Multigrid and Multilevel Methods for Quadratic Spline Collocation

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Abstract

Multigrid methods are developed and analyzed for quadratic spline collocation equations arising from the discretization of one-dimensional second-order differential equations. The rate of convergence of the two-grid method integrated with a damped Richardson relaxation scheme as smoother is shown to be faster than 1/2, independently of the step-size. The additive multilevel versions of the algorithms are also analyzed. The development of quadratic spline collocation multigrid methods is extended to two-dimensional elliptic partial differential equations. Multigrid methods for quadratic spline collocation methods are not straightforward: because the basis functions used with quadratic spline collocation are not nodal basis functions, thus the design of efficient restriction and extension operators is non-trivial. Experimental results, with V-cycle and full multigrid, indicate that suitably chosen multigrid iteration is a very efficient solver for the quadratic spline collocation equations.

1. Introduction

Multigrid methods are considered to be some of the most efficient methods for the iterative solution of linear systems arising from the discretization of partial differential equations. Multigrid methods have been developed and analyzed for linear systems arising from finite difference or Galerkin type finite element discretization of differential equations. The related work for finite element collocation discretization is very limited [Gary81]. This paper develops multigrid methods for quadratic spline collocation. We include an analytic proof of convergence, using Fourier analysis techniques, independent of mesh spacing for a two-level method in one dimension. These results may be extended to two (or more) dimensions. Numerical tests confirm the high-quality convergence.

The development and analysis of efficient solvers for the spline collocation equations are at a beginning level. Some solvers for the quadratic spline collocation equations are being considered in [Chri96] and their parallel performance tested. The present paper contributes to that respect, in the sense that it develops and analyzes iterative methods for the spline collocation equations with rate of convergence independent of the step-size and optimal performance. Developing a general convergence analysis for iterative methods for quadratic spline collocation is more difficult than for Galerkin finite elements and is a much less mature area.

The application of multigrid methods to quadratic spline collocation is nontrivial since quadratic spline collocation uses non-nodal basis functions in the discretization. In most conventional lower order

finite element methods, nodal basis functions are used; hence, the coefficients in the linear system represent values of the function at particular nodes on the grid. (A similar interpretation may also be used for finite difference methods.) This is not the case with quadratic spline collocation. The unknown coefficients do not represent function values hence directly interpolating them between the grid levels in multigrid makes no sense. We use a clever observation about the structure of the quadratic spline collocation basis functions to provide efficient restriction and extension operators, see Section 3.1. These operators can also be used with other finite element methods based on quadratic splines, e.g. quadratic spline Galerkin.

In Section 2, we present a brief overview of the optimal quadratic spline collocation (QSC) method and the spectral properties of the quadratic spline collocation matrix. In Section 3, we develop an extension and restriction operator for quadratic splines and carry out the convergence analysis of the two-grid method for one-dimensional QSC equations. In Section 4, we discuss the implementation of the multigrid method for two-dimensional QSC equations, and in Section 5 we develop alternative restriction operators. In Section 6, we present the results of numerical experiments which indicate that the multigrid solvers are very efficient compared to other solution methods for QSC equations. The summary and conclusions from this study are stated in Section 7.

2. The Optimal Quadratic Spline Collocation Method

We consider a boundary value problem described by the operator equation in a domain Ω (one-dimensional or rectangular multi-dimensional),

$$\mathbf{L}u(x) = g(x) \quad \text{in} \quad \Omega, \tag{2.1}$$

and some boundary conditions defined on the boundary, $\partial\Omega$, of Ω,

$$\mathbf{B}u(x) = \gamma(x) \quad \text{on} \quad \partial\Omega. \tag{2.2}$$

Here $\mathbf{L}$ is a linear elliptic differential operator, $\mathbf{B}$ is a linear boundary differential operator, g and γ are given functions of x (one-dimensional or multi-dimensional), and u is the unknown function of x.

Given a node partition Δ of Ω, we choose a set of basis functions for the quadratic spline space, that is, the space of quadratic piecewise polynomials of continuity $\mathbb{C}^1$ on the nodes of the partition. We also define the set of *data* points or *collocation* points in Ω and on $\partial\Omega$. If Ω is one-dimensional, the collocation points are the midpoints of the subintervals of the partition and the boundary nodes. If Ω is multi-dimensional, the collocation points are tuples of collocation points in each dimension. The collocation method determines the approximation u_Δ to u by requiring that the residuals $\mathbf{L}u_\Delta - g$ and $\mathbf{B}u_\Delta - \gamma$ of the differential operator $\mathbf{L}$ and boundary operator $\mathbf{B}$, respectively, are zero on the data points. If the approximate space is a space of smooth splines, this formulation leads to non-optimal solution approximations, in the sense that the convergence order of the spline collocation approximation is lower than the order of the spline interpolant in the same approximation space. The formulation of the optimal spline collocation methods is based on the construction of appropriate perturbations $\mathbb{P}_\mathbb{L}$ and $\mathbb{P}_\mathbb{B}$ of the operators $\mathbf{L}$ and $\mathbf{B}$ respectively. Two formulations were derived: the *one-step* or *extrapolated* methods, in which u_Δ is determined by requiring that the residuals $(\mathbf{L}+\mathbb{P}_\mathbb{L})u_\Delta - g$ and $(\mathbf{B}+\mathbb{P}_\mathbb{B})u_\Delta - \gamma$ of the perturbed operators $\mathbf{L} + \mathbb{P}_\mathbb{L}$ and $\mathbf{B}+\mathbb{P}_\mathbb{B}$, respectively, are zero on the data points, and the *multiple-step* or *deferred-correction* methods, in which a low (second) order approximation is generated first, and in subsequent steps, higher order approximations are generated, by moving the

perturbation operators to the right hand side of the PDE problem, and applying them to the lower order approximations of the previous step(s). The number of such steps required depends on the order of the BVP and the degree of splines used. For example, for second-order problems and quadratic or cubic splines, two steps suffice. The perturbation operators for quadratic splines and two-dimensional second-order problems are developed in [Chri94].

Both formulations are equivalent with respect to convergence properties, but the deferred-correction methods are more efficient with respect to time and memory requirements [Hous88], [Chri88], because they give rise to a linear system having a smaller bandwidth, with fewer nonzero entries per row than the extrapolated methods. In this paper we consider multigrid methods for the solution of the linear system arising from the two-step QSC equations.

2.1. Spectral Properties of the QSC matrix

Our analysis is applied to the QSC matrix arising from the discretization of the BVP

$$\mathbf{L}u(x) \equiv -u'' = g(x) \quad \text{in} \quad \Omega \equiv (0,1) \tag{2.3}$$

subject to boundary conditions

$$\mathbf{B}u(x) \equiv u = 0 \quad \text{on} \quad x = 0\ ,\ x = 1. \tag{2.4}$$

A set of basis functions for the quadratic spline space constructed on a partition of Ω with uniform step-size h and n subintervals is the set of quadratic B-splines $\{\phi_i^h(x)\}_{i=0}^{n+1}$, with $\phi_i^h(x) = \frac{1}{2}\xi(\frac{x}{h} - i + 2)$ where the quadratic spline function ξ is defined by

$$\xi(x) \equiv \begin{cases} x^2 & \text{if } 0 \le x \le 1 \\ -3 + 6x - 2x^2 & \text{if } 1 \le x \le 2 \\ 9 - 6x + x^2 & \text{if } 2 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases} . \tag{2.5}$$

We note that a quadratic spline basis function ϕ_i^h has support in at most three elements, and that, at any point of the domain (0, 1), there are at most three non-zero basis functions. More specifically, there are exactly three non-zero basis functions on any non-nodal point of (0, 1), and exactly two non-zero basis functions on any node of the partition of (0, 1). Thus, these basis functions are not the usual nodal basis functions we are used to from conventional finite element methods.

Based on the functions $\phi_i^h(x)$ we can construct a set of basis functions $\{\psi_i^h(x)\}_{i=1}^{n}$, so that they satisfy the homogeneous boundary conditions (2.4). They are

$$\psi_1^h(x) = \phi_1^h(x) - \phi_0^h(x) \ \ ; \ \ \psi_i^h(x) = \phi_i^h(x)\ , \ \ i = 2, \cdots, n-1 \ \ ; \ \ \psi_n^h(x) = \phi_n^h(x) - \phi_{n+1}^h(x). \tag{2.6}$$

For problem (2.3)-(2.4), the QSC method implemented with the basis functions defined by (2.6) results in a tridiagonal linear system of the form

$$A\mathrm{x} \equiv \frac{1}{h^2}\begin{bmatrix} 3 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \cdot & \cdot & \cdot & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 3 \end{bmatrix} \mathrm{x} = \mathrm{g} \equiv - \begin{bmatrix} g(h/2) \\ g(3h/2) \\ g(5h/2) \\ \\ g((2n-3)h/2) \\ g((2n-1)h/2) \end{bmatrix}, \tag{2.7}$$

where $h = 1/n$.

