Bi-quartic Spline Collocation Methods for Fourth-order Boundary Value Problems with an Application to the Biharmonic Dirichlet Problem

by

Jingrui Zhang

A thesis submitted in conformity with the requirements
for the degree of Doctor of Philosophy
Graduate Department of Computer Science
University of Toronto

Abstract

Bi-quartic Spline Collocation Methods for Fourth-order Boundary Value Problems with an Application to the Biharmonic Dirichlet Problem

Jingrui Zhang

Doctor of Philosophy

Graduate Department of Computer Science

University of Toronto

2008

We present optimal bi-quartic spline collocation methods for the numerical solution of fourth-order boundary value problems on rectangular domains, and apply a particular instance of these methods to the biharmonic Dirichlet problem.

The methods are based on quartic splines and the collocation discretization methodology with midpoints of a uniform partition as the collocation points. While the standard quartic spline method usually provides second-order approximations, the two bi-quartic spline collocation methods developed in this thesis are observed to produce approximations which are of sixth order at grid points and midpoints, and of fifth order at other points. Both are based on high order perturbations of the differential and boundary conditions operators. The one-step (extrapolated) method forces the approximations to satisfy a perturbed problem, while the three-step (deferred-correction) method proceeds in three steps and perturbs the right sides of the linear equations appropriately.

The three-step bi-quartic spline collocation method is then applied to the biharmonic Dirichlet problem and a fast solver for the resulting linear systems is developed. The fast solver consists of solving an auxiliary biharmonic problem with Dirichlet and second derivative boundary conditions along the two opposite boundaries, and a one-dimensional problem related to the two opposite boundaries. The linear system arising from the bi-quartic spline collocation discretization of the auxiliary problem is solvable by fast

Fourier transforms. The problem related to the two opposite boundaries is solved by preconditioned GMRES (PGMRES). We present a detailed analysis of the performance of PGMRES by studying bounds for the eigenvalues of the preconditioned matrix. We show that the number of PGMRES iterations is independent of the grid size n. As a result, the complexity of the fast solver is $O(n^2 \log(n))$. Numerical experiments from a variety of problems, including practical applications and problems more general than the analysis assumes, verify the accuracy of the discretization scheme and the effectiveness of the fast solver.

Dedication

To my brother Shengli (Victor) Zhang

Acknowledgements

First and foremost, I would especially like to thank my supervisor Professor Christina Christara. I wish I could express my deep appreciation to her. Five years ago, when I had to find a research area and a supervisor for my Ph.D. study, Professor Christara was so kind that she gave me a chance to accept me as her student. Since then, she has always provided me with strong support, invaluable guidance, and unwonted understanding. Without her supervision, this thesis would not have been possible. Her influence is not only on my research skills, but also on my attitude towards life and other people.

I would also like to thank the other members of my committee. I am indebted to Professor Ken Jackson, who has given me a great deal of encouragement and helpful suggestions. I am grateful to Professor Wayne Enright, who carefully read my every checkpoint report and provided many valuable comments on my research and writing skills. I also thank Professor Rudi Mathon for the inspiring discussions. The suggestions by Professors Dhavide Aruliah and Graeme Fairweather on the final version of the thesis are also much appreciated.

Special thanks go to my friends in the department. Their company and encouragement have made my graduate study more enjoyable. They showed me that the best place in the world is where your friends are.

Last but not least, I would like to thank my family: my parents, my three bothers and two sister-in-laws, who always believe in me and love me unconditionally. My mother is my first teacher, who already taught me many characters and basic mathematical calculations before I entered the elementary school. Thanks to her, I have loved study and schools from the very beginning. My brother, Shengli (Victor) Zhang, a role model and a life-time advisor for me, has given me much more than what a brother can give, both spiritually and financially. My sister-in-law, Huiyue Tang, has been a very close friend and a great sister. Finally, I would like to thank Nga Lai Cheung whose love and patience helped me go through the last several busy months of my Ph.D. study.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

We consider the numerical solution of linear fourth-order boundary value problems (BVPs) on rectangular domains described by the operator equation

$$Lu = f, \quad \text{on } \Omega \equiv (a, b) \times (c, d),$$

subject to boundary conditions

$$B_k u = g_k(x, y) \quad \text{on } \partial\Omega,\ k = 1, 2,$$

where L, B_1 and B_2 are differential operators, the detailed forms of which will be given later in the thesis, a, b, c and d are real numbers, $\partial\Omega$ is the boundary of Ω, f, g_1 and g_2 are given functions of variables x and y, and u is the unknown function of x and y. Such problems are used to model some physical phenomena, e.g. the deflection of a clamped plate.

For the discretization of boundary value problems, various numerical methods, such as finite difference, Galerkin, and orthogonal spline collocation methods can be employed. The spline collocation method is an attractive alternative, which is usually computationally less expensive than Galerkin and orthogonal spline collocation [26]. However, standard spline collocation methods usually give rise to second-order approximations. We aim to overcome this limitation (to obtain higher order approximations) by developing optimal spline collocation methods. By "*optimal*" methods, we mean that the order of convergence of the approximations obtained from spline collocation methods is the same as that obtained from the spline interpolation of the exact solution in the same approximation space.

Recently, researchers have developed and analyzed optimal spline collocation methods for certain types of BVPs. These optimal spline collocation methods are based on an appropriate perturbation of the differential equation and boundary condition of the BVP. For the numerical solutions of second-order two-point BVPs in one dimension, Daniel and Swartz developed an optimal cubic spline collocation method in [17], and Houstis, Christara, and Rice presented optimal quadratic spline collocation methods in [25]. Some of these have been extended to two or more dimensions [27, 9]. Both [28] and [13] considered optimal spline collocation methods for fourth-order two-point BVPs. The former used quintic splines, while the latter employed quartic splines.

