore, there exist ν, θ_i, $\xi_{\alpha\beta}$, not all zero, such that for each l and each α,

$$\nu(\frac{\partial V^{(l)}}{\partial t} + \hat{\mathcal{L}} V^{(l)} - rV^{(l)}) + \sum_{i=1}^{n} \theta_i \frac{\partial V^{(l)}}{\partial S_i} + \sum_{\beta=1}^{\ell} \xi_{\alpha\beta} \Delta_{\alpha\beta} V^{(l)} = 0.$$

If each S_i is tradeable, and if there exists an instrument, say the m-th, such that the matrix $[\Delta_{\alpha\beta} V^{(m)}]_{\alpha,\beta=1,\ldots,\ell}$ is non-singular, the coefficient ν of $\frac{\partial V^{(l)}}{\partial t} + \hat{\mathcal{L}} V^{(l)} - rV^{(l)}$ is non-zero. Without loss of generality, we assume $\nu = 1$. Thus,

$$\frac{\partial V^{(l)}}{\partial t} + \hat{\mathcal{L}} V^{(l)} - rV^{(l)} + \sum_{i=1}^{n} \theta_i \frac{\partial V^{(l)}}{\partial S_i} + \sum_{\beta=1}^{\ell} \xi_{\alpha\beta} \Delta_{\alpha\beta} V^{(l)} = 0.$$

Therefore, with $\tau = T - t$, where T the maturity time of the contingent claim, we arrive at the pricing equation

$$\frac{\partial V^{(l)}}{\partial \tau} = \sum_{i,j=1}^{n} \sigma_i \sigma_j \rho_{ij}(\alpha) S_i S_j \frac{\partial^2 V^{(l)}}{\partial S_i S_j} + \sum_{i=1}^{n} (\mu_i + \theta_i) S_i \frac{\partial V^{(l)}}{\partial S_i} \\ - r V^{(l)} + \sum_{\beta=1}^{\ell} \xi_{\alpha\beta} \Delta_{\alpha\beta} V^{(l)}, \tag{2.9}$$

for each l and each α, where $V^{(l)} = V^{(l)}(\tau, \mathbf{S}, \alpha)$, and where we recall that $\mathbf{S}$ is a vector of tradeable non-dividend paying asset prices S_j, and $V^{(1)} = V$. As (2.9) holds for the instrument with value $V = V^{(1)}$, the same equation in particular holds for each S_j (which are also tradeable instruments), so we have

$$0 = 0 + \sum_{i=1}^{n} (\mu_i + \theta_i) S_i \delta_{ij} - r S_j \iff \mu_j + \theta_j = r, \quad \forall j = 1, \ldots, n,$$

with $\delta_{ij} = 0$ for $i \neq j$, and $\delta_{ii} = 1$. In other words, this is same as saying that, in this regime-switching model, the price of the contingent claim does not depend on the asset drift. This is consistent with, for example, Elliott et al., 2007 in their work on regime-switching stochastic volatility processes.

Equation (2.9) holds for $V = V^{(1)}$, which gives the pricing equation

$$\frac{\partial V}{\partial \tau}(\tau, \mathbf{S}, \alpha) = \sum_{i,j=1}^{n} \sigma_i \sigma_j \rho_{ij}(\alpha) S_i S_j \frac{\partial^2 V}{\partial S_i S_j}(\tau, \mathbf{S}, \alpha) + \sum_{i=1}^{n} r S_i \frac{\partial V}{\partial S_i}(\tau, \mathbf{S}, \alpha) \\ - r V(\tau, \mathbf{S}, \alpha) + \sum_{\beta=1}^{k} \xi_{\alpha_\beta} \Delta_{\alpha\beta} V(\tau, \mathbf{S}, \alpha). \tag{2.10}$$

Thus, the price $V(\tau, \mathbf{S}, \alpha)$ is interconnected to all other prices $V(\tau, \mathbf{S}, \beta)$ through a system of ℓ PDEs.