As shown in [Chri94], the eigenvalues of the matrix A in (2.7) are given by

$$\lambda_i = \frac{1}{h^2} 4 \sin^2 \frac{i\pi}{2n}, \tag{2.8}$$

and its orthonormal eigenvectors δ_i, $i = 1, \cdots, n$ are

$$\{\delta_{ij} = \sqrt{2h} \sin\frac{(2j-1)i\pi}{2n}\ ,\ i = 1, \cdots, n-1\ ,\ \text{and}\ \ \delta_{nj} = \sqrt{h} \sin\frac{(2j-1)n\pi}{2n}\},\ j = 1, \cdots, n. \tag{2.9}$$

3. The Multigrid Method for QSC Equations

In this section, we introduce the use of multigrid methods for QSC. We also explain the difficulties that arise when the underlying discretization does not use nodal basis functions and how we deal with this for the QSC method. The extension and restriction operators developed in the next subsection and in sections 4 and 5 are applicable whenever quadratic splines are used as the approximation space, for example, when quadratic spline Galerkin or collocation is considered as the discretization method. The analysis, though, in Section 3.3 uses the eigenvalues of the QSC matrix, therefore, it applies only to the QSC discretization method.

3.1. The Restriction and Extension Operators for QSC Equations

We first develop restriction and extension operators for QSC equations. Let $\phi_i^h(x)$, $i = 0, \cdots, n+1$, be the quadratic spline basis functions constructed with step-size h and $\phi_i^{2h}(x)$, $i = 0, \cdots, n/2+1$, be the quadratic spline basis functions constructed with step-size $2h$. Let also $\psi_i^h(x)$, $i = 1, \cdots, n$ and $\psi_i^{2h}(x)$, $i = 1, \cdots, n/2$, be the respective quadratic spline basis functions that satisfy zero Dirichlet boundary conditions.

Since the basis functions are not nodal, the values of the coefficients do not represent function values at particular grid points. Thus, directly interpolating these coefficients from a coarse to fine grid makes no sense. Instead, naively one must calculate the function values on the coarse grid (from the coefficients), interpolate these to the fine grid, and from these calculate the appropriate coefficients on the fine grid. That final step, however, would require a global linear system solve, which is clearly unacceptable.

Fortunately, in the following lemma, we show that any nodal basis function in the coarse grid can be represented as a particular linear combination of basis functions on the fine grid (i.e. the underlying spaces are nested). From this we can calculate explicit "interpolation-like" formulas to move the coefficients directly from the coarse to fine grid.

Lemma 3.1. If $q(x) = \sum_{i=1}^{n/2} \theta_i^{2h} \psi_i^{2h}(x)$ is the representation of any quadratic spline $q(x)$ with respect to the basis functions ψ_i^{2h} , $i = 1, \cdots, n/2$, then $q(x) = \sum_{i=1}^{n} \theta_i^h \psi_i^h(x)$ is the representation of $q(x)$ with respect to the basis functions ψ_i^h , $i = 1, \cdots, n$, where the following relations hold:

$$\theta_1^h = \frac{1}{2}\theta_1^{2h} \tag{3.1a}$$

$$\theta_{2j}^h = \frac{1}{4}(3\theta_j^{2h} + \theta_{j+1}^{2h}) , \quad j = 1, \cdots, n/2-1 \tag{3.1b}$$

$$\theta_{2j+1}^h = \frac{1}{4}(\theta_j^{2h} + 3\theta_{j+1}^{2h}) , \quad j = 1, \cdots, n/2-1 \tag{3.1c}$$

$$\theta_n^h = \frac{1}{2}\theta_{n/2}^{2h}. \tag{3.1d}$$

Proof: Using (2.5) and the definition of $\phi_i^h(x)$, $i = 0, \cdots, n+1$, as in (2.6), we can easily prove that $\phi_i^h(x)$, $i = 0, \cdots, n+1$ and $\phi_i^{2h}(x)$, $i = 0, \cdots, n/2+1$, are related by

$$\phi_0^{2h} = \frac{1}{4}(3\phi_0^h + \phi_1^h) \tag{3.2a}$$

$$\phi_i^{2h} = \frac{1}{4}(\phi_{2i-2}^h + 3\phi_{2i-1}^h + 3\phi_{2i}^h + \phi_{2i+1}^h) , \quad i = 1, \cdots, n/2 \tag{3.2b}$$

$$\phi_{n+1}^{2h} = \frac{1}{4}(\phi_{2n}^h + 3\phi_{2n+1}^h). \tag{3.2c}$$

The relation $\psi_1^{2h}(x) = \phi_1^{2h}(x) - \phi_0^{2h}(x)$, (3.2a), and (3.2b) with $i = 1$ lead to

$$\psi_1^{2h} = \frac{1}{4}(2\psi_1^h + 3\psi_2^h + \psi_3^h). \tag{3.2d}$$

Similarly, we can get

$$\psi_{n/2}^{2h} = \frac{1}{4}(2\psi_n^h + 3\psi_{n-1}^h + \psi_{n-2}^h). \tag{3.2e}$$

Then

$$\begin{aligned} q(x) &= \sum_{i=1}^{n/2} \theta_i^{2h}\psi_i^{2h}(x) = \\ &= \frac{1}{4}\sum_{i=2}^{n/2-1} \theta_i^{2h}(\psi_{2i-2}^h(x) + 3\,\psi_{2i-1}^h(x) + 3\,\psi_{2i}^h(x) + \psi_{2i+1}^h(x)) + \\ &+ \frac{1}{4}\theta_1^{2h}(2\,\psi_1^h(x) + 3\,\psi_2^h(x) + \psi_3^h(x)) + \\ &+ \frac{1}{4}\theta_{n/2}^{2h}(2\,\psi_n^h(x) + 3\,\psi_{n-1}^h(x) + \psi_{n-2}^h(x)) = \\ &= \frac{1}{4}2\theta_1^{2h}\psi_1^h(x) + \frac{1}{4}(3\theta_1^{2h} + \theta_2^{2h})\psi_2^h(x) + \frac{1}{4}(\theta_1^{2h} + 3\theta_2^{2h})\psi_3^h(x) + \cdots + \\ &+ \frac{1}{4}(3\theta_{n/2-1}^{2h} + \theta_{n/2}^{2h})\psi_{n-2}^h(x) + \frac{1}{4}(\theta_{n/2-1}^{2h} + 3\theta_{n/2}^{2h})\psi_{n-1}^h(x) + \frac{1}{4}2\theta_{n/2}^{2h}\psi_n^h(x). \end{aligned} \tag{3.3}$$

Relations (3.1a-d) are derived directly from Relation (3.3).

□

Relations (3.1a-d) lead to the following extension operator matrix $E \in \mathbb{R}^{n\times\frac{n}{2}}$ for the coefficients

of the finite element representation of a quadratic spline:

$$E = \frac{1}{4}\begin{bmatrix} 2 & & & & & \\ 3 & 1 & & & & \\ 1 & 3 & & & & \\ & 3 & 1 & & & \\ & 1 & 3 & & & \\ & & . & . & & \\ & & . & . & & \\ & & & 3 & 1 & \\ & & & 1 & 3 & \\ & & & & 2 \end{bmatrix}. \tag{3.4a}$$

Thus, if w^{2h} is a vector of size $n/2$ corresponding to step-size $2h$, the respective extended vector w^h corresponding to step-size h is defined by

$$\begin{aligned} w_1^h &= \frac{w_1^{2h}}{2}\ ;\ \ w_{2i+1}^h = \frac{w_i^{2h} + 3w_{i+1}^{2h}}{4}\ ,\ \ i = 1, \cdots, n/2-1 \\ w_{2i}^h &= \frac{3w_i^{2h} + w_{i+1}^{2h}}{4}\ ,\ \ i = 1, \cdots, n/2-1\ ;\ \ w_n^h = \frac{w_{n/2}^{2h}}{2}. \end{aligned} \tag{3.4b}$$

