Quartic spline collocation for second-order boundary value problems

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Abstract— **Collocation methods based on quartic splines are presented for second-order two-point boundary value problems. In order to obtain a uniquely solvable linear system for the degrees of freedom of the quartic spline collocation approximation, in addition to the boundary conditions specified by the problem, extra boundary or near-boundary conditions are introduced. Non-optimal (fourth-order) and optimal (sixth-order) quartic-spline methods are considered. The theoretical behavior of the collocation methods is verified by numerical experiments. The extension of the methods to two-dimensional problems is briefly considered.**

Keywords— **sixth order convergence, quartic splines, second-order BVP, collocation**

I. Introduction

Collocation is a simple discretization methodology for boundary value problems (BVPs), which requires no integration.The most widely used collocation for BVPs is orthogonal piecewise polynomial C^1 collocation at the Gauss points [2]. Splines are piecewise polynomials of degree k and continuity $k-1$ on the nodes of the domain partition. Spline collocation has been shown to be an effective alternative to spline Galerkin or orthogonal collocation. Spline collocation uses only one data point per subinterval of the partition, thus has an advantage over collocation at Gauss points in terms of the size and the complexity of the arising linear systems. When spline collocation is considered for the discretization of second-order BVPs, the typical choice of splines is either quadratic or cubic, while, for fourth-order problems, either quartic or quintic. In the case of quadratic [7] or cubic [5], [1] spline collocation and second-order problems, the computed approximations exhibit up to fourth order convergence, while in the case of quartic [12] or quintic [8] and fourth-order problems, sixth order can be achieved.

In an effort to obtain sixth order convergence with a spline collocation approximation for a second-order problem, we use quartic splines. However, since quartic splines have more degrees of freedom than quadratic and cubic splines for the same partition, the choice of collocation points and equations requires extra care. In developing a quintic spline collocation method for second-order two-point BVPs, Irodotou-Ellina, Houstis and Kim [9] face a similar problem and present a way to overcome it.

Collocation methods based on quartic splines are presented for second-order two-point boundary value problems. In order to obtain a uniquely solvable linear system for the degrees of freedom of the quartic spline collocation approximation, in addition to the boundary conditions specified by the problem, extra boundary or near-boundary conditions are introduced. The straightforward way of forming the quartic spline collocation equations is by satisfying the differential equation, boundary and extra boundary conditions exactly on appropriate sets of collocation points.

The approximations obtained by these equations are fourth order, that is, non-optimal. Two optimal methods, namely the extrapolated (one-step) and the deferred-correction (two-step) methods, are formulated based on appropriate extra boundary conditions and an appropriate perturbation of the operators of the differential equation, boundary conditions and extra boundary conditions. The analysis shows that the maximum discrete error on the nodes and midpoints of a uniform partition is of sixth order, and the maximum global error is of fifth order for the optimal methods. The theoretical behavior of the collocation methods is verified by numerical experiments. The extension of the methods to two-dimensional problems is considered.

II. Problem and discretization

We consider the numerical solution of linear second-order two-point boundary value problems (BVPs), which consist of the operator equation

$$Lu \equiv p(x)u''(x)+q(x)u'(x)+r(x)u(x)=f(x),\ x \in (a,b) \quad (1)$$

subject to boundary conditions

$$Bu \equiv \{ \begin{array}{lcl} B_1 u &=& \alpha_1 u(a) + \beta_1 u'(a) = g_1, \\ B_2 u &=& \alpha_2 u(b) + \beta_2 u'(b) = g_2 \ \}, \end{array} \quad (2)$$

where $u(x)$ is the unknown function, $p(x), q(x), r(x)$ and $f(x)$ are given functions, and α_1, α_2, β_1, β_2, g_1 and g_2 are given scalars. The methods presented can easily be extended to non-separable boundary conditions. For later convenience, extend the notation of the scalars g_1 and g_2 by defining $g(x_0) \equiv g_1$ and $g(x_N) \equiv g_2$.

Let Δ be a partition of the interval $[a,b]$ into subintervals using uniform grid points $x_i = a + i \times h$, where $h = \frac{b-a}{N}$, $i = 0, \cdots, N$. Consider also the set of collocation points

$$T = \{\tau_0 = x_0, \tau_i = \frac{x_{i-1}+x_i}{2}, i = 1, \cdots, N, \tau_{N+1} = x_N\}. \quad (3)$$

Let S^4_Δ be the space of quartic splines with respect to Δ with $C^3[a,b]$ smoothness. Let $\phi_i(x), i = -1, 0, 1, \cdots, N, N+1, N+2$, be the standard basis functions for quartic B-splines. Any quartic spline $S \in S^4_\Delta$ can

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be written as $S(x)=\sum_{i=-1}^{N+2} c_i\phi_i(x)$,for some coefficients (degrees of freedom − DOFs) $c_i, i=-1,\cdots,N+2$. We are seeking an approximation $u_\Delta \in S^4_\Delta$ to the solution u of BVP (1)-(2).

III. Quartic spline interpolation

In this section, we use the definition of a particular quartic spline interpolant from [12], summarize some results shown in [12], and present some more results directly derived from [12]. We denote $u(\tau_i), i=0,\cdots,N+1$ by u_i, and the kth derivative $D^k u(x)$ of $u(x)$ by $u^{(k)}$. For example, $u^{(2)} \equiv u''$. We extend these notations to other functions as well. Let $S \in S^4_\Delta$ satisfy

$$S_i = u_i, \qquad i=1,\cdots,N, \tag{4}$$

$$S_i^{(4)} = u_i^{(4)} - \frac{h^2}{24}u_i^{(6)} + \frac{7h^4}{5760}u_i^{(8)}, i=1,2,N-1,N. \tag{5}$$

A. Quartic spline interpolant error

From [12], if $u \in C^{10}[a,b]$, we have, for $i=1,\cdots,N$,

$$S_i^{(1)} = u_i^{(1)} - \frac{7h^4}{5760}u_i^{(5)} + O(h^6), \tag{6}$$

$$S_i^{(2)} = u_i^{(2)} + \frac{7h^4}{1920}u_i^{(6)} + O(h^6), \tag{7}$$

and, for $i=0,\cdots,N$,

$$S(x_i) = u(x_i) + O(h^6), \tag{8}$$

$$S^{(1)}(x_i) = u^{(1)}(x_i) + \frac{h^4}{720}u^{(5)}(x_i) + O(h^6), \tag{9}$$

$$S^{(2)}(x_i) = u^{(2)}(x_i) - \frac{h^4}{240}u^{(6)}(x_i) + O(h^6). \tag{10}$$

Further, we define the discrete difference operator δ^2 by

$$\delta^2\gamma_i \equiv \gamma_{i-2} - 4\gamma_{i-1} + 6\gamma_i - 4\gamma_{i+1} + \gamma_{i+2}, i=3,\cdots,N-2.$$