On the $\mathbf{S}$-boundaries, the unknown prices $V(\tau, \mathbf{S}, \alpha)$, $\alpha = 1, \ldots, \ell$, satisfy boundary conditions determined by the type of financial derivatives priced. At $\tau = 0$ ($t = T$), for each α, the initial condition for the price $V(\tau, \mathbf{S}, \alpha)$ is the payoff function $g(\mathbf{S})$, which is given by the type of financial derivatives priced.

Equation (2.10) can be cast in vector form as follows. Recall that $\mathbf{V}(\tau, \mathbf{S})$ is the vector whose α-th entry is $V(\tau, \mathbf{S}, \alpha)$, where $\alpha = 1, 2, \ldots, \ell$, and similarly define the vector (of matrices) $\vec{\rho}_{ij} \equiv [\rho_{ij}(\alpha), \alpha = 1, \ldots, \ell]$. Denote $\mathbf{Q}$ to be the $\ell \times \ell$ matrix with matrix entry $\mathbf{Q}_{\alpha\beta} = \xi_{\alpha\beta}$ for $\alpha \neq \beta$, and $\mathbf{Q}_{\alpha\alpha} =$

$-\sum_{\beta=1,\beta\neq\alpha}^{\ell}\xi_{\alpha\beta}$. In vector form (2.10) becomes

$$\frac{\partial \mathbf{V}}{\partial \tau}(\tau,\mathbf{S}) = \sum_{i,j=1}^{n} \sigma_i\sigma_j\vec{\rho}_{ij}S_iS_j\frac{\partial^2 \mathbf{V}}{\partial S_iS_j}(\tau,\mathbf{S}) + \sum_{i=1}^{n} rS_i\frac{\partial \mathbf{V}}{\partial S_i}(\tau,\mathbf{S}) \quad (2.11)$$
$$-r\mathbf{V}(\tau,\mathbf{S}) + \mathbf{Q}\mathbf{V}(\tau,\mathbf{S}),$$

where the multiplication of $\vec{\rho}_{ij}$ with $\frac{\partial \mathbf{V}}{\partial S_iS_j}(\tau,\mathbf{S})$ is component-wise. In the above derivation, the real world transition rates of the underlying Markov chain do not appear in the final pricing equation. The generator matrix $\mathbf{Q}$ is a risk-neutral parameter. Alternatively, one could fix $\mathbf{Q}$ a priori and derive the pricing equation by considering its risk-neutral expectation as the solution to a PDE through the Feynman-Kac theorem. We also note that, we assumed non-dividend paying assets, only for simplicity. If S_i pay dividend rates $\hat{q}_i$, $i = 1,\ldots,n$, respectively, we just adjust the coefficient rS_i of $\frac{\partial \mathbf{V}}{\partial S_i}(\tau,\mathbf{S})$ in (2.11) to $(r-\hat{q}_i)S_i$.

Since, in the numerical experiments, we will be considering the case of two regimes and two-assets contingent claims, we present the system of two PDEs in two space variables in a simplified format. Let $\mathbf{S} = (S_1,S_2)$, $v(\tau,\mathbf{S}) = V(\tau,\mathbf{S},1)$ (regime 1) and $w(\tau,\mathbf{S}) = V(\tau,\mathbf{S},2)$ (regime 2). Let also the notation ρ_{ij} be adjusted so that $\rho_\alpha = \rho_{12}(\alpha) = \rho_{21}(\alpha)$ is the correlation between assets 1 and 2 in regime α, $\alpha = 1,2$. Then (2.11) becomes

$$\frac{\partial v}{\partial \tau} = \frac{\sigma_1^2S_1^2}{2}\frac{\partial^2 v}{\partial S_1^2} + \frac{\sigma_2^2S_2^2}{2}\frac{\partial^2 v}{\partial S_2^2} + \rho_1\sigma_1\sigma_2S_1S_2\frac{\partial^2 v}{\partial S_1\partial S_2} \quad (2.12)$$
$$+(r-\hat{q}_1)S_1\frac{\partial v}{\partial S_1} + (r-\hat{q}_2)S_2\frac{\partial v}{\partial S_2} - rv + q_{12}(w-v) = 0$$
$$\equiv \mathcal{L}(\rho_1)v + q_{12}(w-v) = 0$$