In much of the thesis, we are concerned with the numerical solution of a particular fourth-order partial differential equation (PDE), the biharmonic Dirichlet problem on a rectangular domain Ω, which is given by

$$\begin{aligned} \triangle^2 u &= f \quad \text{in} \quad \Omega, \\ u &= g_1 \quad \text{on} \quad \partial\Omega, \\ \frac{\partial u}{\partial \mathrm{n}} &= g_2 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where $\triangle$ denotes the Laplacian, and $\partial/\partial\mathrm{n}$ is the outer normal derivative on $\partial\Omega$. We remind readers that, for a two-dimensional problem, $\triangle u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ and $\triangle^2 u \equiv \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 y^2} + \frac{\partial^4 u}{\partial y^4}$.

This problem has been attracting intense research interest in both engineering and mathematics for the past several decades. In mathematics, various analytical and numerical methods have employed it as a benchmark problem. In engineering, it is used as a model for problems such as the bending of thin elastic clamped rectangular plate, the equilibrium of an elastic rectangle, the flow of very viscous fluid in a rectangular cavity, etc. [36].

Various numerical methods have been developed for the biharmonic Dirichlet problem. These methods consist of a discretization strategy that converts the continuous problem into a discrete set of linear algebraic equations, and an associated solver for the resulting linear system.

The discretization strategies can be divided essentially into two classes: (a) methods that transform the problem into an equivalent 2×2 system of coupled second order PDEs and apply discretization techniques to the second order derivatives, and (b) methods that treat the biharmonic PDE as one entity and apply discretization directly to the fourth order derivatives.

In the first class of discretization strategies, several researchers [18, 19, 35] use standard five-point finite differences, resulting in second order approximations. The arising linear systems can be solved by fast Poisson solvers iteratively. M. Arad et al. developed a nine-point finite difference scheme whose truncation error is of sixth-order [4].

Other researchers use a finite element discretization of the 2×2 Poisson system. The techniques developed in [15, 14, 20, 38] are based on finite element discretization and iterative methods. In [30], a Galerkin discretization methodology based on Hermite piecewise bicubics is used, resulting in fourth-order accuracy. The Galerkin problem is reduced to a Schur complement problem involving $\triangle u$ on the two vertical boundaries, and an auxiliary biharmonic Galerkin problem with $\triangle u$ instead of $\frac{\partial u}{\partial \mathrm{n}}$ specified on the two vertical boundaries. Fast Fourier Transforms (FFTs) are used for solving the auxiliary biharmonic Galerkin problem, while the preconditioned CG method is used in solving the Schur complement system. Using properties of the matrix and a conjecture, the authors claim that the number of iterations is independent of the gridsize. Therefore, the complexity of the algorithm is $O(n^2 \log_2 n)$ on an $n \times n$ uniform partition. Relatively recently, some researchers have developed and analyzed orthogonal bicubic spline collocation methods for the biharmonic problem [34, 41, 16, 6], and most recently, Abushama and Bialecki have employed the bi-cubic spline collocation method for the biharmonic problem [2]. The global errors of these methods are of fourth order. In [6, 2], Bialecki et al. develop fast solvers for the resulting linear systems. The approach to obtain a Schur complement system is similar to [30]. The PCG method is employed to solve the Schur complement system. With an analysis of the preconditioned matrix and a conjecture, it is shown that the number of iterations is independent of the gridsize n. Then the complexity of the solver is $O(n^2 \log_2 n)$, which is verified by numerical experiments.

For the second class of discretization strategies, that treats the biharmonic as one fourth-order PDE, several finite difference discretizations and associated linear solvers have been proposed. Some methods are based on a thirteen-point stencil discretization, leading to second order approximations. Some efficient algorithms for solving the resulting linear system have been proposed in [8] and [7]. Bjørstad's method [7] has a complexity of $O(n^2 \log_2 n)$ on an $n \times n$ partition.

Certain second and fourth order finite difference approximations for the biharmonic Dirichlet problem on a nine-point compact cell are given in [3, 31, 40]. Unlike the thirteen-point difference scheme, which uses only the values of the solution at grid points, this approach also involves the values of the gradient u_x and u_y at a subset of grid points. The matrix suffers from slow convergence when standard iterative methods such as Gauss-Seidel or SOR are employed. To address this problem, several multigrid algorithms are presented in [3], which run with asymptotic complexity $O(n^2)$.

This thesis introduces and studies a high order discretization strategy for the biharmonic Dirichlet problem and an efficient solver for the resulting linear system. The contributions of this thesis include: (1) the development of two optimal quartic spline collocation methods for two-dimensional, fourth-order partial differential equations, namely, the *one-step* or *extrapolated* method, and the *three-step* or *deferred-correction* method; these methods achieve sixth order of convergence at the grid points and midpoints, and fifth order of convergence at other points; (2) the application of the two methods to the biharmonic Dirichlet problem and the derivation of a fast solver for the linear system arising from the three-step method; (3) a mathematical proof for the observed performance of the fast solver.