In [12], if $u \in C^{10}[a,b]$, from (6)-(10), the following relations are shown for $i=3,\cdots,N-2$:

$$u_i^{(5)} = \frac{\delta^2 S_i^{(1)}}{h^4} + O(h^2), \tag{11}$$

$$u_i^{(6)} = \frac{\delta^2 S_i^{(2)}}{h^4} + O(h^2). \tag{12}$$

If $u \in C^{10}[a,b]$, then the following approximations to $u^{(5)}$ and $u^{(6)}$ are justified in [12] at boundary and near-boundary points $\{x_0, \tau_1, \tau_2, \tau_{N-1}, \tau_N, x_N\}$ for $k=5,6$

$$u^{(k)}(x_0) = \frac{1}{2h^4}(7\delta^2 S_3^{(k-4)} - 5\delta^2 S_4^{(k-4)}) + O(h^2), \tag{13}$$

$$u^{(k)}(\tau_1) = \frac{1}{h^4}(3\delta^2 S_3^{(k-4)} - 2\delta^2 S_4^{(k-4)}) + O(h^2), \tag{14}$$

$$u^{(k)}(\tau_2) = \frac{1}{h^4}(2\delta^2 S_3^{(k-4)} - \delta^2 S_4^{(k-4)}) + O(h^2), \tag{15}$$

$$u^{(k)}(\tau_{N-1}) = \frac{1}{h^4}(2\delta^2 S_{N-2}^{(k-4)} - \delta^2 S_{N-3}^{(k-4)}) + O(h^2), \tag{16}$$

$$u^{(k)}(\tau_N) = \frac{1}{h^4}(3\delta^2 S_{N-2}^{(k-4)} - 2\delta^2 S_{N-3}^{(k-4)}) + O(h^2), \tag{17}$$

$$u^{(k)}(x_N) = \frac{1}{2h^4}(7\delta^2 S_{N-2}^{(k-4)} - 5\delta^2 S_{N-3}^{(k-4)}) + O(h^2). \tag{18}$$

In a similar way, if $u \in C^{10}[a,b]$, we can show, for $k=5,6$

$$u^{(k)}(x_1) = \frac{1}{2h^4}(5\delta^2 S_3^{(k-4)} - 3\delta^2 S_4^{(k-4)}) + O(h^2) \tag{19}$$

$$u^{(k)}(x_{N-1}) = \frac{1}{2h^4}(5\delta^2 S_{N-2}^{(k-4)} - 3\delta^2 S_{N-3}^{(k-4)}) + O(h^2) \tag{20}$$

For the global error bounds, we have Theorem 1 [12].

Theorem 1: Let S be the quartic spline interpolant of $u \in C^{10}[a,b]$ defined by (4)-(5). We have the global error bounds

$$\| S^{(k)} - u^{(k)} \|_\infty = O(h^{5-k}),$$

for $k=0,\cdots,4$.

B. Quartic spline interpolant residual

By (6)-(10), the interpolant S satisfies

$$LS_i = f_i + \frac{7h^4}{1920}p_i u_i^{(6)} - \frac{7h^4}{5760}q_i u_i^{(5)} + O(h^6), \quad i=1,\cdots,N, \tag{21}$$

and

$$B_1 S(x_0) = g_1 + \frac{h^4}{720}\beta_1 u^{(5)}(x_0) + O(h^6), \tag{22}$$

$$B_2 S(x_N) = g_2 + \frac{h^4}{720}\beta_2 u^{(5)}(x_N) + O(h^6). \tag{23}$$

By (8)-(10), S also satisfies

$$\begin{aligned} LS(x_i) &= f(x_i) + \frac{h^4}{240}p(x_i)u^{(6)}(x_i) - \frac{h^4}{720}q(x_i)u^{(5)}(x_i) \\ &+ O(h^6), i=0,\cdots,N, \end{aligned} \tag{24}$$

The approximation of derivatives stated in (11)-(18) allow (21) and (22)-(23) to be written as

$$\begin{aligned} LS_1 &- \frac{7}{1920}p_1(3\delta^2 S_3^{(2)} - 2\delta^2 S_4^{(2)}) + \frac{7}{5760}q_1(3\delta^2 S_3^{(1)} - 2\delta^2 S_4^{(1)}) \\ &= f_1 + O(h^6), \end{aligned} \tag{25}$$

$$\begin{aligned} LS_2 &- \frac{7}{1920}p_2(2\delta^2 S_3^{(2)} - \delta^2 S_4^{(2)}) + \frac{7}{5760}q_2(2\delta^2 S_3^{(1)} - \delta^2 S_4^{(1)}) \\ &= f_2 + O(h^6), \end{aligned} \tag{26}$$

$$\begin{aligned} LS_i &- \frac{7}{1920}p_i\delta^2 S_i^{(2)} + \frac{7}{5760}q_i\delta^2 S_i^{(1)} \\ &= f_i + O(h^6), \quad i=3,\cdots,N-2, \end{aligned} \tag{27}$$

$$\begin{aligned} LS_{N-1} &- \frac{7}{1920}p_{N-1}(2\delta^2 S_{N-2}^{(2)} - \delta^2 S_{N-3}^{(2)}) \\ &+ \frac{7}{5760}q_{N-1}(2\delta^2 S_{N-2}^{(1)} - \delta^2 S_{N-3}^{(1)}) \\ &= f_{N-1} + O(h^6), \end{aligned} \tag{28}$$

$$\begin{aligned} LS_N &- \frac{7}{1920}p_N(3\delta^2 S_{N-2}^{(2)} - 2\delta^2 S_{N-3}^{(2)}) \\ &+ \frac{7}{5760}q_N(3\delta^2 S_{N-2}^{(1)} - 2\delta^2 S_{N-3}^{(1)}) \\ &= f_N + O(h^6), \end{aligned} \tag{29}$$

and

$$B_1 S(x_0) - \frac{1}{1440}\beta_1(7\delta^2 S_3^{(1)} - 5\delta^2 S_4^{(1)}) = g_1 + O(h^6), \tag{30}$$

$$B_2 S(x_N) - \frac{1}{1440}\beta_2(7\delta^2 S_{N-2}^{(1)} - 5\delta^2 S_{N-3}^{(1)}) = g_2 + O(h^6). \tag{31}$$

From (19)-(20), we can write (24) for x_1 and x_{N-1} as

$$\begin{aligned}
LS(x_1) &+ \frac{1}{480}p(x_1)(5\delta^2 S_3^{(2)} - 3\delta^2 S_4^{(2)}) \\
&- \frac{1}{1440}q(x_1)(5\delta^2 S_3^{(1)} - 3\delta^2 S_4^{(1)}) \\
&= f(x_1) + O(h^6), \qquad (32) \\
LS(x_{N-1}) &+ \frac{1}{480}p(x_{N-1})(5\delta^2 S_{N-2}^{(2)} - 3\delta^2 S_{N-3}^{(2)}) \\
&- \frac{1}{1440}q(x_{N-1})(5\delta^2 S_{N-2}^{(1)} - 3\delta^2 S_{N-3}^{(1)}) \\
&= f(x_{N-1}) + O(h^6), \qquad (33)
\end{aligned}$$

and for x_0 and x_N as

$$\begin{aligned}
LS(x_0) &+ \frac{1}{480}p(x_0)(7\delta^2 S_3^{(2)} - 5\delta^2 S_4^{(2)}) \\
&- \frac{1}{1440}q(x_0)(7\delta^2 S_3^{(1)} - 5\delta^2 S_4^{(1)}) \\
&= f(x_0) + O(h^6), \qquad (34) \\
LS(x_N) &+ \frac{1}{480}p(x_N)(7\delta^2 S_{N-2}^{(2)} - 5\delta^2 S_{N-3}^{(2)}) \\
&- \frac{1}{1440}q(x_N)(7\delta^2 S_{N-2}^{(1)} - 5\delta^2 S_{N-3}^{(1)}) \\
&= f(x_N) + O(h^6). \qquad (35)
\end{aligned}$$

Denote by L^* the discrete operator defined by the left side of equations (25)-(29) and (32)-(35). Similarly, denote by B^* the discrete operator defined by the left side of equations (30)-(31). The operators L^* and B^* can be viewed as perturbed operators L and B, respectively, at the associated points. That is, $L^* = L + P_L$ and $B^* = B + P_B$, where P_L and P_B the perturbation terms arising from (25)-(35). Then, we have following lemma.