$$\frac{\partial w}{\partial \tau} = \frac{\sigma_1^2S_1^2}{2}\frac{\partial^2 w}{\partial S_1^2} + \frac{\sigma_2^2S_2^2}{2}\frac{\partial^2 w}{\partial S_2^2} + \rho_2\sigma_1\sigma_2S_1S_2\frac{\partial^2 w}{\partial S_1\partial S_2} \quad (2.13)$$
$$+(r-\hat{q}_1)S_1\frac{\partial w}{\partial S_1} + (r-\hat{q}_2)S_2\frac{\partial w}{\partial S_2} - rw + q_{21}(v-w) = 0$$
$$\equiv \mathcal{L}(\rho_2)w + q_{21}(v-w) = 0.$$

3. Numerical methods

In this section, we present the discretization of (2.12)-(2.13). We first truncate the semi-infinite space domain $[0,\infty)\times[0,\infty)$ to $[0,S_{1,\max}]\times[0,S_{2,\max}]$, for appropriately large $S_{1,\max}$ and $S_{2,\max}$. Then, discretize $[0,S_{1,\max}]\times[0,S_{2,\max}]$

using a rectangular partition with N and M subintervals in the S_1- and S_2-dimensions, respectively.

We use standard second-order finite differences for the space discretization of (2.12)-(2.13). For the time-stepping, we use the θ-method, which, for $\theta = \frac{1}{2}$ gives Crank-Nicolson (CN), and, for $\theta = 1$, gives Backward Euler (BE or fully implicit method). While it is easy to formulate the discretization with non-uniform space and/or time stepsizes, in the numerical experiments, we consider uniform stepsizes in both space and time. Also, for the numerical experiments, we set $\theta = \frac{1}{2}$ for all timesteps, except the first four, for which we set $\theta = 1$ and pick half time stepsize. Thus, we are using the typical CN-Rannacher timestepping Rannacher, 1984, which is known to smoothen out the propagation of the initial conditions discontinuity.

Let $A_{\mathcal{L}}(\rho)$ be the matrix arising from the (spatial) discretization of $\mathcal{L}(\rho)$. Then the matrix arising from the spatial discretization of the system of PDEs (2.12)-(2.13) is

$$A = \begin{bmatrix} A_{\mathcal{L}}(\rho_1) - q_{12}\mathbf{I} & q_{12}\mathbf{I} \\ q_{21}\mathbf{I} & A_{\mathcal{L}}(\rho_2) - q_{21}\mathbf{I} \end{bmatrix}, \tag{3.1}$$

where $\mathbf{I}$ is the $(N-1)(M-1) \times (N-1)(M-1)$ identity matrix. Then, with $\mathbf{I}_2$ being the $2(N-1)(M-1) \times 2(N-1)(M-1)$ identity matrix, and time stepsize $\Delta\tau$, the θ-timestepping at the kth timestep, $k = 1, \ldots, N_t$, for the system of PDEs (2.12)-(2.13) becomes

$$(\mathbf{I}_2 - \theta\Delta\tau A) \begin{bmatrix} v^k \\ w^k \end{bmatrix} = (\mathbf{I}_2 + (1-\theta)\Delta\tau A) \begin{bmatrix} v^{k-1} \\ w^{k-1} \end{bmatrix} + \mathbf{b} \tag{3.2}$$

with appropriate adjustment to the boundary equations, and with $\mathbf{b}$ a vector arising from the boundary conditions.

Note that the system of linear equations that needs to be solved at each timestep is of size $2(N-1)(M-1)$ (assuming appropriate Dirichlet conditions). The matrix A as presented in (3.1) (and therefore the matrix $\mathbf{I}_2 - \theta\Delta\tau A$ as well) has semi-bandwidth $(N-1)(M-1)$, where we have assumed that we first order the equations arising from the discretization of (2.12) (in natural ordering), then those from (2.13). We can easily reduce the semi-bandwidth to $2\min\{N-1, M-1\}$, by changing the ordering of the equations to alternating between those arising from (2.12) and (2.13). In this case, the linear system has bandwidth just twice as large as a linear system arising from the discretization of a scalar two-dimensional PDE.