An outline of the thesis is as follows. Bi-Quartic SC concepts and a bi-quartic spline interpolant are introduced in Chapter 2. In the same chapter, the expansions of the errors of the interpolant at the collocation points are derived. Chapter 3 describes the formulations of two optimal bi-quartic spline collocation methods, namely, the one-step method and the three-step method. Several biharmonic problems are introduced in Chapter 4. In particular, two special biharmonic problems with homogeneous boundary conditions are discussed. The first one is the standard biharmonic Dirichlet problem with homogeneous boundary conditions. The other one is an auxiliary biharmonic problem, which has the Dirichlet boundary condition, the Neumann condition along the two horizontal boundaries, and the second derivative condition along the two vertical boundaries. These biharmonic problems are discretized by the bi-quartic spline collocation methods and the resulting linear systems are presented. In Chapter 5, an FFT-based solver for the resulting linear system from the auxiliary problem is described. We then present a fast solver for solving the Bi-Quartic SC equations of the biharmonic Dirichlet problem. The fast solver is based on FFTs applied to the auxiliary problem, and on the preconditioned GMRES (PGMRES) method applied to a problem related to the two vertical boundaries. In Chapter 6, we present the theoretical analysis of the complexity of the solver. The complexity is governed by the performance of PGMRES, for which we show that the convergence rate is independent of the problem size. Chapter 7 presents our attempts to prove the convergence orders of the three-step method and the one-step method. The proof of the orders of convergence is based partly on mathematical arguments, and partly on numerically supported conjectures. Results from numerical experiments are given in Chapter 8 to demonstrate the performance of the discretization methods and of the fast solver. Chapter 9 summarizes the contributions of this thesis and discusses possible future work.

Chapter 2

Bi-Quartic Spline Interpolation

In this chapter, we introduce quartic splines and a certain bi-quartic spline interpolant S for a given function u. We then derive the error formulas for S, which lay out the foundation for the optimal bi-quartic spline collocation methods.

Throughout the formulation and analysis of the methods, without explicit mention, we assume that the two-dimensional function $u(x, y)$ satisfies $u \in C^{10}$.

2.1 Bi-Quartic Spline Space

We first introduce quartic splines and some notation. Without loss of generality, we consider a rectangular domain $\bar{\Omega} \equiv \Omega \cup \partial\Omega \equiv [0,1] \times [0,1]$ and let $\Delta_x \equiv \{x_i = i/n, i = 0, \cdots, n\}$ and $\Delta_y \equiv \{y_j = j/n, j = 0, \cdots, n\}$ be uniform partitions of $(0,1)$ with a stepsize $h = \frac{1}{n}$. We denote by $\widehat{S}_{\Delta_x}$ and $\widehat{S}_{\Delta_y}$ the quartic spline spaces with respect to partitions Δ_x and Δ_y, respectively. We choose as basis functions for $\widehat{S}_{\Delta_x}$ and $\widehat{S}_{\Delta_y}$ the sets $\{\widehat{\phi}_i^x(x)\}_{i=-1}^{n+2}$ and $\{\widehat{\phi}_j^y(y)\}_{j=-1}^{n+2}$, respectively, where $\widehat{\phi}_i^x(x) \equiv \phi(\frac{x}{h} - i + 3)$ and $\widehat{\phi}_j^y(y) \equiv \phi(\frac{y}{h} - j + 3)$ are the standard quartic B-splines generated from the model quartic spline $\phi(x)$ defined by

$$\phi(x) = \frac{1}{24}\begin{cases} x^4 & \text{for } 0 \le x \le 1 \\ x^4 - 5(x-1)^4 & \text{for } 1 \le x \le 2 \\ x^4 - 5(x-1)^4 + 10(x-2)^4 & \text{for } 2 \le x \le 3 \\ x^4 - 5(x-1)^4 + 10(x-2)^4 - 10(x-3)^4 & \text{for } 3 \le x \le 4 \\ x^4 - 5(x-1)^4 + 10(x-2)^4 - 10(x-3)^4 + 5(x-4)^4 & \text{for } 4 \le x \le 5 \\ 0 & \text{elsewhere.} \end{cases}$$

Let $\widehat{S}_\Delta \equiv \widehat{S}_{\Delta_x} \otimes \widehat{S}_{\Delta_y}$ be the bi-quartic spline space with respect to partitions Δ_x and Δ_y. The set of basis functions for $\widehat{S}_\Delta$ is chosen to be $\{\widehat{\phi}_i^x(x)\widehat{\phi}_j^y(y)\}_{i=-1,j=-1}^{n+2,n+2}$, the tensor product of the basis functions of $\widehat{S}_{\Delta_x}$ and $\widehat{S}_{\Delta_y}$. Any bi-quartic spline $S \in \widehat{S}_\Delta$ can be written as

$$S(x,y) = \sum_{i=-1}^{n+2}\sum_{j=-1}^{n+2} c_{i,j}\widehat{\phi}_i^x(x)\widehat{\phi}_j^y(y),$$

where the scalars $c_{i,j}$ are called the *coefficients* or *degrees of freedom*. Clearly, $(n+4)^2$ linearly independent conditions are required to uniquely determine a bi-quartic spline.

2.2 Bi-Quartic Spline Interpolation

In this section we introduce a bi-quartic spline interpolant S and prove some formulas for the errors of S and its derivatives at certain points. Please note that some of the results presented in this section are trivial extensions of those in [43]. We decided to include them for completeness. These formulas will be used later in the formulation and analysis of optimal bi-quartic spline collocation methods.