Lemma 1: Let S be the quartic spline interpolant of the solution $u \in C^{10}[a,b]$ to (1)-(2), defined by (4)-(5). Then S satisfies the relations

$$\begin{aligned}
LS(\tau_i) &= f(\tau_i) + O(h^4), \quad i = 1, \cdots, N, \\
BS(x_i) &= g(x_i) + O(h^4), \quad i = 0, N, \qquad (36) \\
LS(x_i) &= f(x_i) + O(h^4), \quad i = 0, 1, N-1, N,
\end{aligned}$$

$$\begin{aligned}
L^*S(\tau_i) &= f(\tau_i) + O(h^6), \quad i = 1, \cdots, N, \\
B^*S(x_i) &= g(x_i) + O(h^6), \quad i = 0, N, \qquad (37) \\
L^*S(x_i) &= f(x_i) + O(h^6), \quad i = 0, 1, N-1, N.
\end{aligned}$$

Note that, since $L^* = L + P_L$ and $B^* = B + P_B$, relations (37) can be equivalently written as

$$\begin{aligned}
LS(\tau_i) &= f(\tau_i) - P_L S(\tau_i) + O(h^6), \ i = 1, \cdots, N, \\
BS(x_i) &= g(x_i) - P_B S(x_i) + O(h^6), \ i = 0, N, \qquad (38) \\
LS(x_i) &= f(x_i) - P_L S(x_i) + O(h^6), \ i = 0, 1, N-1, N.
\end{aligned}$$

IV. Quartic spline collocation

Let $u_\Delta(x) = \sum_{i=-1}^{N+2} c_i \phi_i(x)$ be a quartic spline collocation approximation to $u(x)$. It is clear that $N+4$ linearly independent conditions are required in order to uniquely determine a quartic spline. The straightforward formulation of collocation applies the operator and boundary operator equations (1)-(2) at a certain prescribed set of collocation points. By applying collocation to (1) and (2) at the points of T, we have $N+2$ linearly independent conditions. In order to uniquely define the quartic spline $u_\Delta(x)$, another two conditions are required, referred to as *extra boundary* or *near-boundary conditions.* The choice of the extra conditions and of the points at which they are applied is critical in formulating any quartic spline collocation method and, in particular, an optimal quartic spline collocation method.

Let E_1 and E_2 be the operators of the extra conditions, and let t_1 and t_2 be the collocation points, where these extra conditions are applied. A set of $N+4$ collocation conditions is then

$$\begin{aligned}
Lu_\Delta(\tau_i) &= f(\tau_i), \quad i = 1, \cdots, N, \\
Bu_\Delta(x_i) &= g(x_i), \quad i = 0, N, \\
E_i u_\Delta(t_i) &= e_i, \qquad i = 1, 2,
\end{aligned}$$

where the scalars e_1 and e_2 are determined with respect to the choice of E_1 and E_2. In the following, we discuss possible choices for E_1, E_2, t_1 and t_2.

A. Extra conditions $Lu = f$ at x_1 and x_{N-1}

One choice of extra conditions which leads to a uniquely defined quartic spline collocation approximation, as well as to an optimal quartic spline collocation method is to apply the operator L at the grid points, x_1 and x_{N-1}, that is, the nodes near the boundary points. This implies that $E_1 = E_2 = L$, $t_1 = x_1$, $t_2 = x_{N-1}$, $e_1 = f(x_1)$, and $e_2 = f(x_{N-1})$.

A quartic spline collocation method for BVP (1)-(2) determines $v \in S_\Delta^4$ by the equations

$$\begin{aligned}
Lv(\tau_i) &= f(\tau_i), \quad i = 1, \cdots, N, \qquad (39) \\
Bv(x_i) &= g(x_i), \quad i = 0, N, \qquad (40) \\
Lv(x_i) &= f(x_i), \quad i = 1, N-1. \qquad (41)
\end{aligned}$$

The approximation v determined by the above equations turns out to be of fourth order, that is, sub-optimal.

In one formulation of the optimal quartic spline collocation method, we determine a quartic spline $u_\Delta \in S_\Delta^4$ that satisfies the relations

$$\begin{aligned}
L^*u_\Delta(\tau_i) &= f(\tau_i), \quad i = 1, \cdots, N, \qquad (42) \\
B^*u_\Delta(x_i) &= g(x_i), \quad i = 0, N, \qquad (43) \\
L^*u_\Delta(x_i) &= f(x_i), \quad i = 1, N-1. \qquad (44)
\end{aligned}$$

We refer to this method as the *one-step* or *extrapolated* quartic spline collocation method.

In an alternative formulation of the optimal quartic spline collocation method, we determine a quartic spline $u_\Delta \in S_\Delta^4$ through a *two-step* quartic spline collocation method. In *Step 1*, $v \in S_\Delta^4$ is determined by equations (39)-(41). In *Step 2*, $u_\Delta \in S_\Delta^4$ is determined by equations

$$\begin{aligned}
Lu_\Delta(\tau_i) &= f(\tau_i) - P_L v(\tau_i), \quad i = 1, \cdots, N, \qquad (45) \\
Bu_\Delta(x_i) &= g(x_i) - P_B v(x_i), \quad i = 0, N, \qquad (46) \\
Lu_\Delta(x_i) &= f(x_i) - P_L v(x_i), \quad i = 1, N-1. \qquad (47)
\end{aligned}$$

In [6], it is shown that the one-step quartic spline collocation equations (42)-(44) are uniquely solvable for the BVP problem (1)-(2), where $p(x) = 1, q(x) = r(x) = 0, \alpha_1 = \alpha_2 = 1, \beta_1 = \beta_2 = 0, g_1 = g_2 = 0$. Similar results can be shown for the two-step method.

B. Extra conditions $Lu = f$ at x_0 and x_N

A second choice of extra boundary conditions is to apply L at the boundary points x_0 and x_N. This implies that $E_1 = E_2 = L$, $t_1 = x_0$, $t_2 = x_N$, $e_1 = f(x_0)$, and $e_2 = f(x_N)$.