For the purpose of carrying out the analysis of the two-grid method for QSC equations, we define the restriction operator matrix R to be the transpose of E, scaled by 1/2, namely,

$$R = \frac{1}{2}E^T. \tag{3.5a}$$

Thus, if w^h is a vector of size n corresponding to step-size h, the respective restricted vector w^{2h} corresponding to step-size $2h$ is defined by

$$\begin{aligned} w_1^{2h} &= \frac{2w_1^h + 3w_2^h + w_3^h}{4}\ ;\ \ w_{n/2}^{2h} = \frac{w_{n-2}^h + 3w_{n-1}^h + 2w_n^h}{4} \\ w_{i/2}^{2h} &= \frac{w_{i-2}^h + 3w_{i-1}^h + 3w_i^h + w_{i+1}^h}{4}\ ,\ \ i = 4,6, \cdots, n-2. \end{aligned} \tag{3.5b}$$

The restriction operator is applied to residual values on the data points. Note that the restriction operator R of (3.5) gives rise to the following interpolation formula, which uses the values of a function u at points $h/2$, $3h/2$, $5h/2$, and $7h/2$ to compute an approximation of u at point $2h$:

$$u(2h) \approx \frac{1}{8}(u(\frac{h}{2}) + 3u(\frac{3h}{2}) + 3u(\frac{5h}{2}) + u(\frac{7h}{2})). \tag{3.6}$$

It is worth noticing that the above formula is exact for constant and linear polynomials, but not for quadratic ones. It is also interesting to note that the quadratic spline collocation stiffness matrix, 2.7, very closely resembles the matrix arising from centered finite difference, while our extension, 3.4a, and corresponding restriction operators are quite different from the standard linear ones used with finite differences.

3.2. The Coarse Grid Correction Scheme for QSC Equations

Having defined the restriction and extension operators, at any iteration, we have the following coarse grid correction scheme (or two-grid scheme) for QSC equations:
Step 1: Apply the restriction operator to the residual to obtain a restricted residual vector corresponding to the coarse grid.
Step 2: Solve the coarse grid QSC system with right-side vector the restricted residual from Step 1. The result is the vector of coefficients of the finite element representation of the error corresponding to the restricted residual.
Step 3: Apply the extension operator to the vector of coefficients from Step 3 to obtain a coarse grid corrected residual corresponding to the fine grid.

We emphasize that the extension operator is applied to the **coefficients of the finite element representation** of the error correction and not to the **components of the error correction** itself. This is because there is more than one non-zero quadratic spline basis function on the data points. Thus the values of a function written as a linear combination of the quadratic spline basis functions on the data points are different from the coefficients of the linear combination. This is true for any finite element method considered, e.g. collocation or Galerkin, as long as it is based on quadratic splines.

3.3. Convergence Analysis of the Two-Grid Method for QSC Equations

In this section, we study the behavior of the error at any iteration of the two-grid method for QSC equations. We consider two types of two-grid methods, which turn out to be almost equivalent as far as convergence rate is concerned. Both methods consist of a simple iterative scheme, like Richardson's iteration, integrated with a coarse grid correction scheme. They correspond to multiplicative and additive algorithms in the literature of multilevel methods. We note that the multiplicative version can be applied directly as a classical multigrid method. The additive variant, in general, must be accelerated with a Krylov subspace method.

For the first method we assume that, at some iteration k, given an approximate solution $\mathrm{x}^{(k)}$, we first apply a simple relaxation scheme, to get an approximation $\hat{\mathrm{x}}^{(k)}$ and the respective residual $\hat{r}^{(k)}$. Then we apply the coarse grid correction scheme, as described in Section 3.2, to $\hat{r}^{(k)}$, to obtain the preconditioned residual $\hat{s}^{(k)}$. The approximation $\mathrm{x}^{(k+1)}$ is obtained by $\mathrm{x}^{(k+1)} = \hat{\mathrm{x}}^{(k)} + \hat{s}^{(k)}$.

The effect of the coarse grid correction scheme on the error can be expressed as the result of the application of the operator

$$M_c = \mathbb{I} - E \cdot A'^{-1} \cdot R \cdot A \tag{3.7}$$

to the error vector, where $\mathbb{I}$ is the identity operator of appropriate dimension; A and A' are the QSC matrices for step-sizes h and $2h$, respectively, as defined in (2.7); and E and R are the extension and restriction operators respectively, as defined in (3.4)-(3.5).

As a simple relaxation scheme, we choose Richardson's iteration, damped by the factor $\frac{4}{h^2}$. Thus, the effect of one iteration of the relaxation scheme on the error can be expressed as the result of the application of the operator

$$M_r = \mathbb{I} - \frac{h^2}{4} \cdot A. \tag{3.8}$$

Then, the effect of one iteration of the two-grid method on the error is expressed as the result of the application of the operator

$$M = M_c \cdot M_r. \tag{3.9}$$

Theorem 3.1. Let an iteration of the two-grid method for QSC equations (2.7) consist of a Richardson iteration damped by $\frac{4}{h^2}$ and a coarse grid correction scheme characterized by the extension operator (3.4) and the restriction operator (3.5), and applied to the residual of the damped Richardson iteration approximation. Then the two-grid method converges with rate bounded by 1/2 and a contraction factor less than 1/2 in the Euclidean norm, independently of the step-size.

Proof: In our analysis, we follow [Hack94]. Let δ_i , $i = 1, \cdots, n$, and δ'_i , $i = 1, \cdots, n/2$, be the eigenvectors of A and A', respectively. Let Q be the matrix with its columns formed by the eigenvectors of A in the following order:

$$Q \equiv [\delta_n, \delta_1, \delta_{n-1}, \delta_2, \delta_{n-2}, \cdots, \delta_{n/2-1}, \delta_{n/2+1}, \delta_{n/2}]. \tag{3.10}$$

Let Q' be the matrix with its columns formed by the eigenvectors of A' in the natural order:

$$Q' \equiv [\delta'_1, \delta'_2, \cdots, \delta'_{n/2}]. \tag{3.11}$$

Let $s_i \equiv \sin\frac{i\pi}{2n}$ and $c_i \equiv \cos\frac{i\pi}{2n}$, $i = 1, \cdots, n$.

Consider the transformed matrices

$$\tilde{M}_r \equiv Q^T \cdot M_r \cdot Q \ , \quad \tilde{M}_c \equiv Q^T \cdot M_c \cdot Q \quad \text{and} \quad \tilde{M} \equiv Q^T \cdot M \cdot Q = \tilde{M}_c \cdot \tilde{M}_r = (\mathbb{I} - \tilde{E} \cdot \tilde{A}'^{-1} \cdot \tilde{R}\tilde{A})(\mathbb{I} - \frac{h^2}{4}\tilde{A})$$

with

$$\tilde{E} \equiv Q^T \cdot E \cdot Q' \ , \quad \tilde{A}'^{-1} \equiv Q'^T \cdot A'^{-1} \cdot Q' \ , \quad \tilde{R} \equiv Q'^T \cdot R \cdot Q \quad \text{and} \quad \tilde{A} \equiv Q^T \cdot A \cdot Q.$$

Taking into account that $A\delta_i = \lambda_i \delta_i$, $i = 1, \cdots, n$, we get a block-diagonal structure for the transformed matrices. More specifically,

$$\tilde{A} = \frac{4}{h^2} blockdiag\{A_0, A_1, \cdots, A_{n/2}\}$$

with

$$A_0 = [1] \ ; \ A_i = \begin{bmatrix} s_i^2 & 0 \\ 0 & c_i^2 \end{bmatrix}, \ i = 1, \cdots, n/2-1 \ ; \ A_{n/2} = [1/2]$$

where the notation $blockdiag\{B_1, \cdots, B_n\}$ denotes a block-diagonal matrix with blocks $B_1, \cdots, B_n$ on the diagonal. Also,

$$\tilde{M}_r = blockdiag\{M_{r0}, M_{r1}, \cdots, M_{rn/2}\}$$

with

$$M_{r0} = [0] \ ; \ M_{ri} = \begin{bmatrix} c_i^2 & 0 \\ 0 & s_i^2 \end{bmatrix}, \ i = 1, \cdots, n/2-1 \ ; \ M_{rn/2} = [1/2]$$