In the general case of ℓ regimes, the system of linear equations that needs to be solved at each timestep is of size $\ell(N-1)(M-1)$, and the equations

$S_{i,\max}$	T	σ_1	σ_2	ρ_1	ρ_2	q_{12}	q_{21}	$\hat{q}_1$	$\hat{q}_2$
500	1	0.20	0.30	0.1	0.7			0	0

Table 4.1. Numerical and market parameters for exchange option; parameters q_{12} and q_{21} are set as indicated in each experiment

can be ordered so that the matrix A (and therefore the matrix $\mathbf{I}_\ell - \theta\Delta\tau A$ as well) has semi-bandwidth $\ell \min\{N-1, M-1\}$. Therefore, both the number of equations and the semi-bandwidth increase by a factor of ℓ compared to the constant correlation case. The bandwidth $\ell \min\{N-1, M-1\}$ is the smallest bandwidth that can be obtained for a two-asset problem with ℓ regimes, assuming a simple direct linear solver is used at each timestep. Clearly, other more efficient linear solvers can be used, and other timestepping techniques can be employed. For example, Alternating Direction Implicit (ADI) timestepping methods could result in solving linear systems of semi-bandwidth ℓ, independently of N or M.

4. Numerical results

In this section, we present numerical results that demonstrate that our numerical model converges in second order, and study numerically the effect of certain parameters to the price functions.

The first test, while financially uninteresting, is a typical numerical test of the accuracy and convergence of our numerical model. We consider an exchange option, which has payoff

$$g(S_1, S_2) = \max\{S_1(T) - S_2(T), 0\}.$$

For such an option, assuming constant correlation, the exact price is known and given in analytic form by Margrabe's formula Margrabe, 1978. For our regime-switching PDE model (2.12)-(2.13) and its numerical solution, we pick the parameter values given in Table 4.1, and we intentionally set $q_{12} = 0$, which means that we are always in regime 1, and, therefore, we have a "degenerate" regime switching case with constant correlation ρ_1. We run our numerical method (3.2) for this problem and compare our results with Margrabe's formula (using ρ_1 as correlation), by calculating the exact error and respective order of convergence. The results are shown in Table 4.2. We notice a straight second order convergence for the price and the Greeks. Similar orders have been obtained on other points of the domain.

$N=M$	N_t	V	error	order	$\frac{\partial V}{\partial S_2}$	error	order	$\frac{\partial^2 V}{\partial S_2^2}$	error	order
50	25	18.4115	1.42e-01		-5.5669e-01	3.02e-03		1.2936e-02	-1.49e-04	
100	50	18.5182	3.51e-02	2.02	-5.5442e-01	7.42e-04	2.02	1.2823e-02	-3.61e-05	2.04
200	100	18.5445	8.74e-03	2.00	-5.5386e-01	1.85e-04	2.01	1.2796e-02	-8.97e-06	2.01
400	200	18.5510	2.18e-03	2.00	-5.5372e-01	4.62e-05	2.00	1.2789e-02	-2.24e-06	2.00
800	400	18.5527	5.46e-04	2.00	-5.5369e-01	1.15e-05	2.00	1.2788e-02	-5.60e-07	2.00

Table 4.2. Values, errors and orders of convergence for a European exchange option under a degenerate "two-regime" correlation model with the parameters in Table 4.1, and $q_{12} = 0$, at point (100, 90). Margrabe's formula gives price 18.5532.

Next, we consider again an exchange option, with the same parameter values given in Table 4.1, and with $q_{12} = 0.3$ and $q_{21} = 1.5$. In this test, we have a two-regime correlation problem and do not have exact solution values, so we approximate the error of our numerical method, by the change of the values obtained by a refinement (doubling) of the grid. The order of convergence is calculated based on the approximate error. The results are shown in Table 4.3, where we again notice a stable second order of convergence for the price and the Greeks. We also note that once we use at least 400 spatial points and 200 timesteps, there is good agreement of the PDE results with the Monte Carlo (MC) simulation results, where we have used 50000 simulations with a timestep of 10^{-3}.