A fourth-order quartic spline collocation method for BVP (1)-(2) determines $v \in S^4_\Delta$ by the equations (39)-(40) and

$$Lv(x_i) \quad = \quad f(x_i), \quad i = 0, N. \qquad (48)$$

The optimal one-step quartic spline collocation method determines $u_\Delta \in S^4_\Delta$ by the equations (42)-(43) and

$$L^* u_\Delta(x_i) \quad = \quad f(x_i), \quad i = 0, N. \qquad (49)$$

The optimal two-step quartic spline collocation method determines $u_\Delta \in S^4_\Delta$ in two steps. First $v \in S^4_\Delta$ is determined by equations (39)-(40) and (48), then $u_\Delta \in S^4_\Delta$ is determined by equations (45)-(46) and

$$Lu_\Delta(x_i) \quad = \quad f(x_i) - P_L v(x_i), \quad i = 0, N. \qquad (50)$$

In [6], it is shown that the one-step quartic spline collocation equations (42)-(43) and (49) are uniquely solvable for the BVP problem (1)-(2), where $p(x) = 1, q(x) = r(x) = 0, \alpha_1 = \alpha_2 = 1, \beta_1 = \beta_2 = 0, g_1 = g_2 = 0$. Similar results can be shown for the two-step method.

In [6], other types of extra conditions are also presented. Some conditions resemble the natural cubic spline interpolant end-conditions or variations thereof. They lead to uniquely determined, however sub-optimal, quartic spline collocation approximations.

V. Convergence Analysis

In this section, we present the convergence analysis of the two-step quartic spline collocation method (39)-(40) and (48), (45)-(46) and (50). The analysis of the one-step method is found in [6].

We consider the BVP (1)-(2) with $p(x) = 1$ and homogeneous Dirichlet boundary conditions $g_1 = g_2 = 0$. We assume that u, u_Δ and v satisfy the boundary conditions. Assume that the BVP $u'' = 0$, $Bu = 0$ has a unique solution. Then [11] there is a Green's function $G(x,t)$ for that problem. Let $y \equiv u''$, $y_\Delta \equiv u''_\Delta$, and $s \equiv v''$. Then u, u_Δ, v and their first derivatives can be obtained by

$$u(x) = \int_a^b G(x,t) y(t) dt, \;\; u'(x) = \int_a^b G_x(x,t) y(t) dt,$$

$$u_\Delta(x) = \int_a^b G(x,t) y_\Delta(t) dt, \;\; u'_\Delta(x) = \int_a^b G_x(x,t) y_\Delta(t) dt,$$

$$v(x) = \int_a^b G(x,t) s(t) dt, \;\; v'(x) = \int_a^b G_x(x,t) s(t) dt.$$

Introduce the operator $K : C^1(a,b) \to C(a,b)$ defined by

$$Kz(x) = q(x) \int_a^b G_x(x,t) z(t) dt + r(x) \int_a^b G(x,t) z(t) dt, \qquad (51)$$

and the linear projection P_Δ that maps $L_2(a,b)$ to S^2_Δ by piecewise quadratic interpolation at the midpoints $\{\tau_i\}_1^N$ and grid points x_0, x_N.

With the notations introduced, we can rewrite equations (39), (48) and (45), (50), respectively as

$$P_\Delta(s + Ks) \quad = \quad P_\Delta f, \qquad (52)$$

$$P_\Delta(y_\Delta + Ky_\Delta) \quad = \quad P_\Delta \bar{f}, \qquad (53)$$

where $\bar{f}$ the discrete function defined by $\bar{f}(x) \equiv f(x) - P_L v(x)$, for $x = \tau_i, i = 1, \cdots, N$, or $x = x_i, i = 0, N$. Since $P_\Delta s = s$ and $P_\Delta y_\Delta = y_\Delta$, we simplify (52) and (53) as

$$(I + P_\Delta K)s \quad = \quad P_\Delta f, \qquad (54)$$

$$(I + P_\Delta K) y_\Delta \quad = \quad P_\Delta \bar{f}. \qquad (55)$$

Equation (1) can be rewritten as

$$y + Ky = f. \qquad (56)$$

Assumption (a2) implies that (56) is uniquely solvable for any f, therefore the $(I + K)^{-1}$ exists and is bounded. By the definition of P_Δ, $\| P_\Delta y - y \|_\infty$ converges to zero as h approaches zero for a continuous function y. The complete continuity of K [11], implies that $\| P_\Delta K - K \|_\infty$ converges to zero as h converges to zero. Therefore, by Neumann's theorem [10], we conclude that the operators $(I + P_\Delta K)^{-1}$ exist, and are uniformly bounded for sufficient small h.

Theorem 2: We assume that
(a1) $q(x)$, $r(x)$ and $f(x)$ are continuous on $[a,b]$,
(a2) the BVP $Lu = f$, $Bu = 0$ has a unique solution in $C^{10}(a,b)$, and
(a3) the BVP $u'' = 0$, $Bu = 0$ has a unique solution.
Then we conclude that
(b1) the collocation approximation $v \in S^4_\Delta$ defined by (39)-(40) and (48) in step 1 exists,
(b2) the global error $u - v$ satisfies

$$\| u - v \|_\infty \quad = \quad O(h^4), \qquad (57)$$

$$\| (u - v)^{(k)} \|_\infty \quad = \quad O(h^{5-k}), \quad k = 1, 2, \qquad (58)$$

and
(b3) the error $u - v$ at the midpoints satisfies

$$| (u - v)^{(k)}(\tau_i) | = O(h^4), \quad k = 0, 1, 2, \; i = 1, \cdots, N. \quad (59)$$

Proof: From the existence and uniform boundedness of $(I + P_\Delta K)^{-1}$, the solvability of (54) and (55) follows, hence the unique existence of v follows.

Recall the quartic spline interpolant S of u in (4)-(5). By (22)-(23) we have shown that $BS = O(h^4)$. Note that there exists a linear function w such that $Bw = BS = O(h^4)$ because of assumption (a3), and the fact that any problem with non-homogeneous boundary conditions can be converted to one with homogeneous ones. It can be further shown that $\| w \|_\infty = O(h^4)$ and $\| w' \|_\infty = O(h^4)$. It is clear that the problem $(S - w)'' = S''$, $B(S - w) = 0$ is solvable. Then (21) and (24) can be rewritten in the operator notations introduced as

$$(I + P_\Delta K) S'' = P_\Delta f + O(h^4),$$

in which S'' is substituted by $(S-w)''$ to yield

$$(I+P_\Delta K)(S''-w'')=P_\Delta f+O(h^4). \qquad (60)$$

Subtracting (54) and (60) we have

$$(I+P_\Delta K)(S''-w''-v'')=O(h^4).$$

From the uniform boundedness of $(I+P_\Delta K)^{-1}$, we obtain

$$\| S''-w''-v'' \|_\infty = O(h^4). \qquad (61)$$

Since the unique solvability of $(S-w-v)''=0$, $B(S-w-v)=0$ is ensured by assumption (a3), we obtain using the Green's function

$$\begin{aligned}(S-w-v)'(x) &= \int_a^b G_x(x,t)(S''-w''-v'')(t)dt,\\ (S-w-v)(x) &= \int_a^b G(x,t)(S''-w''-v'')(t)dt.\end{aligned}$$

These imply that

$$\begin{aligned}\| S-w-v \|_\infty &= O(h^4), &(62)\\ \| S'-w'-v' \|_\infty &= O(h^4). &(63)\end{aligned}$$

By (61), (62)-(63), the definition and properties of w, and the use of the triangle inequality we obtain

$$\| S^{(k)}-v^{(k)} \|_\infty \le \| S^{(k)}-w^{(k)}-v^{(k)} \|_\infty + \| w^{(k)} \|_\infty = O(h^4), \qquad (64)$$

for $k=0,1,2$. We can establish the error bounds (57)-(58) and (59) from equations (6)-(7), Theorem 1, relation (64), and the use of the triangle inequality. This completes the proof.