$$\tilde{A}' = \frac{4}{h^2} blockdiag\{A'_1, A'_2, \cdots, A'_{n/2}\}$$

with

$$A'_i = [s_i^2 \cdot c_i^2].$$

Noting that

$$\sin\frac{(2j-1)i\pi}{2n} + 3\sin\frac{(2j+1)i\pi}{2n} + 3\sin\frac{(2j+3)i\pi}{2n} + \sin\frac{(2j+5)i\pi}{2n} = 8\cdot\cos^3\frac{i\pi}{2n}\cdot\sin\frac{(2j'-1)i\pi}{2n}$$

$$\text{with } \; j' = (j+1)/2 \; , \;\; j = 1,3,5, \cdots, n-1 \; , \;\; i = 1, \cdots, n/2,$$

we also get

$$Q'^T \cdot R \cdot Q = \sqrt{2}\;\; blockdiag\{[0],[c_1^3 \quad s_1^3], \cdots, [c_{n/2-1}^3 \quad s_{n/2-1}^3], \sqrt{2}\,[c_{n/2}^3]\}$$

and

$$Q^T \cdot E \cdot Q' = 2(Q'^T \cdot R \cdot Q)^T.$$

Thus,

$$\tilde{M}_c = blockdiag\{M_{c\,0}, M_{c\,1}, \cdots, M_{cn/2}\}$$

with

$$M_{c\,0} = [1] \; ; \;\; M_{ci} = \begin{bmatrix} 1-c_i^4 & -s_i c_i^3 \\ -s_i^3 c_i & 1-s_i^4 \end{bmatrix} , \;\; i = 1, \cdots, n/2-1 \; ; \;\; M_{cn/2} = [1/2],$$

and

$$\tilde{M} = blockdiag\{M_0, M_1, \cdots, M_{n/2}\}$$

with

$$M_0 = [0] \; ; \;\; M_i = \begin{bmatrix} (1-c_i^4)c_i^2 & -s_i^3 c_i^3 \\ -s_i^3 c_i^3 & (1-s_i^4)s_i^2 \end{bmatrix} , \;\; i = 1, \cdots, n/2-1 \; ; \;\; M_{n/2} = [1/4].$$

It is easy to prove that the blocks M_i , $i = 0, \cdots, n/2$ are symmetric positive definite and so is $\tilde{M}$. Therefore the Euclidean norms of $\tilde{M}$ and of the blocks M_i , $i = 0, \cdots, n/2$ are the same as the respective spectral radii. According to Lemma 10.3.1 in [Hack94], the Euclidean norm of the matrix $\tilde{M}$ is the maximum of the Euclidean norms of the blocks M_i , $i = 0, \cdots, n/2$, that make up $\tilde{M}$.

It is easy to see that the eigenvalues of the blocks M_i , $i = 1, \cdots, n/2-1$ are $\mu_{i\,1} = s_i^2 c_i^2$ and $\mu_{i\,2} = 2\, s_i^2 c_i^2$. Then, $\max_{i=1}^{n/2-1}\{\,|\,\mu_{i\,1}\,|\,,\,|\,\mu_{i\,2}\,|\,\} = \max_{i=1}^{n/2-1}\{2\, s_i^2 c_i^2\} = \max_{i=1}^{n/2-1}\{2\,\frac{1}{4}\sin^2\frac{2i\pi}{2n}\} < \frac{1}{2}$. Thus, $S(\tilde{M}) < \max\{0, 1/2, 1/4\} = 1/2$. So the Euclidean (spectral) norm of $\tilde{M}$ is bounded by 1/2, independently of the step-size. This proves the convergence of the two-grid method for QSC equations corresponding to the model problem (2.3)-(2.4).

□

We now consider the determination of the optimal damping factor in the Richardson relaxation and the effect of having two or more relaxation sweeps per iteration. Let ω' denote the damping factor, with $\omega' = 4\omega / h^2$, and ν denote the number of relaxation sweeps per iteration. Then the matrices M_i as computed above are given by

$$M_0 = [1-\frac{1}{\omega}]^\nu \; ; \;\; M_i = \begin{bmatrix} (1-c_i^4)(1-\frac{s_i^2}{\omega})^\nu & -s_i c_i^3 (1-\frac{c_i^2}{\omega})^\nu \\ -s_i^3 c_i (1-\frac{s_i^2}{\omega})^\nu & (1-s_i^4)(1-\frac{c_i^2}{\omega})^\nu \end{bmatrix} , \;\; i = 1, \cdots, n/2-1 \; ; \;\; M_{n/2} = [\frac{1}{2}(1-\frac{1}{2\omega})^\nu].$$

The optimal damping factor corresponds to that ω that minimizes the maximum eigenvalue of M_i for all i, i.e. for all $0 \le c_i < 1$. For one smoothing step the optimal ω is approximately .75 and the spectral radius of the error iteration matrix is approximately .33. In Table 3.1, we give the approximate convergence rate ρ_1 for the natural damping factor (that is, with $\omega = 1$) and for the optimal damping factor for 1 to 5 smoothing steps.

Table 3.1. Convergence rates of the two-grid method for QSC equations with the natural and the optimal damping factors. The damping factor is $\omega' = 4\omega / h^2$, with the natural ω being 1 and the optimal as shown in the table. The number of relaxation sweeps per iteration is denoted by ν.

ν	ρ_1	ω optimal	ρ_ω
1	0.50	0.75	0.33
2	0.26	0.70	0.19
3	0.19	0.67	0.13
4	0.15	0.64	0.10
5	0.12	0.63	0.08

For the second method we assume that, at some iteration k, given an approximate solution $x^{(k)}$ and the respective residual $r^{(k)}$, we apply a simple relaxation scheme, to get an approximation $\hat{x}^{(k)}$. We also apply the coarse grid correction scheme, as described in Section 3.2, to $r^{(k)}$, to obtain the preconditioned residual $s^{(k)}$. The approximation $x^{(k+1)}$ is obtained by $x^{(k+1)} = \hat{x}^{(k)} + s^{(k)}$. This is known as an **additive** multilevel scheme, whereas the first method can be interpreted as a **multiplicative** multilevel scheme.

In this case, the effect of the coarse grid correction scheme on the error is expressed as the result of the application of the operator

$$N = \mathbb{I} - \frac{h^2}{4} \cdot A - E \cdot A'^{-1} \cdot R \cdot A. \tag{3.12}$$

Theorem 3.2. Let an iteration of the two-grid method for QSC equations (2.7) consist of a Richardson iteration damped by $\frac{4}{h^2}$ and a coarse grid correction scheme characterized by the extension operator (3.4) and the restriction operator (3.5), and applied to the residual of the previous two-grid method iteration approximation. Then the two-grid method converges with a contraction factor less than 1/2 in the Euclidean norm, independently of the step-size.

Proof: The proof is similar to the proof of Theorem 3.1. The transformed matrix $\tilde{N}$ is now

$$\tilde{N} \equiv Q^T \cdot N \cdot Q = [\mathbb{I} - \tilde{E} \cdot \tilde{A}'^{-1} \cdot \tilde{R}\tilde{A} - \frac{h^2}{4}\tilde{A}]$$

where Q, $\tilde{E}$, $\tilde{A}'$, $\tilde{R}$ and $\tilde{A}$ are the same as in the proof of Theorem 3.1. We can show that

$$\tilde{N} = blockdiag\{N_0, N_1, \cdots, N_{n/2}\}$$

with

$$N_0 = [0]\ ;\ \ N_i = \begin{bmatrix} 1-c_i^4-s_i^2 & -s_i c_i^3 \\ -s_i^3 c_i & 1-s_i^4-c_i^2 \end{bmatrix},\ \ i = 1, \cdots, n/2-1\ ;\ \ N_{n/2} = [0].$$

Note that the blocks N_i , $i = 0, \cdots, n/2$ are not symmetric positive definite. The Euclidean norm of N_i is easily shown to be $s_i c_i$, therefore, $\| \tilde{N} \|_2 = \max\{0, \max_{i=1}^{n/2-1}\{s_i c_i\}\} = \frac{1}{2}\cos\frac{\pi}{n} < \frac{1}{2}$. This proves the convergence of the additive two-grid method for QSC equations corresponding to the model problem (2.3)-(2.4).

□

It is certainly possible to use a Gauss-Seidel smoother, rather then the Jacobi-Richardson scheme used in the analysis above. However, the technical details in calculating the eigenvalues of the error propagation operator are much more tedious.

4. Extension to Two-Dimensional Problems

We consider the extension of the two-grid method described and analyzed in Section 3 to a general two-dimensional linear elliptic BVP defined in a rectangular domain Ω. We first define two-dimensional extension and restriction operators by relations similar to (3.4b) and (3.5b).