Let $\tau_i^x = (x_{i-1} + x_i)/2$, $i = 1, \cdots, n$, and $\tau_j^y = (y_{j-1} + y_j)/2$, $j = 1, \cdots, n$. For convenience, we extend the notation so that $\tau_0^x \equiv x_0$, $\tau_{n+1}^x \equiv x_n$, $\tau_0^y \equiv y_0$, and $\tau_{n+1}^y \equiv y_n$. For later use we adopt a classification from [9] to define the following sets of points: the set of collocation points of $\bar{\Omega}$

$$\Pi \equiv \{(\tau_i^x, \tau_j^y), i = 0, \cdots, n+1, j = 0, \cdots, n+1\},$$

the set of interior collocation points in Ω

$$\Pi_i \equiv \{(\tau_i^x, \tau_j^y), i = 3, \cdots, n-2, j = 3, \cdots, n-2\},$$

the set of boundary collocation points on $\partial\Omega$

$$\Pi_\partial \equiv \Pi \cap \partial\Omega \equiv \Pi_{ud}^\partial \cup \Pi_{lr}^\partial \cup \Pi_{cy}^\partial \cup \Pi_{cx}^\partial \cup \Pi_c^\partial,$$

where

$$\begin{aligned}
\Pi_{ud}^\partial &\equiv \{(\tau_i^x, y_j), i = 3, \cdots, n-2, j = 0, n\}, \\
\Pi_{lr}^\partial &\equiv \{(x_i, \tau_j^y), i = 0, n, j = 3, \cdots, n-2\}, \\
\Pi_{cy}^\partial &\equiv \{(\tau_i^x, y_j), i = 1, 2, n-1, n, j = 0, n\}, \\
\Pi_{cx}^\partial &\equiv \{(x_i, \tau_j^y), i = 0, n, j = 1, 2, n-1, n\}, \\
\Pi_c^\partial &\equiv \{(x_i, y_j), i = 0, n, j = 0, n\},
\end{aligned}$$

and the set of interior-boundary collocation points of Ω

$$\Pi_{i\partial} \equiv \Pi - (\Pi_i \cup \Pi_\partial) \equiv \Pi_{ud} \cup \Pi_{lr} \cup \Pi_c,$$

where

$$\begin{aligned}
\Pi_{ud} &\equiv \{(\tau_i^x, \tau_j^y), i = 3, \cdots, n-2, j = 1, 2, n-1, n\}, \\
\Pi_{lr} &\equiv \{(\tau_i^x, \tau_j^y), i = 1, 2, n-1, n, j = 3, \cdots, n-2\}, \\
\Pi_c &\equiv \{(\tau_i^x, \tau_j^y), i = 1, 2, n-1, n, j = 1, 2, n-1, n\}.
\end{aligned}$$

The kth derivative operator with respect to the variable z is denoted by D_z^k. When $k = 1$, we will ignore the superscript and use D_z. We denote by $\mathbb{I}_x$ the one-dimensional quartic spline interpolation operator

$$\mathbb{I}_x : \mathbb{C}([0,1]) \to \widehat{S}_{\Delta_x}, \tag{2.1}$$

defined by the interpolation conditions

$$(\mathbb{I}_x u)(\tau_i^x) = u(\tau_i^x), \quad \text{for } i = 1, \cdots, n \tag{2.2}$$

$$D_x^4(\mathbb{I}_x u)(\tau_i^x) = D_x^4 u(\tau_i^x) - \frac{h^2}{24} D_x^6 u(\tau_i^x) + \frac{7h^4}{5760} D_x^8 u(\tau_i^x), \quad \text{for } i = 1, 2, n-1, n, \tag{2.3}$$

where $u(\cdot)$ is a function of one variable. The y-direction quartic spline interpolation operator $\mathbb{I}_y$ is defined in a similar way. The existence and uniqueness of $\mathbb{I}_x u$ is shown in [43], but due to a typographical error, the proof is incomplete. Therefore, we give a complete proof later in this thesis when we discuss the existence and uniqueness of the respective bi-quartic spline interpolation operator. For a two-variable function u, let $u_{i,j} \equiv u(\tau_i^x, \tau_j^y)$, $i, j = 0, \cdots, n+1$. We denote by $\mathbb{I}_{xy}$ the two-dimensional bi-quartic spline interpolation operator

$$\mathbb{I}_{xy} : \mathbb{C}(\bar{\Omega}) \to \widehat{S}_\Delta, \tag{2.4}$$

defined by the interpolation conditions

$$\begin{aligned}
&(\mathbb{I}_{xy}u)_{i,j} = u_{i,j}, \quad \text{for } i, j = 1, \cdots, n, &(2.5)\\
&D_x^4(\mathbb{I}_{xy}u)_{i,j} = (D_x^4 - \frac{h^2}{24}D_x^6 + \frac{7h^4}{5760}D_x^8)u_{i,j}, \quad \text{for } i = 1, 2, n-1, n, \quad j = 1, \cdots, n, &(2.6)\\
&D_y^4(\mathbb{I}_{xy}u)_{i,j} = (D_y^4 - \frac{h^2}{24}D_y^6 + \frac{7h^4}{5760}D_y^8)u_{i,j}, \quad \text{for } i = 1, \cdots, n, \quad j = 1, 2, n-1, n, &(2.7)\\
&D_x^4 D_y^4(\mathbb{I}_{xy}u)_{i,j} = [D_x^4 D_y^4 - \frac{h^2}{24}(D_x^4 D_y^6 + D_x^6 D_y^4) + \frac{h^4}{24^2}D_x^6 D_y^6 + \frac{7h^4}{5760}(D_x^4 D_y^8 + D_x^8 D_y^4)]u_{i,j}\\
&\qquad - \frac{7h^6}{24 \times 5760}(D_x^8 D_y^6 + D_x^6 D_y^8)u_{i,j} + \frac{49h^8}{5760^2}D_x^8 D_y^8 u_{i,j}, \ i, j = 1, 2, n-1, n, &(2.8)
\end{aligned}$$