Before proceeding to the analysis of the optimal quartic spline collocation method, we make an assumption. We assume that the expansion of the error $S^{(k)}-v^{(k)}$, $k=0,1,2$, at the collocation points (which was shown to be $O(h^4)$ in Theorem 2) is smooth enough, so that we have the relation

$$\delta^2 S_i^{(k)} = \delta^2 v_i^{(k)} + O(h^6), \qquad (65)$$

for $k=0,1,2$ and $i=2,\cdots,N-1$. This relation is verified in the numerical results of Table IV.

Theorem 3: Under the hypotheses of Theorem 2, we conclude that
(b1) the collocation approximation $u_\Delta \in S_\Delta^4$ defined by (45)-(46) and (50) in step 2 exists,
(b2) the global error $u-u_\Delta$ satisfies

$$\| (u-u_\Delta)^{(k)} \|_\infty = O(h^{5-k}), \quad k=0,1,2, \qquad (66)$$

and
(b3) the error $u-u_\Delta$ at the nodes and midpoints satisfies

$$\begin{aligned}| (u-u_\Delta)(x_i) &= O(h^6),\ i=0,\cdots,N, &(67)\\ | (u-u_\Delta)(\tau_i) &= O(h^6),\ i=1,\cdots,N, &(68)\\ | (u-u_\Delta)^{(k)}(\tau_i) &= O(h^4),\ k=1,2,\ i=1,\cdots,N &(69)\end{aligned}$$

Proof: Using (65), equations (25)-(31) and (34)-(35) can be rewritten as

$$\begin{aligned}LS_i &= \bar f_i + O(h^6), & i=1,\cdots,N,\\ LS(x_i) &= \bar f(x_i)+O(h^6), & i=0,N,\\ B_kS(x_i) &= \bar g_k + O(h^6), & i=0,N,\quad k=1,2.\end{aligned}$$

Therefore, for u_Δ defined in (45)-(46), (50), we have

$$\begin{aligned}L(S-u_\Delta)(\tau_i) &= O(h^6), & i=1,\cdots,N, \quad (70)\\ L(S-u_\Delta)(x_i) &= O(h^6), & i=0,N, \quad (71)\\ B(S-u_\Delta) &= O(h^6). & (72)\end{aligned}$$

Again note that there exists a linear function w such that $Bw=B(S-u_\Delta)=O(h^6)$, and $\| w \|_\infty = O(h^6)$, $\| w' \|_\infty = O(h^6)$. It is clear that the problem $(S-w-u_\Delta)''=(S-u_\Delta)''$, $B(S-w-u_\Delta)=0$ is solvable due to the assumption (a3). Then we can rewrite equations (70)-(72) as

$$(I+P_\Delta K)(S''-w''-u''_\Delta)=O(h^6).$$

Using the same arguments in the proof of Theorem 2, we can obtain the bounds

$$\| S^{(k)}-w^{(k)}-u_\Delta^{(k)} \|_\infty = O(h^6), \qquad k=0,1,2.$$

By the triangle inequality

$$\| S^{(k)}-u_\Delta^{(k)} \|_\infty \le \| S^{(k)}-w^{(k)}-u_\Delta^{(k)} \|_\infty + \| w^{(k)} \|_\infty,$$

it follows that

$$\| S^{(k)}-u_\Delta^{(k)} \|_\infty = O(h^6), \qquad k=0,1,2. \qquad (73)$$

The error bounds (66) and (67)-(69) follow from equations (6)-(10), Theorem 1, relation (73), and the use of the triangle inequality. This completes the proof.

VI. Numerical results

In this section, we present numerical results to demonstrate the performance of the quartic spline collocation methods. More results can be found in [6]. In the tables of this section, we present, for several problems, the maximum in absolute value errors of the quartic spline collocation approximations and derivatives at the midpoints (τ_i), grid points (x_i), Gauss points (λ_i), and at a large set of uniformly distributed points (∞) (e.g. 1002 points). These points are referred to as *global points*, and the maximum error at these points is considered as an approximation to the infinity norm of the error. We consider two sets of boundary conditions

$$\begin{aligned}u(a) &= g_1, \quad u(b)=g_2, \quad \text{and} &(74)\\ u(a)-u'(a) &= g_1, \quad u(b)-u'(b)=g_2. &(75)\end{aligned}$$

A. Extra conditions $Lu=f$ at x_1 and x_{N-1}

We apply the optimal two-step quartic spline collocation method (39)-(41), (45)-(47) to the problems indicated and report the results.

Problem 3 Consider the differential equation

$$Lu = u''(x) + u'(x) + u(x) = f(x), \quad x \in [0,1] \tag{76}$$

with boundary conditions (75). We consider three instances of this problem, with the function $f(x)$, g_1 and g_2 chosen so that $u(x) = e^x \sin(\pi x)$, $u(x) = x^{\frac{11}{2}}$, and $u(x) = x^{\frac{13}{2}}$, respectively.

Tables I-III show the errors and the orders of convergence for the indicated exact solutions $u(x)$. In Tables I and III, the order of convergence of the midpoints and global errors are about 6, while in Table II they are about 5. In Tables I and III, we obtain superconvergence for the approximation to $u^{(2)}(x)$ at midpoints and grid points (order 4), for the approximation to $u^{(4)}(x)$ at midpoints (order 2), and for the approximation to $u^{(3)}(x)$ at Gauss points (order 3). These superconvergence orders are degraded by one half in Table II. This indicates that C^6 continuity for $u(x)$ is necessary to obtain the optimal orders of convergence, including superconvergence. We emphasize that the condition $u \in C^{10}$, mentioned in the formulation and analysis of the quartic spline collocation methods, is only a sufficient and not a necessary condition. Also, these results indicate that the fact that L contains all the terms up to second order and the fact that the boundary conditions (75) involve the derivative of $u(x)$ do not affect the optimal orders of convergence, or the superconvergence.

TABLE I

Errors, orders of conv. for Problem 3 $u(x) = e^x \sin(\pi x)$.