If w^{2h} is a vector of size $nm/4$ corresponding to step-size $2h$ in both dimensions, the respective extended vector w^h corresponding to step-size h in both dimensions is defined by the following relations. The components of w^h corresponding to points close to the corners are given by

$$w^h_{1,1} = \frac{w^{2h}_{1,1}}{2}\ ;\ \ w^h_{1,m} = \frac{w^{2h}_{1,m/2}}{2}\ ;\ \ w^h_{n,1} = \frac{w^{2h}_{n/2,1}}{2}\ ;\ \ w^h_{n,m} = \frac{w^{2h}_{n/2,m/2}}{2}. \tag{4.1a}$$

The components of w^h corresponding to points close to the $y = 0$ boundary are given by

$$w^h_{2i+1,1} = \frac{w^{2h}_{i,1} + 3w^{2h}_{i+1,1}}{4}\ ;\ \ w^h_{2i,1} = \frac{3w^{2h}_{i,1} + w^{2h}_{i+1,1}}{4}\ ,\ \ i = 1, \cdots, n/2-1 \tag{4.1b}$$

and similarly for the other points close to the boundary. At the rest of the points, we have for $i = 1, \cdots, n/2-1$, $j = 1, \cdots, m/2-1$,

$$w^h_{2i,2j} = \frac{9w^{2h}_{i,j} + 3w^{2h}_{i,j+1} + 3w^{2h}_{i+1,j} + w^{2h}_{i+1,j+1}}{16}\ ;\ \ w^h_{2i,2j+1} = \frac{3w^{2h}_{i,j} + 9w^{2h}_{i,j+1} + w^{2h}_{i+1,j} + 3w^{2h}_{i+1,j+1}}{16} \tag{4.1c}$$

$$w^h_{2i+1,2j} = \frac{3w^{2h}_{i,j} + w^{2h}_{i,j+1} + 9w^{2h}_{i+1,j} + 3w^{2h}_{i+1,j+1}}{16}\ ;\ \ w^h_{2i+1,2j+1} = \frac{w^{2h}_{i,j} + 3w^{2h}_{i,j+1} + 3w^{2h}_{i+1,j} + 9w^{2h}_{i+1,j+1}}{16}.$$

If w^h is a vector of size nm corresponding to step-size h in both dimensions, the respective restricted vector w^{2h} corresponding to step-size $2h$ in both dimensions is defined by the following relations. The component of w^{2h} corresponding to the point close to the (0, 0) corner is given by

$$w^{2h}_{1,1} = \frac{4w^h_{1,1} + 6w^h_{1,2} + 2w^h_{1,3} + 6w^h_{2,1} + 9w^h_{2,2} + 3w^h_{2,3} + 2w^h_{3,1} + 3w^h_{3,2} + w^h_{3,3}}{16} \tag{4.2a}$$

and similarly for the other points close to the corners. The components of w^{2h} corresponding to points close to the $y = 0$ boundary are given by

$$w^{2h}_{i/2,1} = \frac{2w^h_{i-2,1} + 6w^h_{i-1,1} + 6w^h_{i,1} + 2w^h_{i+1,1} + 3w^h_{i-2,2} + 9w^h_{i-1,2} + 9w^h_{i,2} + 3w^h_{i+1,2} + w^h_{i-2,3} + 3w^h_{i-1,3} + 3w^h_{i,3} + w^h_{i+1,3}}{16} \tag{4.2b}$$

for $i = 4,6, \cdots, n-2$, and similarly for the other points close to the boundary. At the rest of the points, we have for $i = 4,6, \cdots, n-2$, $j = 4,6, \cdots, m-2$,

$$w^{2h}_{i/2,j/2} = (w^h_{i-2,j-2} + 3w^h_{i-2,j-1} + 3w^h_{i-2,j} + w^h_{i-2,j+1} + 3w^h_{i-1,j-2} + 9w^h_{i-1,j-1} + 9w^h_{i-1,j} + 3w^h_{i-1,j+1} + \\ + 3w^h_{i,j-2} + 9w^h_{i,j-1} + 9w^h_{i,j} + 3w^h_{i,j+1} + w^h_{i+1,j-2} + 3w^h_{i+1,j-1} + 3w^h_{i+1,j} + w^h_{i+1,j+1})/16. \quad (4.2c)$$

Given the relations (4.1), it is easily seen that the computational cost of the application of the extension operator in two dimensions is three floating-point operations (flops) per (non-boundary) component, i.e. asymptotically, a total of $3n^2$ flops for an $(n+1)\times(n+1)$ grid (i.e. n subintervals in each dimension).

Similarly, given the relations (4.2), it is easily seen that the computational cost of the application of the restriction operator in two dimensions is 15 floating-point operations (flops) per (non-boundary) component, i.e. asymptotically, a total of $15\frac{n^2}{4}$ flops for an $(n+1)\times(n+1)$ grid.

5. Alternative Coarse Grid Methods

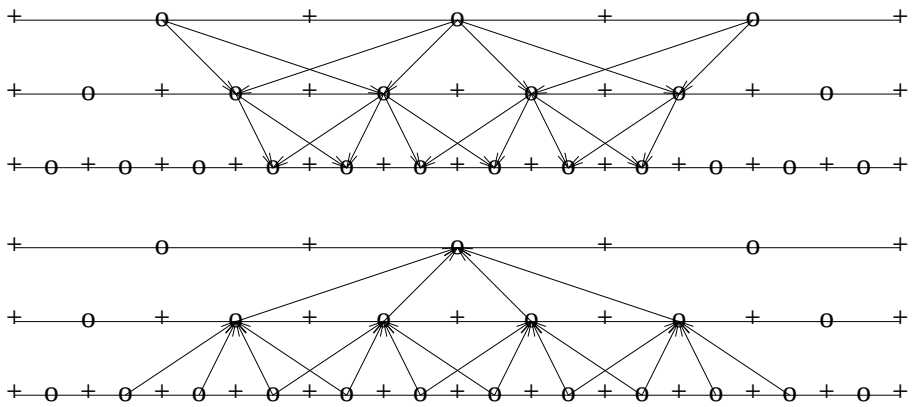
5.1. The Two-Grid Method with Grids of Step-Size Ratio 4

In this section an alternative two-grid method is presented. We will assume that the fine grid has step-size h, while the coarse one $4h$. A possible advantage of such a two-grid method is that the coarse problem is of smaller size. In the one-dimensional case it is of size 1/4 of the fine one, instead of 1/2, but in the two-dimensional case it is of size 1/16, instead of 1/4. A coarse grid problem of step-size $4h$ also has the consequence that in the case of a multigrid method (i.e. a recursive application of the two-grid method until a fixed size coarse grid is reached) the coarsest grid will be reached with fewer steps. More specifically, $\log_4 \frac{n}{n_c}$ steps, instead of $\log_2 \frac{n}{n_c}$ steps, where n_c is the coarsest grid size. In addition, the restricted vectors are of smaller size, so fewer components need to be computed.

In order to use a coarse grid of step-size $4h$, extension and restriction relations between levels 1 and 3 are needed. If the extension and restriction relations (3.4b) and (3.5b) (or (4.1) and (4.2) in two dimensions) are applied once between levels 1 and 2 (step-sizes h and $2h$ respectively), and once more between levels 2 and 3 (step-sizes $2h$ and $4h$ respectively), extension and restriction relations directly between levels 1 and 3 arise. A clear disadvantage in this approach is that the extension and restriction relations between levels 1 and 3 require more flops than those between levels 1 and 2. For example, the one-dimensional restriction relation (3.5b) between levels 1 and 2 requires 3 flops (using 4 components), while the respective relation between levels 1 and 3 requires 9 flops (using 10 components). Table 5.1 lists the flops required for the extension and restriction operators in one and two dimensions when the step-sizes have ratio 2 and 4. Figure 5.1 shows the dependency of components of levels 1, 2 and 3.

Table 5.1. Floating-point operations required to apply the extension and restriction operators (3.4b) and (3.5b) in one dimension and the respective ones (4.1) and (4.2) in two dimensions, when the step-sizes have ratio 2 and 4.

	step-sizes h and $2h$		step-sizes h and $4h$	
	1D	2D	1D	2D
extension	n	$3n^2$	$1.5n$	$5.25n^2$
restriction	$1.5n$	$3.75n^2$	$2.25n$	$6.1875n^2$



(a)

(b)

Figure 5.1. The dependency of vector components corresponding to grids with step-sizes h, $2h$ and $4h$ when applying (a) the extension operator (3.4b), and (b) the restriction operator (3.5b). A "+" denotes a grid point, while a "o" a collocation point (midpoint).