where $u(\cdot, \cdot)$ is a function of two variables. In the above formulas, $D_x^4(\mathbb{I}_{xy}u)_{i,j}$ denotes $(D_x^4(\mathbb{I}_{xy}u))(\tau_i^x, \tau_j^y)$. $D_x^4(\mathbb{I}_{xy}u)_{i,j}$ and $D_x^4 D_y^4(\mathbb{I}_{xy}u)_{i,j}$ follow the same notation. It is trivial to show that $\mathbb{I}_{xy} = \mathbb{I}_x(\mathbb{I}_y u(\cdot, y))(x, \cdot) \equiv \mathbb{I}_x \otimes \mathbb{I}_y$. For convenience, let $S \equiv \mathbb{I}_{xy}u$. Next we will show the existence and uniqueness of S.

Throughout this thesis, we use I and $\mathbf{0}$ to denote a generic identity matrix and a generic zero matrix, respectively, whose sizes depend on the context. In some cases we also use I_n to emphasize that the identity matrix is of size $n \times n$. When presenting a matrix, the zero elements or blocks are not shown explicitly, except when we want to emphasize that certain elements or blocks are zero.

Lemma 2.2.1. *Let A be a square matrix which takes the 3×3 block form*

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & \boldsymbol{0} \\ \boldsymbol{0} & K & \boldsymbol{0} \\ \boldsymbol{0} & A_{3,2} & A_{3,3} \end{bmatrix},$$

where $A_{1,1}$, K, and $A_{3,3}$ are square and invertible. Then A^{-1} exists and

$$A^{-1} = \begin{bmatrix} (A_{1,1})^{-1} & \widetilde{A}_{1,2} & \boldsymbol{0} \\ \boldsymbol{0} & K^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \widetilde{A}_{3,2} & (A_{3,3})^{-1} \end{bmatrix},$$

where $\widetilde{A}_{1,2} = -(A_{1,1})^{-1}A_{1,2}K^{-1}$, $\widetilde{A}_{3,2} = -(A_{3,3})^{-1}A_{3,2}K^{-1}$.

Proof. The lemma can be easily verified through the direct multiplication $AA^{-1} = I$. $\square$

For convenience, in the rest of this thesis we employ the notation *blockdiag*$(D_{1,1}, D_{2,2}, \cdots)$ to denote a block diagonal matrix whose diagonal blocks are as listed.

Lemma 2.2.2. *The bi-quartic spline interpolant S defined by equations (2.5)-(2.8) exists and is uniquely defined.*

Proof. Let S be written as

$$S(x, y) = \sum_{i=-1}^{n+2} \sum_{j=-1}^{n+2} c_{i,j} \widehat{\phi}_i^x(x) \widehat{\phi}_j^y(y).$$

Taking into account the one-dimensional interpolation matrix in [43], it is easy to see that the interpolation equations (2.5)-(2.8) can take the form of a linear system

$$G\bar{c} = \bar{u},$$

where $\bar{c}$ is an $(n+4)^2$ vector with $\bar{c}_k = c_{i,j}$ with $k = (i+1)(n+4) + j + 2$ for $i, j = -1, \cdots, n+2$, and $\bar{u}$ is an appropriate vector so that G is given by

$$G = T \otimes T,$$

where T is an $(n+4) \times (n+4)$ matrix of the form

$$T = \begin{bmatrix} 1 & -4 & 6 & -4 & 1 & & & & \\ & 1 & -4 & 6 & -4 & 1 & & & \\ 1 & 76 & 230 & 76 & 1 & & & & \\ & 1 & 76 & 230 & 76 & 1 & & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & & 1 & 76 & 230 & 76 & 1 & \\ & & & & 1 & 76 & 230 & 76 & 1 \\ & & & 1 & -4 & 6 & -4 & 1 & \\ & & & & 1 & -4 & 6 & -4 & 1 \end{bmatrix}.$$

The existence and uniqueness of S can be proved by showing G is nonsingular. If we show T is nonsingular, the non-singularity of G follows. Note that, in [43], the proof that T is nonsingular is incomplete due to a typographical error. Therefore, we give a complete proof here. Simple calculations can verify that

$$P_1 T = Q,$$

where P_1 and Q are $(n+4) \times (n+4)$ matrices of the forms

$$P_1 = \mathrm{blockdiag}(P_{1,1}, I, P_{3,3}), P_{1,1} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ -1 & -80 & 1 & \\ & -1 & & 1 \end{bmatrix}, \text{ and } P_{3,3} = \begin{bmatrix} 1 & & -1 & \\ & 1 & -80 & -1 \\ & & 1 & \\ & & & 1 \end{bmatrix},$$

and

$$Q = \begin{bmatrix} 1 & -4 & 6 & -4 & 1 & & & & & & & & \\ & 1 & -4 & 6 & -4 & 1 & & & & & & & \\ & & 544 & -400 & 320 & -80 & & & & & & & \\ & & 80 & 224 & 80 & & & & & & & & \\ & & 1 & 76 & 230 & 76 & 1 & & & & & & \\ & & & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & \\ & & & & 1 & 76 & 230 & 76 & 1 & & & & \\ & & & & & & 80 & 224 & 80 & & & & \\ & & & & & -80 & 320 & -400 & 544 & & & & \\ & & & & & 1 & -4 & 6 & -4 & 1 & & & \\ & & & & & & 1 & -4 & 6 & -4 & 1 & & \end{bmatrix}.$$