	N	τ_i	x_i	∞	λ_i
u_Δ	8	1.3-04	1.3-04	1.3-04	1.3-04
	16	2.6-06 5.63	2.6-06 5.63	2.7-06 5.57	2.7-06 5.57
	32	3.6-08 6.15	3.7-08 6.13	4.0-08 6.07	4.0-08 6.07
	64	4.4-10 6.37	4.5-10 6.35	5.5-10 6.18	5.5-10 6.19
	128	5.6-12 6.31	5.8-12 6.28	8.6-12 6.00	8.6-12 6.01
u'_Δ	8	1.2-04	3.1-04	3.1-04	1.6-04
	16	1.2-05 3.31	1.4-05 4.45	1.4-05 4.45	3.7-06 5.42
	32	7.8-07 3.99	8.8-07 4.03	8.8-07 4.04	1.6-07 4.58
	64	4.9-08 4.01	5.5-08 4.00	5.5-08 3.99	9.3-09 4.08
	128	3.0-09 4.00	3.5-09 4.00	3.4-09 4.00	5.8-10 4.01
u''_Δ	8	2.5-03	5.2-03	1.1-02	1.1-02
	16	1.8-04 3.81	1.9-04 4.76	1.3-03 3.02	1.3-03 3.02
	32	1.1-05 3.97	1.3-05 3.91	1.6-04 3.01	1.6-04 3.01
	64	7.2-07 3.99	8.1-07 3.96	2.0-05 3.00	2.0-05 3.00
	128	4.5-08 3.99	5.2-08 3.97	2.6-06 3.00	2.6-06 3.00
$u^{(3)}_\Delta$	8	4.4-01	8.7-01	8.7-01	1.3-01
	16	1.1-01 2.01	2.2-01 2.00	2.1-01 2.02	1.2-02 3.38
	32	2.7-02 2.00	5.4-02 2.00	5.3-02 2.02	1.6-03 2.94
	64	6.8-03 2.00	1.4-02 2.00	1.3-02 2.04	2.0-04 2.97
	128	1.7-03 2.00	3.4-03 2.00	3.3-03 1.95	2.5-05 2.98
$u^{(4)}_\Delta$	8	2.2+00	4.1+01	4.1+01	2.4+01
	16	4.8-01 2.22	2.1+01 0.98	2.1+01 1.00	1.2+01 1.01
	32	1.3-01 1.91	1.0+01 1.00	1.0+01 1.01	6.0+00 1.00
	64	3.3-02 1.95	5.2+00 1.00	5.1+00 1.00	3.0+00 1.00
	128	8.5-03 1.98	2.6+00 1.00	2.6+00 0.99	1.5+00 1.00

Table IV presents the errors and convergence of the 5-th and 6-th derivatives of the approximation v in the first step of the collocation method for $u(x) = e^x \sin(\pi x)$. We denote by $\| \cdot \|_b$ the (maximum) errors at the boundary points x_0, x_N, and near-boundary points x_1, x_{N-1}. We also denote $\frac{\delta^2 v^{(1)}}{h^4}$ by $v^{(5)}$ and $\frac{\delta^2 v^{(2)}}{h^4}$ by $v^{(6)}$, for simplicity. The results of Table IV indicate that the errors $\| u^{(5)} - \frac{\delta^2 v_i^{(1)}}{h^4} \|$ and $\| u^{(6)} - \frac{\delta^2 v_i^{(2)}}{h^4} \|$ are of second order, i.e. they verify assumption (65). Note that, by (11)-(12) and (65), we have for $k = 1, 2$:

$$u_i^{(k+4)} = \frac{\delta^2 S_i^{(k)}}{h^4} + O(h^2) = \frac{\delta^2 v_i^{(k)}}{h^4} + O(h^2).$$

TABLE II

Errors, orders of conv. for Problem 3, $u(x) = x^{\frac{11}{2}}$.

	N	τ_i	x_i	∞	λ_i
u_Δ	8	9.8-06	9.9-06	1.2-05	1.2-05
	16	2.5-07 5.32	2.5-07 5.32	3.1-07 5.26	3.1-07 5.26
	32	9.9-09 4.63	1.0-08 4.57	1.2-08 4.72	1.2-08 4.72
	64	4.5-10 4.44	4.7-10 4.48	4.8-10 4.60	4.8-10 4.60
	128	2.0-11 4.48	2.1-11 4.50	2.1-11 4.54	2.1-11 4.54
u'_Δ	8	9.7-05	1.0-04	1.0-04	2.1-05
	16	6.1-06 3.98	6.7-06 3.95	6.7-06 3.95	1.1-06 4.30
	32	3.8-07 4.00	4.2-07 3.99	4.2-07 3.99	6.6-08 4.00
	64	2.4-08 4.00	2.6-08 3.99	2.6-08 3.99	4.2-09 3.98
	128	1.5-09 4.00	1.7-09 3.99	1.7-09 3.99	2.6-10 3.98
u''_Δ	8	3.7-04	2.0-03	4.9-03	4.9-03
	16	3.2-05 3.55	1.8-04 3.50	6.3-04 2.97	6.3-04 2.97
	32	2.8-06 3.51	1.6-05 3.50	7.9-05 3.00	7.9-05 2.99
	64	2.4-07 3.50	1.4-06 3.50	9.8-06 3.00	9.9-06 3.00
	128	2.1-08 3.50	1.2-07 3.50	1.2-06 3.00	1.2-06 3.00
$u^{(3)}_\Delta$	8	2.0-01	4.3-01	4.3-01	3.9-02
	16	5.2-02 1.97	1.1-01 2.01	1.1-01 2.01	6.9-03 2.51
	32	1.3-02 1.99	2.6-02 2.00	2.6-02 2.00	1.2-03 2.50
	64	3.3-03 1.99	6.6-03 2.00	6.6-03 2.00	2.2-04 2.50
	128	8.2-04 2.00	1.7-03 2.00	1.7-03 2.00	3.8-05 2.50
$u^{(4)}_\Delta$	8	5.8-01	2.0+01	2.0+01	1.2+01
	16	2.1-01 1.50	1.0+01 0.99	1.0+01 0.99	5.8+00 0.99
	32	7.3-02 1.50	5.1+00 1.00	5.1+00 1.00	2.9+00 1.00
	64	2.6-02 1.50	2.5+00 1.00	2.5+00 1.00	1.5+00 1.00
	128	9.1-03 1.50	1.3+00 1.00	1.3+00 1.00	7.3-01 1.00

Problem 4

Consider the differential equation

$$Lu = e^x u''(x) + e^x u'(x) - \frac{1}{1+x} u(x) = f(x), \quad x \in [0,1] \tag{77}$$

with boundary conditions (74). The function $f(x)$, g_1 and g_2 are chosen so that $u(x) = x^{\frac{13}{2}}$. Table V shows the errors and the orders of convergence.We notice that we achieve the optimal orders of convergence, including superconvergence, for the approximation and its derivatives, even with a general operator with variable coefficients.

B. Extra conditions $Lu = f$ at x_0 and x_N

We apply the optimal two-step quartic spline collocation method to Problem 3, and we use the extra boundary conditions (48) instead of (41). The function $f(x)$, g_1 and g_2

TABLE III

Errors, orders of conv. for Problem 3, $u(x) = x^{\frac{13}{2}}$.