5.2. Alternative Restriction Relations

In this section alternative restriction relations are developed. These are advantageous from the computational complexity point of view when step-sizes of ratio 4 are used. They are also more accurate, as we shall show.

Recall that the restriction operator (3.5) gives rise to the interpolation formula (3.6), which is exact for polynomials of degree at most one. Consider now the following interpolation formula:

$$u(2h) \approx \frac{1}{16}(-u(\frac{h}{2}) + 9u(\frac{3h}{2}) + 9u(\frac{5h}{2}) - u(\frac{7h}{2})). \tag{5.1}$$

This formula is exact for polynomials of degree at most three and uses four components and three flops.

Cubic interpolation can be used to develop restriction relations between either levels 1 and 2, or levels 1 and 3, or any two levels. For example, if w^h is a vector of size n corresponding to step-size h, the respective restricted vector w^{2h} corresponding to step-size $2h$ is defined by

$$w_1^{2h} = \frac{5w_1^h + 15w_2^h - 5w_3^h + w_4^h}{16}, \quad w_{n/2}^{2h} = \frac{w_{n-3}^h - 5w_{n-2}^h + 15w_{n-1}^h + 5w_n^h}{16}$$

$$w_{i/2}^{2h} = \frac{-w_{i-2}^h + 9w_{i-1}^h + 9w_i^h - w_{i+1}^h}{16}, \quad i = 4,6,\cdots,n-2. \tag{5.2}$$

Also, if w^h is a vector of size n corresponding to step-size h, the respective restricted vector w^{4h} corresponding to step-size $4h$ is defined by

$$w_{i/4}^{4h} = \frac{-w_{i-3}^h + 9w_{i-2}^h + 9w_{i-1}^h - w_i^h}{16}, \quad i = 4,8, \cdots, n. \tag{5.3}$$

Thus, in the one-dimensional case, only 3 flops are required for the computation of each of the components of the restricted vectors, either at level 2 or at level 3.

Relations (5.2) and (5.3) can be extended to two dimensions in a natural way. Table 5.2 lists the flops required for this type of restriction operator in one and two dimensions when the step-sizes have ratio 2 and 4. Figure 5.2 shows the dependency of components of levels 1, 2 and 3.

Table 5.2. Floating-point operations required to apply the restriction operators (5.2) and (5.3) in one dimension and the respective ones in two dimensions.

	step-sizes h and $2h$		step-sizes h and $4h$	
	1D	2D	1D	2D
restriction	$1.5n$	$3.75n^2$	$0.75n$	$0.9375n^2$

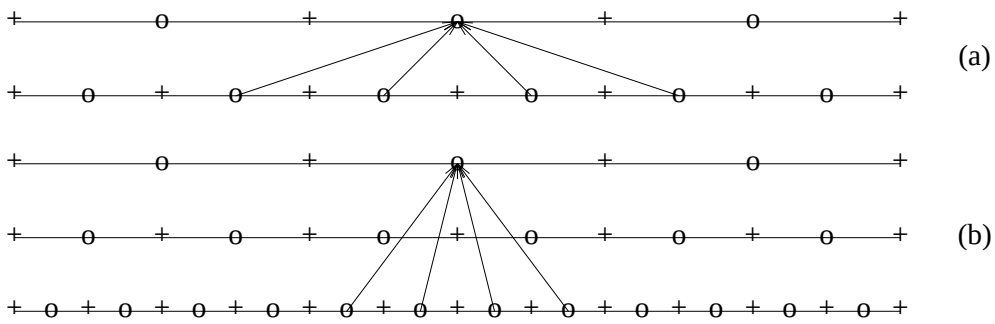


Figure 5.2. The dependency of vector components corresponding to grids with step-sizes h, $2h$ and $4h$ when applying (a) the restriction operator (5.2), and (b) the restriction operator (5.3). A "+" denotes a grid point, while a "o" denotes a collocation point (midpoint).

6. Numerical results.

In this section results from numerical experiments that demonstrate the convergence and computational efficiency of the multigrid method for quadratic spline collocation (QSC) equations are presented.

First, some results that demonstrate the effect of the restriction operators (3.5b), (5.2) and (5.3) on the convergence of the two-grid method are presented. The performance of the V-cycle and the full multigrid methods using the same three restriction operators are also tested. The V-cycle multigrid method consists of the recursive application of the two-grid method until a certain coarsest grid level has been reached. The full multigrid method constructs an initial approximation for each V-cycle starting from the coarsest grid. The interpolation operators used in the full multigrid method are identical to those discussed above.

These tests were applied to Problem 1, listed in the Appendix, which gives rise to symmetric linear system if the set of basis functions $\{\psi_i^h(x)\}_{i=1}^n$ is used. Diagonal (Jacobi) preconditioning was

used as a relaxation scheme. No acceleration method was used. At each iteration, we calculated the residual of the linear system and the maximum error of the collocation approximation on a constant 8×8 grid. The latter can be considered as an approximation to the maximum norm of the error of the collocation approximation. These experiments were carried out using MATLAB.

Table 6.1 shows the error and the residual for 5 iterations of the two-grid and the V-cycle and full multigrid methods with three restriction operators for a 257×257 discretization grid. The coarsest grid in the case of a multigrid method was 17×17. One relaxation iteration was applied at each level. The initial solution vector was chosen to be the zero vector and the respective error and residual are shown as "iteration 0" error and residual, respectively.

The experiments show that the full multigrid method with restriction operator (5.2) in 2D reaches the discretization error in 1 iteration. The full multigrid method with restriction operator (4.2) is just a bit behind and needs two iterations to reach the discretization error, while the full multigrid method with restriction operator (5.3) in 2D needs more than 5 iterations to reach the discretization error. The two-grid methods with restriction operators (4.2) and (5.2) in 2D require 2-3 iterations to reach the discretization error, while the respective V-cycle methods take 4-5 iterations. As far as the residual convergence rate is concerned the full multigrid method with restriction operator (5.2) in 2D is again the fastest method.

Table 6.1. Observed error of the QSC approximation and residual of the QSC linear system (both in exponential format) for 5 iterations of the two-grid, the V-cycle and the full multigrid methods corresponding to Problem 1 for a 257×257 grid size and for the first step of the QSC method.

method	full multigrid			V-cycle multigrid			two-grid		
restr. oper.	(4.2)	(5.2)	(5.3)	(4.2)	(5.2)	(5.3)	(4.2)	(5.2)	(5.3)
iteration				error					
0	3.2e-4	3.2e-4	3.2e-4	3.2e-4	3.2e-4	3.2e-4	3.2e-4	3.2e-4	3.2e-4
1	1.0e-7	2.5e-8	5.9e-6	1.2e-5	3.5e-6	7.5e-6	1.7e-7	6.6e-8	4.7e-7
2	2.2e-8	4.0e-8	1.0e-6	5.9e-7	9.2e-7	8.6e-6	2.2e-8	3.3e-8	5.5e-7
3	2.2e-8	1.6e-8	3.0e-7	2.1e-7	7.6e-8	2.6e-6	2.2e-8	4.0e-8	2.5e-7
4	2.2e-8	2.4e-8	1.7e-7	9.6e-8	3.1e-8	1.7e-6	2.2e-8	1.6e-8	1.4e-7
5	2.2e-8	2.1e-8	1.0e-7	4.8e-8	2.4e-8	3.6e-7	2.2e-8	2.4e-8	6.6e-8
				residual					
0	1.8e+0	1.8e+0	1.8e+0	1.8e+0	1.8e+0	1.8e+0	1.8e+0	1.8e+0	1.8e+0
1	5.1e-2	1.7e-2	8.5e-2	4.3e-1	5.2e-1	5.4e-1	1.6e-1	1.5e-1	2.3e-1
2	6.6e-3	6.0e-3	2.5e-2	1.6e-1	1.7e-1	2.8e-1	3.8e-2	2.8e-2	6.3e-2
3	1.7e-3	1.7e-3	1.1e-2	6.1e-2	3.4e-2	1.5e-1	1.0e-2	7.6e-3	3.0e-2
4	5.1e-4	5.5e-4	6.1e-3	2.5e-2	7.1e-3	8.8e-2	3.2e-3	1.5e-3	1.8e-2
5	1.6e-4	1.8e-4	3.8e-3	1.1e-2	2.0e-3	5.6e-2	9.4e-4	4.8e-4	1.2e-2

Another parameter in the implementation of the two-grid or multigrid methods is the number of Jacobi iterations (diagonal preconditioning) applied before the coarse grid correction scheme is applied. By varying this number we found that, in some cases, it is beneficial from the computational performance point of view to apply a few Jacobi iterations before the coarse grid correction scheme is applied. Figures 6.1(a) and 6.1(b) plot the error and the residual, respectively, versus the number of floating-point operations (flops) measured by MATLAB for the full multigrid method with three restriction operators on a 257×257 discretization grid. The number of relaxation iterations applied at each level is shown on the figure.