It is easy to verify that $P_{1,1}$ and $P_{3,3}$ are non-singular. Applying Lemma 2.2.1 to the matrix P_1, we can see that P_1 is nonsingular. Consider the $n \times n$ matrix

$$M = \begin{bmatrix} 544 & -400 & 320 & -80 & & & & \\ 80 & 224 & 80 & & & & & \\ 1 & 76 & 230 & 76 & 1 & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & 1 & 76 & 230 & 76 & 1 & \\ & & & & 80 & 224 & 80 & \\ & & & -80 & 320 & -400 & 544 & \end{bmatrix},$$

and notice that

$$Q = \begin{bmatrix} Q_{1,1} & Q_{1,2} & \mathbf{0} \\ \mathbf{0} & M & \mathbf{0} \\ \mathbf{0} & Q_{3,2} & Q_{3,3} \end{bmatrix}, \quad \text{where } Q_{1,1} = (Q_{3,3})^T = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix},$$

$$Q_{1,2} = \begin{bmatrix} 6 & -4 & 1 & 0 & 0 & \cdots & 0 \\ -4 & 6 & -4 & 1 & 0 & \cdots & 0 \end{bmatrix}, \quad \text{and } Q_{3,2} = \begin{bmatrix} 0 & \cdots & 0 & 1 & -4 & 6 & -4 \\ 0 & \cdots & 0 & 0 & 1 & -4 & 6 \end{bmatrix},$$

and $Q_{1,2}$ and $Q_{3,2}$ are $2 \times n$. It is also worth noticing that

$$P_2 M = N,$$

where P_2 and N are $n \times n$ matrices of the forms

$$P_2 = \begin{bmatrix} 1 & 1.5 & & & \\ & 1 & & & \\ & & \cdot & & \\ & & & 1 & \\ & & & 1.5 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 664 & -64 & 440 & -80 & & & & \\ 80 & 224 & 80 & & & & & \\ 1 & 76 & 230 & 76 & 1 & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & 1 & 76 & 230 & 76 & 1 \\ & & & & 80 & 224 & 80 \\ & & & -80 & 440 & -64 & 664 \end{bmatrix}.$$

Clearly, N is a strictly diagonally dominant matrix, thus N^{-1} exists. In addition, by Lemma 2.2.1 it can be shown that P_2 is nonsingular. Then, M is nonsingular and its inverse is defined by

$$M^{-1} = N^{-1} P_2.$$

The non-singularity of M, Q_{11} and Q_{33}, and Lemma 2.2.1 imply that the inverse of Q exists and takes the block form

$$Q^{-1} = \begin{bmatrix} (Q^{-1})_{1,1} & (Q^{-1})_{1,2} & \mathbf{0} \\ \mathbf{0} & (Q^{-1})_{2,2} & \mathbf{0} \\ \mathbf{0} & (Q^{-1})_{3,2} & (Q^{-1})_{3,3} \end{bmatrix}$$

where

$$(Q^{-1})_{1,1} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \quad (Q^{-1})_{3,3} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}, \quad (Q^{-1})_{2,2} = M^{-1},$$

$$(Q^{-1})_{1,2} = \begin{bmatrix} 10 & -20 & 15 & -4 \\ 4 & -6 & 4 & -1 \end{bmatrix} \begin{bmatrix} M^{-1}_{1,1} & M^{-1}_{1,2} & \cdots & M^{-1}_{1,n-1} & M^{-1}_{1,n} \\ M^{-1}_{2,1} & M^{-1}_{2,2} & \cdots & M^{-1}_{2,n-1} & M^{-1}_{2,n} \\ M^{-1}_{3,1} & M^{-1}_{3,2} & \cdots & M^{-1}_{3,n-1} & M^{-1}_{3,n} \\ M^{-1}_{4,1} & M^{-1}_{4,2} & \cdots & M^{-1}_{4,n-1} & M^{-1}_{4,n} \end{bmatrix},$$

$$(Q^{-1})_{3,2} = \begin{bmatrix} -1 & 4 & -6 & 4 \\ -4 & 15 & -20 & 10 \end{bmatrix} \begin{bmatrix} M^{-1}_{n-3,1} & M^{-1}_{n-3,2} & \cdots & M^{-1}_{n-3,n-1} & M^{-1}_{n-3,n} \\ M^{-1}_{n-2,1} & M^{-1}_{n-2,2} & \cdots & M^{-1}_{n-2,n-1} & M^{-1}_{n-2,n} \\ M^{-1}_{n-1,1} & M^{-1}_{n-1,2} & \cdots & M^{-1}_{n-1,n-1} & M^{-1}_{n-1,n} \\ M^{-1}_{n,1} & M^{-1}_{n,2} & \cdots & M^{-1}_{n,n-1} & M^{-1}_{n,n} \end{bmatrix}.$$

Recalling that $P_1T = Q$ and P_1 is nonsingular, we conclude that

$$T^{-1} = Q^{-1}P_1.$$

Therefore, $G^{-1} = (Q^{-1}P_1) \otimes (Q^{-1}P_1)$. □

Recall that we denote by S the two-dimensional bi-quartic spline interpolant $\mathbb{I}_{x,y}u$. We now discuss the error bounds that S satisfies. We start with the errors at the midpoints (τ_i^x, τ_j^y), $i, j = 1, \cdots, n$. In the following, $D_xS_{i,j}$ denotes $(D_xS)(\tau_i^x, \tau_j^y)$, and other partial derivatives follow the same notation.