We also consider an exchange option with the same parameters as in Table 4.1, but now q_{12} varies as shown in Table 4.4, and $q_{21} = 1.5$. We only show the results with $N = M = 800$ and $N_t = 400$. We notice that the value at a point decreases as q_{12} increases. This is financially expected, as increasing transition rate from regime 1 to 2 means that the likelihood of the correlation going from $\rho_1 = 0.0$ to $\rho_2 = 0.7$ is higher, and therefore the two assets are likely to be higher correlated, which implies a drop in the exchange option value, as the payoff depends on the difference $S_1(T) - S_2(T)$.

We next consider a basket put option, which has payoff

$$g(S_1, S_2) = \max\{K - (S_1(T) + S_2(T)), 0\},$$

with K the strike price. Since the payoff of a basket put involves the sum $S_1(T) + S_2(T)$ and not the difference $S_1(T) - S_2(T)$ as in the case of the exchange option, we expect that an increase of correlation between the two assets would imply an increase in the value of the option. In Table 4.6, we present results of the application of our numerical method to a European dividend-paying basket put with the parameters of Table 4.5, $q_{21} = 1.5$, and with q_{12} varying as shown in Table 4.5. We notice that as q_{12} increases, which means

$N=M$	N_t	V	change	order	$\frac{\partial V}{\partial S_2}$	change	order	$\frac{\partial^2 V}{\partial S_2^2}$	change	order
50	25	18.0423			-5.6336e-01			1.3379e-02		
100	50	18.1522	1.10e-01		-5.6100e-01	2.36e-03		1.3256e-02	-1.22e-04	
200	100	18.1793	2.71e-02	2.02	-5.6042e-01	5.78e-04	2.03	1.3227e-02	-2.95e-05	2.05
400	200	18.1861	6.75e-03	2.00	-5.6028e-01	1.44e-04	2.01	1.3219e-02	-7.31e-06	2.01
800	400	18.1878	1.69e-03	2.00	-5.6024e-01	3.59e-05	2.00	1.3218e-02	-1.82e-06	2.00

Table 4.3. Values, changes and orders of convergence for a European exchange option in regime 1, under a two-regime correlation model with the parameters in Table 4.1, $q_{12} = 0.3$ and $q_{21} = 1.5$, at point (100, 90). Monte Carlo price is 18.1872, and the 95% confidence interval is (18.1798, 18.1945).

(S_1, S_2)	$q_{12}=0$	$q_{12}=0.1$	$q_{12}=0.2$	$q_{12}=0.3$
(90, 100)	8.5528	8.4252	8.3037	8.1879
(100, 100)	13.6365	13.4939	13.3580	13.2285
(110, 100)	19.8601	19.7175	19.5817	19.4523

Table 4.4. Values of a European exchange option in regime 1, under a two-regime correlation model with the parameters in Table 4.1, with $q_{21} = 1.5$, and q_{12} varying as indicated.

$S_{i,\max}$	T	K	r	σ_1	σ_2	ρ_1	ρ_2	q_{12}	q_{21}	$\hat{q}_1$	$\hat{q}_2$
500	1	200	0.05	0.20	0.30	0.1	0.7		1.5	0.03	0.02

Table 4.5. Numerical and market parameters for basket put option; parameter q_{12} is set as indicated in each experiment

that the likelihood of passing from regime 1 with $\rho_1 = 0.1$ to regime 2 with $\rho_2 = 0.7$ increases, the value of the option also increases, as expected.

In the next experiment, we study the effect of maturity on the prices of options either under regime switching or under a constant correlation. We consider an exchange option with parameters as in Table 4.1, except that T

(S_1, S_2)	$q_{12}=0$	$q_{12}=0.1$	$q_{12}=0.2$	$q_{12}=0.3$
(90, 100)	17.0331	17.1327	17.2272	17.3169
(100, 100)	12.2263	12.3313	12.4309	12.5255
(110, 100)	8.4790	8.5815	8.6788	8.7712

Table 4.6. Values of a European basket put option in regime 1, under a two-regime correlation model with the parameters in Table 4.5, and q_{12} varying as indicated.