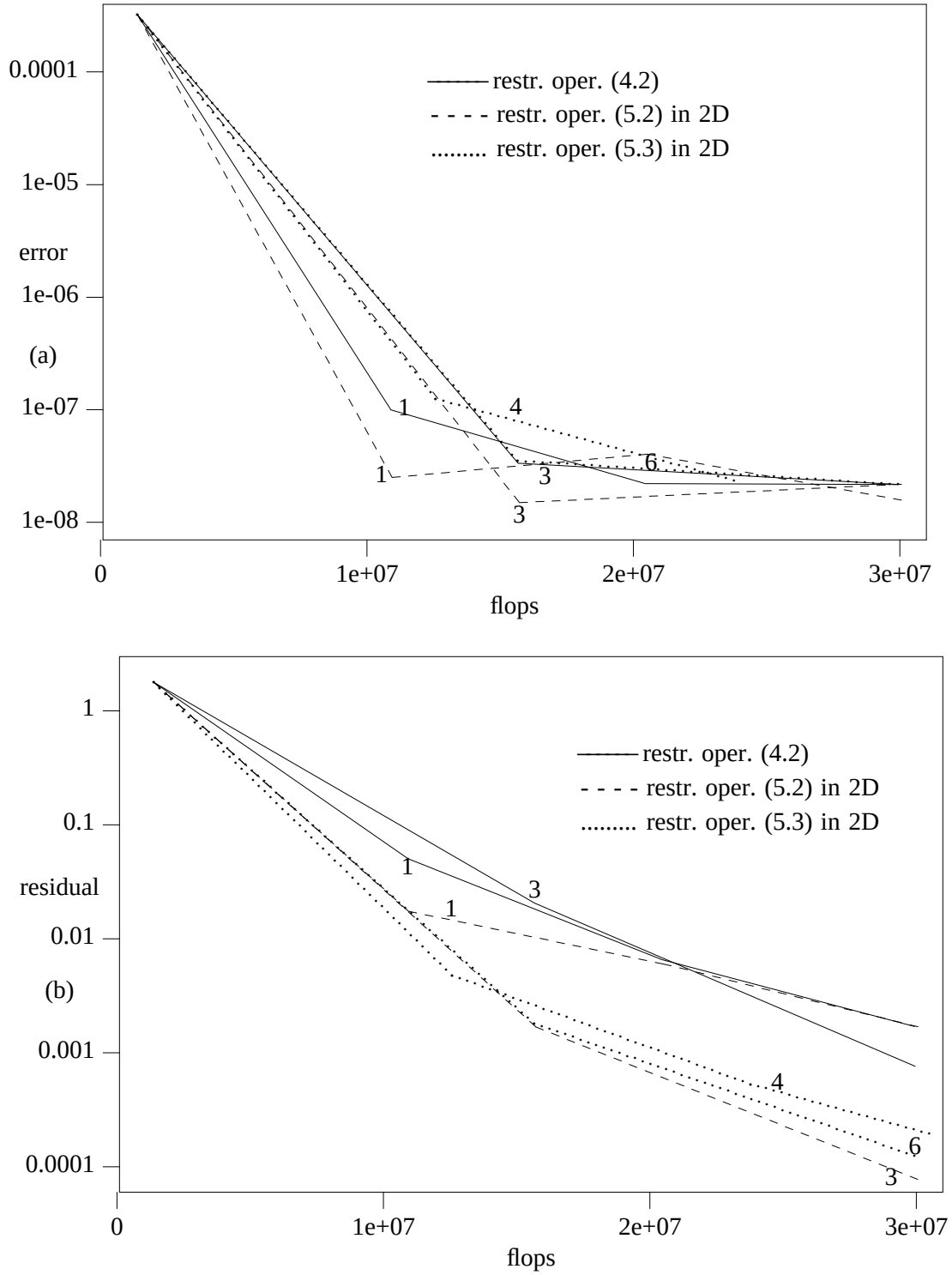


Figure 6.1. Plots of the number of floating-point operations (flops) performed by the full multigrid method with three restriction operators versus the error (in log scale) of the QSC approximation (a) and the residual (in log scale) of the linear system (b), respectively, corresponding to Problem 1 with grid size 257×257 and for the first step of the QSC method. The numbers shown on each line indicate the number of relaxation iterations performed at each level of the multigrid method.

In order to view the details of the performance of the (5.2) restriction operator with 3 relaxations per level and of the (5.3) restriction operator with 6 relaxations per level we have included Table 6.2.

Table 6.2. Observed residual of the QSC linear system, error of the QSC approximation (both in exponential format) and flops (in thousands) required for 5 iterations of the full multigrid method with restriction operators (5.2) and (5.3), with 3 and 6 relaxations per level respectively, corresponding to Problem 1 for a 257×257 grid size and for the first step of the QSC method.

restr. oper.	(5.2)			(5.3)		
no. of. relax.	3			6		
iteration	residual	error	flops	residual	error	flops
0	1.8e+00	3.2e-04	1388	1.8e+00	3.2e-04	1388
1	1.7e-03	1.5e-08	15722	1.8e-03	3.5e-08	15665
2	7.7e-05	2.2e-08	30055	1.2e-04	2.2e-08	29941
3	2.6e-06	2.2e-08	44389	1.9e-05	2.2e-08	44218
4	1.5e-07	2.2e-08	58722	3.3e-06	2.2e-08	58494
5	1.1e-08	2.2e-08	71816	6.9e-07	2.2e-08	71532

Our experiments show that when the performance criterion is the number of flops to reach the discretization error the best method is undoubtedly the full multigrid method with restriction operator (5.2) in 2D and one relaxation iteration per level. When the performance criterion is the slope of the number of flops versus residual line the best method is the full multigrid method with restriction operator (5.2) in 2D and three relaxation iterations per level, followed closely by the full multigrid method with restriction operator (5.3) in 2D and six relaxation iterations per level. The above results do not contradict [Bran77], where it is stated that, for the standard finite difference discretisation scheme, a simple injection restriction operator and linear extension operator, the optimal step-size ratio between the fine and the coarse grids is 2. It is worth noting, though, that for the QSC matrix and the operators developed in this paper, the observed performance of the multigrid methods with grids of ratio 4 does not fall so much behind that of methods with grids of ratio 2.

Next, the convergence of the two-grid method on PDE problems with various operators and boundary conditions is tested. Both steps of the QSC method [Chri94] are considered. In some of the BVPs considered the solution function is not necessarily zero on the boundary. Therefore, the basis functions used for QSC are the set $\{\phi_i^h(x)\}_{i=0}^{n+1}$ defined in Section 2. This set of functions gives rise to unsymmetric linear systems, even for the Laplace operator, because of collocation of the boundary operator on the boundary collocation points. For this reason and for faster convergence, Bi-CGStab, a nonsymmetric acceleration method, was used. For the implementation of the Bi-CGStab acceleration the KSP package [Grop93] was used. Due to the use of the basis functions $\{\phi_i^h(x)\}_{i=0}^{n+1}$, the extension and restriction operators are adjusted on the boundary points.

Diagonal (Jacobi) preconditioning was used as a relaxation scheme. Additional preconditioning was provided by a coarse grid correction scheme, characterized by the restriction operator (5.3) extended to two dimensions (with a coarse grid 1/4 the fine grid size in each dimension). The coarse grid problem, which was 1/16 the fine grid problem size, was solved by a direct band solver without pivoting. The two preconditioned residuals, one from diagonal preconditioning and the second from coarse-grid preconditioning, were added. This method is referred to as MGJ-BCGS. The stopping criterion used was the relative Euclidean norm of the residual and the tolerance was set to 10^{-11} for the first step of the QSC equations and to 10^{-9} for the second step. For the first step of the QSC equations the initial guess vector was the zero vector, while for the second step the solution vector computed in

the first step. The solution vector computed was compared with that resulting from a direct solver to ensure that the quality of approximation produced by the iterative solver was similar to that produced by a direct solver. Note that the stopping criterion and the tolerance chosen may force an iterative method to perform more iterations than those needed to reach the discretization error. We believe, though, that the relative residual is a realistic and commonly used stopping criterion, and the tolerance chosen, though tough, ensures the equivalence of the iterative method with a direct one. Thus it is fair to test iterative methods with this stopping criterion and tolerance.