	N	τ_i	x_i	∞	λ_i
	8	2.6-05	2.8-05	3.3-05	3.3-05
	16	2.4-07 6.75	2.6-07 6.73	4.5-07 6.20	4.4-07 6.21
u_Δ	32	2.7-09 6.45	2.9-09 6.50	8.9-09 5.64	8.9-09 5.65
	64	3.8-11 6.16	3.1-11 6.56	3.0-10 4.91	2.9-10 4.91
	128	1.9-12 4.35	1.7-12 4.16	1.0-11 4.88	1.0-11 4.88
	8	3.9-04	4.8-04	4.8-04	1.2-04
	16	2.5-05 3.95	3.0-05 4.01	3.0-05 4.01	6.3-06 4.26
u'_Δ	32	1.6-06 3.97	1.9-06 4.00	1.9-06 4.00	3.6-07 4.16
	64	1.0-07 3.98	1.2-07 4.00	1.2-07 4.00	2.1-08 4.09
	128	6.3-09 3.99	7.3-09 4.00	7.3-09 4.00	1.3-09 4.05
	8	1.8-03	2.1-03	2.1-02	2.1-02
	16	1.2-04 4.00	1.3-04 3.97	2.7-03 2.97	2.7-03 2.97
u''_Δ	32	7.3-06 3.99	8.4-06 3.97	3.4-04 2.98	3.4-04 2.98
	64	4.6-07 4.00	5.3-07 4.00	4.2-05 3.00	4.3-05 2.99
	128	2.9-08 4.00	3.3-08 4.00	5.3-06 3.00	5.4-06 2.99
	8	8.3-01	1.8+00	1.8+00	6.4-02
	16	2.2-01 1.93	4.6-01 1.99	4.6-01 1.99	8.1-03 2.98
$u^{(3)}_\Delta$	32	5.6-02 1.97	1.1-01 2.00	1.1-01 2.00	1.0-03 2.98
	64	1.4-02 1.98	2.9-02 2.00	2.9-02 2.00	1.3-04 3.00
	128	3.6-03 1.99	7.2-03 2.00	7.2-03 2.00	1.6-05 3.00
	8	1.3+00	8.5+01	8.5+01	4.9+01
	16	3.4-01 1.91	4.3+01 0.98	4.3+01 0.98	2.5+01 0.97
$u^{(4)}_\Delta$	32	8.5-02 1.98	2.2+01 0.99	2.2+01 0.99	1.3+01 0.99
	64	2.1-02 1.99	1.1+01 0.99	1.1+01 0.99	6.3+00 0.99
	128	5.4-03 2.00	5.5+00 1.00	5.5+00 1.00	3.2+00 1.00

TABLE V

Errors, orders of conv. for Problem 4, $u(x) = x^{\frac{13}{2}}$.

	N	τ_i	x_i	∞	λ_i
	8	2.7-06	3.2-07	9.8-06	9.4-06
	16	4.0-08 6.09	1.5-08 4.36	2.9-07 5.09	2.8-07 5.07
u_Δ	32	6.2-10 5.99	3.4-10 5.52	8.7-09 5.04	8.6-09 5.03
	64	9.8-12 5.98	6.2-12 5.75	2.7-10 5.02	2.7-10 5.01
	128	1.5-13 5.99	9.5-14 6.04	8.4-12 5.01	8.3-12 5.01
	8	3.8-04	4.9-04	4.9-04	1.3-04
	16	2.5-05 3.93	3.0-05 4.03	3.0-05 4.03	6.4-06 4.32
u'_Δ	32	1.6-06 3.97	1.9-06 4.00	1.9-06 4.00	3.6-07 4.17
	64	1.0-07 3.98	1.2-07 4.00	1.2-07 4.00	2.1-08 4.10
	128	6.3-09 3.99	7.3-09 4.00	7.3-09 4.00	1.3-09 4.05
	8	1.9-03	2.0-03	2.1-02	2.1-02
	16	1.2-04 4.01	1.3-04 3.96	2.7-03 2.97	2.7-03 2.97
u''_Δ	32	7.3-06 3.99	8.4-06 3.97	3.4-04 2.98	3.4-04 2.98
	64	4.6-07 4.00	5.3-07 4.00	4.2-05 3.00	4.3-05 2.99
	128	2.9-08 4.00	3.3-08 4.00	5.3-06 3.00	5.4-06 2.99
	8	8.3-01	1.8+00	1.8+00	6.4-02
	16	2.2-01 1.93	4.6-01 1.99	4.6-01 1.99	8.1-03 2.98
$u^{(3)}_\Delta$	32	5.6-02 1.97	1.1-01 2.00	1.1-01 2.00	1.0-03 2.98
	64	1.4-02 1.98	2.9-02 2.00	2.9-02 2.00	1.3-04 3.00
	128	3.6-03 1.99	7.2-03 2.00	7.2-03 2.00	1.6-05 3.00
	8	1.3+00	8.5+01	8.5+01	4.9+01
	16	3.4-01 1.91	4.3+01 0.98	4.3+01 0.98	2.5+01 0.97
$u^{(4)}_\Delta$	32	8.5-02 1.98	2.2+01 0.99	2.2+01 0.99	1.3+01 0.99
	64	2.1-02 1.99	1.1+01 0.99	1.1+01 0.99	6.3+00 0.99
	128	5.4-03 2.00	5.5+00 1.00	5.5+00 1.00	3.2+00 1.00

TABLE IV

Errors and orders of convergence for the 5-th and 6-th derivatives of $u(x) = e^x\sin(\pi x)$.

N	$\|u^{(5)}-v^{(5)}\|_{\tau_i}$	$\|u^{(6)}-v^{(6)}\|_{\tau_i}$	$\|u^{(5)}-v^{(5)}\|_b$	$\|u^{(6)}-v^{(6)}\|_b$
8	3.6+02	7.9+02	5.3+02	1.1+03
16	8.8+01 2.02	1.6+02 2.29	1.3+02 2.03	2.3+02 2.29
32	1.9+01 2.25	4.0+01 2.01	2.8+01 2.22	6.3+01 1.87
64	3.5+00 2.41	1.2+01 1.73	5.6+00 2.33	1.9+01 1.75
128	6.5-01 2.43	3.0+00 2.01	8.8-01 2.66	4.8+00 1.98

are chosen so that $u(x) = e^x\sin(\pi x)$.The errors and the orders of convergence are presented in Table VI.

Comparing Table VI withTable I,we see that the quartic spline collocation method with extra conditions (48) has the same optimal orders of convergence, including superconvergence, for this problem, as the method with extra conditions (41). In general, the two optimal quartic spline collocation methods with extra boundary conditions (41) and (48) seem to be equivalent in orders of convergence and accuracy, with the method using (41) being only slightly more accurate in the approximation of $u(x)$.

C. One-step method

We apply the optimal one-step quartic spline collocation method to Problem 3 with extra conditions (44). The function $f(x)$, g_1 and g_2 are chosen so that $u(x) = e^x\sin(\pi x)$. Table VII shows the errors and the orders of convergence.

Comparing the results in Table VII with those in Table I obtained by the two-step collocation method, we can see that the numerical results by the two-step collocation method are slightly better than those by the one-step collocation method, specifically, for the approximation to $u(x)$ at the midpoints, the grid points, the global points and Gauss points. We should though emphasize that this is not always the case. We applied the one-step quartic spline collocation method to several problems [6] and we found that, in general, the one-step quartic spline collocation method gives almost equivalent results as the two-step method. There are a few problems where the one-step method gives slightly better results than the two-step method, but more problems where the opposite happens.

VII. Extension to two dimensions, other

A natural extension of the methods presented in this paper is an optimal bi-quartic spline collocation method for elliptic partial differential equations. The method for two-point BVPs that uses extra boundary conditions at x_1 and x_{N-1} can be easily extended to two dimensions. In this case, collocation of the operator L takes place on the points that are Cartesian products of $\{\tau^x_1, x_1, \tau^x_i, i = 2, \cdots, N-1, x_{N-1}, \tau^x_N\}$, with $\{\tau^y_1, y_1, \tau^y_j, j = 2, \cdots, M-1, y_{M-1}, \tau^y_M\}$, where τ^x_i, τ^y_j and y_j are the $x-$midpoints, $y-$midpoints and $y-$ grid points, respectively. This gives $n_L = (N+2)(M+2)$ equations, including the extra boundary conditions. The operator B is collocated at

TABLE VI

Errors, orders of conv. for Problem 3, $u(x) = e^x \sin(\pi x)$.