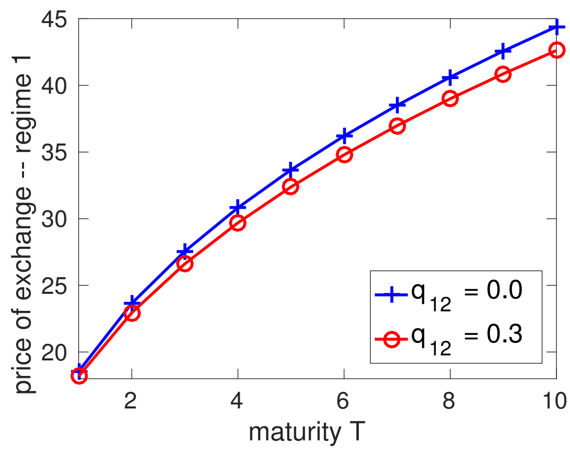

Figure 4.1. Plot of European exchange option price at (100, 90) versus T with a constant correlation and the two-regime correlation model (2.12)-(2.13), and parameters in Table 4.1, with $q_{21} = 1.5$, while T and q_{12} are as indicated.

varies from 1 to 10 years. In Figure 4.1, we plot the value of the exchange option versus the maturity T, with $q_{12} = 0$ (constant correlation, no regime switching), and with $q_{12} = 0.3$ (regime switching). We notice that the values increase with maturity, as expected. Furthermore, the difference between the no regime and the regime switching values also increases with T. These results highlight the importance of appropriate modelling of correlation in pricing long maturity financial derivatives.

We finally do a comparison between the results from the stochastic correlation model in Leung, 2017, Leung et al., 2016, and the regime-switching correlation model in this paper. The model in Leung, 2017, Leung et al., 2016 assumes that correlation is a stochastic variable with

$$d\rho(t) = \hat{\alpha}(t, \cdot)dt + \hat{\beta}(t, \cdot)dW_\rho(t), \tag{4.1}$$

where W_ρ is independent of W_i, $i = 1, 2, \ldots, n$. The model for $\hat{\alpha}$ and $\hat{\beta}$ considered in Leung, 2017, Leung et al., 2016 is a mean-reverting model, with the risk-neutral specification

$$\hat{\alpha}(\rho(t)) = \kappa(\eta - \rho(t)),\ \hat{\beta}(\rho(t)) = \sigma_\rho\sqrt{1 - \rho(t)^2}, \tag{4.2}$$

where $\kappa > 0$ is the mean reversion rate of correlation and $\eta \in (-1, 1)$ is the mean reversion level of correlation. For the case of two assets with prices S_1 and S_2, evolving as (2.1)-(2.2), the PDE that the price $V = V(\tau, \mathbf{S}, \rho)$ of a

$S_{i,\max}$	T	σ_1	σ_2	η	κ	σ_ρ
500	1	0.20	0.30	0.1	1.2	1.0

Table 4.7. Numerical and market parameters for exchange option under stochastic correlation model (4.1)-(4.2)

European contingent claim on $\mathbf{S} = (S_1, S_2)$ with the stochastic correlation model (4.1)-(4.2) satisfies, as derived in Leung, 2017, Leung et al., 2016, is

$$\begin{aligned}\frac{\partial V}{\partial \tau} \quad = \quad & \frac{\sigma_1^2 S_1}{2}\frac{\partial^2 V}{\partial S_2^2} + \frac{\sigma_2^2 S_2}{2}\frac{\partial^2 V}{\partial S_2^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 S_2} + \frac{\hat{\beta}^2}{2}\frac{\partial^2 V}{\partial \rho^2} \\ & + (r-\hat{q}_1)S_1\frac{\partial V}{\partial S_1} + (r-\hat{q}_2)S_2\frac{\partial V}{\partial S_2} + \hat{\alpha}\frac{\partial V}{\partial \rho} - rV,\end{aligned} \tag{4.3}$$