Theorem 2.2.3. *If $u \in C^{10}(\bar{\Omega})$, then, for $i, j = 1, \cdots, n$,*

$$\begin{aligned} D_xS_{i,j} &= (D_x - \frac{7h^4}{5760}D_x^5)u_{i,j} + O(h^6), && (2.9) \\ D_yS_{i,j} &= (D_y - \frac{7h^4}{5760}D_y^5)u_{i,j} + O(h^6), && (2.10) \\ D_x^2S_{i,j} &= (D_x^2 + \frac{7h^4}{1920}D_x^6)u_{i,j} + O(h^6), && (2.11) \\ D_y^2S_{i,j} &= (D_y^2 + \frac{7h^4}{1920}D_y^6)u_{i,j} + O(h^6), && (2.12) \end{aligned}$$

$$\begin{aligned} D_x^3S_{i,j} &= (D_x^3 + \frac{h^2}{24}D_x^5 - \frac{7h^4}{1920}D_x^7)u_{i,j} + O(h^6), && (2.13) \\ D_y^3S_{i,j} &= (D_y^3 + \frac{h^2}{24}D_y^5 - \frac{7h^4}{1920}D_y^7)u_{i,j} + O(h^6), && (2.14) \\ D_x^4S_{i,j} &= (D_x^4 - \frac{h^2}{24}D_x^6 + \frac{7h^4}{5760}D_x^8)u_{i,j} + O(h^6), && (2.15) \\ D_y^4S_{i,j} &= (D_y^4 - \frac{h^2}{24}D_y^6 + \frac{7h^4}{5760}D_y^8)u_{i,j} + O(h^6). && (2.16) \end{aligned}$$

The theorem is a trivial extension of Theorem 1 in [43] (page 16). The next theorem extends the above results to the cross-derivatives.

Theorem 2.2.4. *If $u \in C^{10}(\bar{\Omega})$, then, for $i, j = 1, \cdots, n$,*

$$D_x D_y S_{i,j} = (D_x D_y - \frac{7h^4}{5760} D_x^5 D_y - \frac{7h^4}{5760} D_x D_y^5) u_{i,j} + O(h^6), \tag{2.17}$$
$$D_x D_y^2 S_{i,j} = (D_x D_y^2 - \frac{7h^4}{5760} D_x^5 D_y^2 + \frac{7h^4}{1920} D_x D_y^6) u_{i,j} + O(h^6), \tag{2.18}$$
$$D_x^2 D_y S_{i,j} = (D_x^2 D_y - \frac{7h^4}{5760} D_y^5 D_x^2 + \frac{7h^4}{1920} D_y D_x^6) u_{i,j} + O(h^6), \tag{2.19}$$
$$D_x^2 D_y^2 S_{i,j} = (D_x^2 D_y^2 + \frac{7h^4}{1920} D_x^2 D_y^6 + \frac{7h^4}{1920} D_y^2 D_x^6) u_{i,j} + O(h^6), \tag{2.20}$$
$$D_x D_y^3 S_{i,j} = (D_x D_y^3 + \frac{h^2}{24} D_x D_y^5 - \frac{7h^4}{1920} D_x D_y^7 - \frac{7h^4}{5760} D_x^5 D_y^3) u_{i,j} + O(h^6), \tag{2.21}$$
$$D_x^3 D_y S_{i,j} = (D_x^3 D_y + \frac{h^2}{24} D_y D_x^5 - \frac{7h^4}{1920} D_y D_x^7 - \frac{7h^4}{5760} D_y^5 D_x^3) u_{i,j} + O(h^6). \tag{2.22}$$

Proof. Following techniques similar to those in [9], we first give the proof of (2.21). By the definition of $\mathbb{I}_{xy}$, we have

$$\begin{aligned}
&D_x D_y^3 \mathbb{I}_{xy} u_{i,j} - D_x D_y^3 u_{i,j} = D_x \mathbb{I}_x D_y^3 \mathbb{I}_y u_{i,j} - D_x D_y^3 u_{i,j} \\
&= D_x \mathbb{I}_x (D_y^3 \mathbb{I}_y u_{i,j} - D_y^3 u_{i,j}) + D_y^3 (D_x \mathbb{I}_x u_{i,j} - D_x u_{i,j}) \\
&= D_x \mathbb{I}_x w_{i,j} - D_x w_{i,j} + D_x (D_y^3 \mathbb{I}_y u_{i,j} - D_y^3 u_{i,j}) + D_y^3 (D_x \mathbb{I}_x u_{i,j} - D_x u_{i,j}),
\end{aligned}$$

where $w = D_y^3 \mathbb{I}_y u - D_y^3 u$. From Theorem 2.2.3, we know $w_{i,j} = \frac{h^2}{24} D_y^5 u_{i,j} - \frac{7h^4}{1920} D_y^7 u_{i,j} + O(h^6)$. Therefore,

$$D_x \mathbb{I}_x w_{i,j} - D_x w_{i,j} = -\frac{7h^4}{5760} D_x^5 (\frac{h^2}{24} D_y^5 u_{i,j} - \frac{7h^4}{1920} D_y^7 u_{i,j}) + O(h^8).$$