	N	τ_i	x_i	∞	λ_i
u_Δ	8	1.3-04	1.3-04	1.3-04	1.3-04
	16	2.4-06 5.75	2.5-06 5.73	2.6-06 5.69	2.6-06 5.68
	32	3.7-08 6.06	3.7-08 6.05	4.0-08 5.99	4.0-08 5.99
	64	5.3-10 6.11	5.4-10 6.10	6.5-10 5.97	6.5-10 5.97
	128	9.1-12 5.87	9.3-12 5.86	1.2-11 5.70	1.2-11 5.71
u'_Δ	8	1.7-04	2.8-04	2.8-04	1.5-04
	16	1.3-05 3.70	1.4-05 4.30	1.4-05 4.30	3.5-06 5.44
	32	7.8-07 4.02	8.8-07 4.00	8.8-07 4.01	1.6-07 4.51
	64	4.9-08 4.01	5.5-08 3.99	5.5-08 3.99	9.3-09 4.07
	128	3.0-09 4.00	3.5-09 4.00	3.4-09 4.00	5.7-10 4.01
u''_Δ	8	2.5-03	3.4-03	1.1-02	1.1-02
	16	1.8-04 3.79	2.3-04 3.91	1.3-03 3.03	1.3-03 3.03
	32	1.1-05 3.97	1.4-05 4.06	1.6-04 3.01	1.6-04 3.01
	64	7.2-07 3.99	8.4-07 4.03	2.0-05 3.00	2.0-05 3.00
	128	4.5-08 3.99	5.2-08 4.01	2.6-06 3.00	2.6-06 3.00
$u^{(3)}_\Delta$	8	4.4-01	8.7-01	8.7-01	9.6-02
	16	1.1-01 2.01	2.2-01 2.00	2.1-01 2.02	1.3-02 2.85
	32	2.7-02 2.00	5.4-02 2.00	5.3-02 2.02	1.7-03 3.02
	64	6.8-03 2.00	1.4-02 2.00	1.3-02 2.04	2.1-04 3.01
	128	1.7-03 2.00	3.4-03 2.00	3.3-03 1.95	2.6-05 3.00
$u^{(4)}_\Delta$	8	1.9+00	4.1+01	4.1+01	2.4+01
	16	5.3-01 1.82	2.1+01 0.98	2.1+01 1.00	1.2+01 1.01
	32	1.3-01 1.98	1.0+01 1.00	1.0+01 1.01	6.0+00 1.00
	64	3.4-02 1.99	5.2+00 1.00	5.1+00 1.00	3.0+00 1.00
	128	8.5-03 1.99	2.6+00 1.00	2.6+00 0.99	1.5+00 1.00

TABLE VII

Errors, orders of conv. for Problem 3, $u(x) = e^x \sin(\pi x)$ for the one-step method.

	N	τ_i	x_i	∞	λ_i
u_Δ	8	1.3-04	1.3-04	1.3-04	1.3-04
	16	2.4-06 5.76	2.4-06 5.74	2.5-06 5.69	2.5-06 5.69
	32	3.6-08 6.07	3.6-08 6.06	4.0-08 6.00	4.0-08 6.00
	64	5.1-10 6.14	5.2-10 6.13	6.3-10 5.99	6.3-10 5.99
	128	7.4-12 6.12	7.5-12 6.11	1.1-11 5.82	1.1-11 5.82
u'_Δ	8	1.7-04	2.8-04	2.8-04	1.5-04
	16	1.3-05 3.71	1.4-05 4.30	1.4-05 4.30	3.5-06 5.44
	32	7.8-07 4.02	8.8-07 4.00	8.8-07 4.01	1.6-07 4.51
	64	4.9-08 4.01	5.5-08 3.99	5.5-08 3.99	9.3-09 4.07
	128	3.0-09 4.00	3.5-09 4.00	3.5-09 4.00	5.8-10 4.01
u''_Δ	8	2.5-03	3.4-03	1.1-02	1.1-02
	16	1.8-04 3.79	2.3-04 3.91	1.3-03 3.03	1.3-03 3.03
	32	1.1-05 3.97	1.4-05 4.06	1.6-04 3.01	1.6-04 3.01
	64	7.2-07 3.99	8.4-07 4.03	2.0-05 3.00	2.0-05 3.00
	128	4.5-08 3.99	5.2-08 4.01	2.6-06 3.00	2.6-06 3.00
$u^{(3)}_\Delta$	8	4.4-01	8.7-01	8.7-01	9.6-02
	16	1.1-01 2.01	2.2-01 2.00	2.1-01 2.02	1.3-02 2.85
	32	2.7-02 2.00	5.4-02 2.00	5.3-02 2.02	1.6-03 3.02
	64	6.8-03 2.00	1.4-02 2.00	1.3-02 2.04	2.1-04 3.01
	128	1.7-03 2.00	3.4-03 2.00	3.3-03 1.95	2.6-05 3.00
$u^{(4)}_\Delta$	8	1.9+00	4.1+01	4.1+01	2.4+01
	16	5.3-01 1.82	2.1+01 0.98	2.1+01 1.00	1.2+01 1.01
	32	1.3-01 1.98	1.0+01 1.00	1.0+01 1.01	6.0+00 1.00
	64	3.4-02 1.99	5.2+00 1.00	5.1+00 1.00	3.0+00 1.00
	128	8.5-03 1.99	2.6+00 1.00	2.6+00 0.99	1.5+00 1.00

the points that are Cartesian products of $\{x_0, x_N\}$ with $\{y_0, \tau_1^y, y_1, \tau_j^y, j = 2, \cdots, M-1, y_{M-1}, \tau_M^y, y_M\}$, and of $\{y_0, y_M\}$ with $\{\tau_1^x, x_1, \tau_j^x, i = 2, \cdots, N-1, x_{N-1}, \tau_N^x\}$. This gives another $n_B = 2(M+4) + 2(N+2)$ equations, to a total of $n_L + n_B = (N+4)(M+4)$ equations, which is the dimension of the bi-quartic spline space. Note that the method that uses x_0 and x_N as collocation points for the extra boundary conditions may require careful development of extra boundary conditions at the four corners of the two-dimensional domain.

Problems with layers and generally rough behaviour of the solution function usually require adaptive (non-uniform) grids. The development of appropriate perturbations of the differential, boundary and extra boundary operators for quartic splines on non-uniform grids is a difficult and interesting task. Furthermore, the development of gridsize and error estimators is a necessary companion of any adaptive grid technique [3], [4]. In [6], we make preliminary tests of some grid and error estimators for quartic spline collocation and get satisfactory results.

VIII. Conclusions

We have presented quartic spline collocation methods of optimal orders of convergence (up to sixth order) for two-point BVPs. Besides their high order of convergence, the methods are also efficient, in the sense that there is only one point/equation/unknown for each subinterval of the domain partition.

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