where, now, $\rho \in [-1, 1]$ is an extra variable. Clearly, the stochastic correlation PDE model (4.1)-(4.2)-(4.3) is different from the two-regime correlation PDE model (2.12)-(2.13). However, in Figures 4.2 and 4.3, we plot the prices of European exchange options versus S_2 for $S_1 = 100$, computed with the two-regime correlation model in this paper, and with the stochastic correlation model in Leung, 2017, Leung et al., 2016, respectively. For the two-regime correlation model we set the parameters as in Table 4.1, with $q_{12} = 0.03$ and $q_{21} = 1.3$, and plot the prices in both regimes ($\rho_1 = 0.1$ and $\rho_2 = 0.7$). For the stochastic correlation model we set the parameters as in Table 4.7, and plot the prices for two values of ρ, namely $\rho = 0.1$ and $\rho = 0.7$. At least qualitatively speaking, the two Figures indicate considerable agreement in the general behaviour of the prices. Clearly, it is important to note that, while the stochastic correlation PDE model (4.1)-(4.2)-(4.3) gives solutions that can be evaluated at any ρ, we plotted the prices that correspond to $\rho = \rho_1$ and $\rho = \rho_2$. Finally, it is worth emphasizing that, in this experiment, the mean reversion level η of the stochastic correlation PDE model is set to the "primary" correlation value ρ_1 of the two-regime model.

5. Conclusions

We have developed a regime-switching correlation PDE model for valuing European options on multiple assets. The transition rates between regimes are assumed to be given. The model involves a system of PDEs, as many as the number of regimes, with the component functions interconnected to each other through the transition rate matrix applied to the no-derivative term. We formulated a standard second-order finite differences method for its solution. We

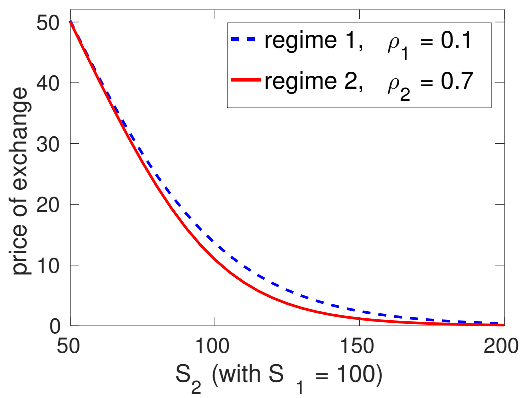

Figure 4.2. Plot of European exchange option price versus S_2 with two-regime correlation model (2.12)-(2.13), and parameters in Table 4.1, $q_{12} = 0.03$ and $q_{21} = 1.3$

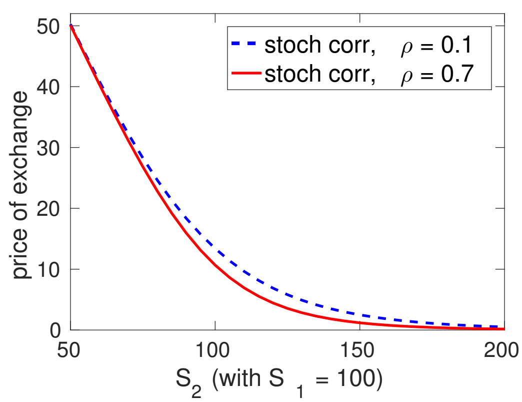

Figure 4.3. Plot of European exchange option price versus S_2 with stochastic correlation model (4.1)-(4.2)-(4.3) Leung, 2017, Leung et al., 2016, and parameters in Table 4.7

tested the model and numerical method on certain options on two assets, under a two-regime setting. We numerically verified second order convergence for the solution and its derivatives (Greeks), as well as good agreement with MC simulations. We studied the effect of the transition rates on the computed prices and showed that the model and method respond faithfully to certain financial facts. We also compared the results from the two-regime correlation model with those from a full stochastic correlation model in Leung, 2017, Leung et al., 2016, and noted interesting similarities. The advantage of the regime-switching correlation model in this paper compared to the model in Leung, 2017, Leung et al., 2016 is that it gives rise to smaller size and bandwidth linear systems to be solved at each timestep, since it does not introduce another dimension, but only increases the number of equations and bandwidth by a factor equal to the number of regimes. On the other hand, the advantage of the model in Leung, 2017, Leung et al., 2016 is that it can incorporate more flexible correlation behaviours and allow the prices to be evaluated at any correlation level in $[-1, 1]$, at the expense of introducing extra dimensions and, therefore, considerably increasing the bandwidth and size of linear systems to be solved at each timestep.

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