The test problems used are listed in the Appendix. Table 6.3 shows the number of iterations required for convergence of the two-grid method for Problems 2, 3, 4 and 5 for several grid sizes. These experiments were carried out using FORTRAN on a Sparcstation1 in double precision.

Table 6.3. Number of iterations required for convergence of the two-grid method for quadratic spline collocation equations for several grid sizes and for both steps of the QSC method.

	Problem 2		Problem 3		Problem 4		Problem 5	
grid size	step 1	step 2	step 1	step 2	step 1	step 2	step 1	step 2
33×33	31	22	38	29	46	37	94	91
65×65	31	19	44	28	51	39	172	65
129×129	30	20	40	35	47	37	40	26

For Problem 2, the number of iterations is almost insensitive to the grid size. This behavior agrees with that predicted by Theorem 3.2. Note that Theorem 3.2 applies to a symmetric linear system arising from the set of basis functions $\{\psi_i^h(x)\}_{i=1}^n$, while in the experiments the matrix solved is unsymmetric and its eigenvalues are not known in closed form.

Problem 3 has Neumann conditions on one side of the boundary which make the matrix even more unsymmetric. Again, the number of iterations, although larger than for the Dirichlet condition case, because of the difficulty of the problem, did not significantly vary with the problem size.

Problem 4 has a cross-derivative term, which is of the same order as the second derivative terms u_{xx} and u_{yy}. Thus, the symmetry of most of the matrix rows is significantly affected. The number of iterations, although slightly larger than that for Problem 3, varies only slightly with the problem size.

Problem 5 has a variable coefficient for the first derivative (convection) terms. The coefficients are chosen so that they become relatively large on some parts of the domain. Thus, in the matrix rows corresponding to these parts of the domain, the first order terms, which are very unsymmetric, are dominant, if the step-size is not very small. The approximation obtained in this case was of reasonable quality, but the number of iterations was large, unless the step-size was small. The poor, inconsistent convergence for this problem is not unexpected. In fact, similar behavior is noted for finite differences and finite elements and special techniques must be derived to deal with them. The construction of fast solvers for convection-dominated problems is still in its infancy [Yser93].

Note that the QSC matrix arising from PDE problems with first order derivative terms when using the basis functions $\{\phi_i^h(x)\}_{i=0}^{n+1}$ cannot be written as the sum of a symmetric positive-definite matrix corresponding to the even order derivative terms and a non-symmetric or indefinite matrix (corresponding to the first order derivative terms). The dominating terms of the PDE do not give rise to a symmetric positive-definite matrix. Therefore, the techniques described in [Yser88] or [XuJi92] are not applicable.

Finally, the performance of the two-grid method for QSC equations is compared with that of other solvers [Chri96]. Figure 6.2 shows graphically the observed computational efficiency of several solvers applied to QSC equations arising from Problem 1. The solvers considered are:

Band-LU	Banded Gauss elimination without pivoting.
Rich-CG	Conjugate Gradient algorithm without preconditioning.
SC-PCG	Domain decomposition method with tridiagonally preconditioned CG solution of the Schur complement system [Chri90].
SS-GMRES-MG	Domain decomposition method with overlapping subdomains (Schwarz splitting), accelerated by GMRES and an additional coarse grid correction scheme similar to that used for method MGJ-BCGS.
IF1m-CG	Incomplete factorization preconditioned CG algorithm with level 1 for fill-in and row equality modification.
MGJ-BCGS	The solver described in this paper. (Jacobi relaxation with a coarse grid correction scheme accelerated by Bi-CGStab.)

Certain solvers, e.g. Rich-CG, are applicable only to symmetric systems. Problem 1 has homogeneous boundary conditions, allowing the use of the set of basis functions $\{\psi_i^h(x)\}_{i=1}^n$, which, for the Laplace operator, give rise to a symmetric linear system. This set of basis functions was used for those solvers applicable only to symmetric systems.

From the slopes of the time versus grid size lines plotted in Figure 6.2, it becomes clear that MGJ-BCGS is an optimal method, with respect to asymptotic computational efficiency, and the best in both absolute and relative terms compared to the solvers considered.

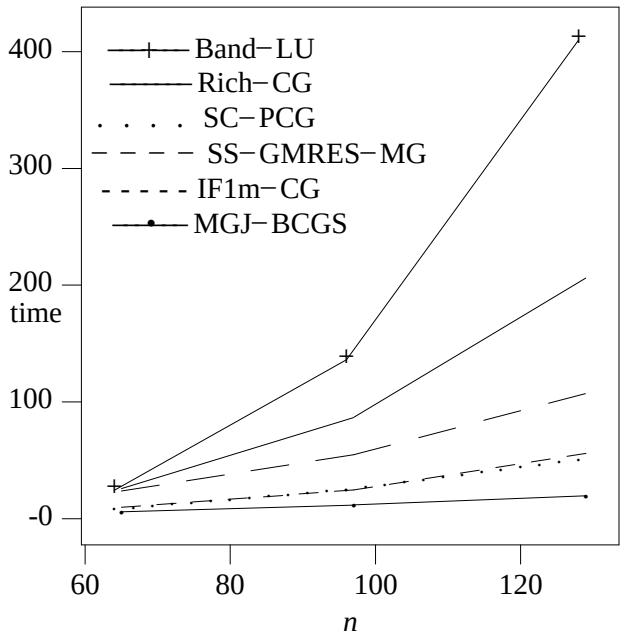


Figure 6.2. Plot of the time in seconds taken by several methods applied to the system of QSC equations corresponding to Problem 1 versus the grid size n in one dimension and for the first step of the QSC method. The slopes in a log-log plot are: Band−LU 4.09; Rich−CG 3.05; SC−PCG 2.71; SS−GMRES−MG 2.19; IF1m−CG 2.57; MGJ−BCGS 1.75

7. Summary and Conclusions

This paper describes the formulation and analysis of multigrid methods for QSC equations. Extension and restriction operators for quadratic splines are developed. The analysis is carried out for a model 1D problem. Alternative restriction operators are developed, for which the ratio of step-sizes of the coarse grid problem over the fine grid one may be 2 or 4. Two-grid methods as well as multigrid methods are tested. Experimental results show that the behavior of the methods agrees well with that predicted by the analysis, even for problems with more general PDE operators and boundary conditions than those assumed in the analysis. The asymptotic computational behavior of the methods is optimal.

Appendix

Problem 1.

$$u_{xx} + u_{yy} = g \quad \text{in} \quad (0,1)\times(0,1)$$
$$u = 0 \quad \text{on} \quad x = 0,\ x = 1,\ y = 0,\ y = 1$$

The function g is chosen so that the exact solution to the problem is $u = x^{\frac{9}{2}}(x-1)^2 y^{\frac{9}{2}}(y-1)^2$.

Problem 2.

$$u_{xx} + u_{yy} = g \quad \text{in} \quad (0,1)\times(0,1)$$
$$u = \gamma \quad \text{on} \quad x = 0,\ x = 1,\ y = 0,\ y = 1$$

Problem 3.

$$u_{xx} + u_{yy} = g \quad \text{in} \quad (0,1)\times(0,1)$$
$$u = \gamma \quad \text{on} \quad x = 1$$
$$u_n = \delta \quad \text{on} \quad x = 0,\ y = 0,\ y = 1$$

Problem 4.

$$u_{xx} + u_{xy} + u_{yy} + u_x + u_y + u = g \quad \text{in} \quad (0,1)\times(0,1)$$
$$u = \gamma \quad \text{on} \quad x = 0,\ x = 1,\ y = 0,\ y = 1$$

Problem 5.

$$u_{xx} + u_{yy} + \frac{1}{x^2 + 10^{-2}} u_x + \frac{1}{y + 10^{-2}} u_y = g \quad \text{in} \quad (0,1)\times(0,1)$$
$$u = \gamma \quad \text{on} \quad x = 0,\ x = 1,\ y = 0,\ y = 1$$

In Problems 2, 3, 4 and 5 the functions g, γ and δ (whenever applicable) are chosen so that the exact solution to the problems is $u = e^{x+y}$.

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