Hence,

$$\begin{aligned}
&D_x D_y^3 \mathbb{I}_{xy} u_{i,j} - D_x D_y^3 u_{i,j} \\
&= D_x \mathbb{I}_x w_{i,j} - D_x w_{i,j} + D_x (D_y^3 \mathbb{I}_y u_{i,j} - D_y^3 u_{i,j}) + D_y^3 (D_x \mathbb{I}_x u_{i,j} - D_x u_{i,j}) \\
&= -\tfrac{7h^4}{5760} D_x^5 (\tfrac{h^2}{24} D_y^5 u_{i,j} - \tfrac{7h^4}{1920} D_y^7 u_{i,j}) + D_x (\tfrac{h^2}{24} D_y^5 u_{i,j} - \tfrac{7h^4}{1920} D_y^7 u_{i,j}) - D_y^3 (\tfrac{7h^4}{5760} D_x^5 u_{i,j}) + O(h^6) \\
&= \tfrac{h^2}{24} D_x D_y^5 u_{i,j} - \tfrac{7h^4}{1920} D_x D_y^7 u_{i,j} - \tfrac{7h^4}{5760} D_x^5 D_y^3 u_{i,j} + O(h^6).
\end{aligned}$$

The rest of the relations can be shown similarly. □

The preceding two theorems describe the errors of S at the midpoints. Next, we give two theorems that discuss the errors of S at the gridpoints.

Theorem 2.2.5. *If $u \in C^{10}(\bar{\Omega})$, then, at the gridpoints (x_i, y_j), for $i, j = 0, \cdots, n$,*

$$\begin{aligned}
S(x_i, y_j) &= u(x_i, y_j) + \frac{h^6}{3072}(D_x^6 + D_y^6)u(x_i, y_j) + O(h^8), && (2.23)\\
D_x S(x_i, y_j) &= (D_x + \frac{h^4}{720}D_x^5)u(x_i, y_j) + O(h^6), && (2.24)\\
D_y S(x_i, y_j) &= (D_y + \frac{h^4}{720}D_y^5)u(x_i, y_j) + O(h^6), && (2.25)\\
D_x^2 S(x_i, y_j) &= (D_x^2 - \frac{h^4}{240}D_x^6)u(x_i, y_j) + O(h^6), && (2.26)\\
D_y^2 S(x_i, y_j) &= (D_y^2 - \frac{h^4}{240}D_y^6)u(x_i, y_j) + O(h^6), && (2.27)\\
D_x^3 S(x_i, y_j) &= (D_x^3 - \frac{h^2}{12}D_x^5 + \frac{h^4}{240}D_x^7)u(x_i, y_j) + O(h^6), && (2.28)\\
D_y^3 S(x_i, y_j) &= (D_y^3 - \frac{h^2}{12}D_y^5 + \frac{h^4}{240}D_y^7)u(x_i, y_j) + O(h^6). && (2.29)
\end{aligned}$$

This theorem is a straightforward extension of Theorem 2 in [43] (page 18). We next extend the theorem to the cross-derivatives.

Theorem 2.2.6. *If $u \in C^{10}(\bar{\Omega})$, then, at the gridpoints (x_i, y_j), for $i, j = 0, \cdots, n$,*

$$\begin{aligned}
D_x D_y S(x_i, y_j) &= (D_x D_y + \frac{h^4}{720}D_x^5 D_y + \frac{h^4}{720}D_x D_y^5)u(x_i, y_j) + O(h^6), && (2.30)\\
D_x D_y^2 S(x_i, y_j) &= (D_x D_y^2 - \frac{h^4}{240}D_x D_y^6 + \frac{h^4}{720}D_x^5 D_y^2)u(x_i, y_j) + O(h^6), && (2.31)\\
D_x^2 D_y S(x_i, y_j) &= (D_x^2 D_y - \frac{h^4}{240}D_y D_x^6 + \frac{h^4}{720}D_y^5 D_x^2)u(x_i, y_j) + O(h^6), && (2.32)\\
D_x^2 D_y^2 S(x_i, y_j) &= (D_x^2 D_y^2 - \frac{h^4}{240}D_x^2 D_y^6 - \frac{h^4}{240}D_x^6 D_y^2)u(x_i, y_j) + O(h^6), && (2.33)\\
D_x D_y^3 S(x_i, y_j) &= (D_x D_y^3 - \frac{h^2}{12}D_x D_y^5 + \frac{h^4}{240}D_x D_y^7 + \frac{h^4}{720}D_x^5 D_y^3)u(x_i, y_j) + O(h^6), && (2.34)\\
D_y D_x^3 S(x_i, y_j) &= (D_y D_x^3 - \frac{h^2}{12}D_y D_x^5 + \frac{h^4}{240}D_y D_x^7 + \frac{h^4}{720}D_y^5 D_x^3)u(x_i, y_j) + O(h^6). && (2.35)
\end{aligned}$$

The proof of these relations is similar to that of Theorem 2.2.4. Besides the error results above, we also need the error equations for some other specific points.

Theorem 2.2.7. *If $u \in C^{10}(\bar{\Omega})$, then, at the points (x_i, τ_j^y), (τ_i^x, y_j) we have* *for* $i = 0, \cdots, n,\ j = 1, \cdots, n,$

$$S(x_i, \tau_j^y) = u(x_i, \tau_j^y) + \frac{h^6}{3072} D_x^6 u(x_i, \tau_j^y) + O(h^8), \tag{2.36}$$

$$D_x S(x_i, \tau_j^y) = (D_x + \frac{h^4}{720} D_x^5) u(x_i, \tau_j^y) + O(h^6), \tag{2.37}$$

$$D_x^2 S(x_i, \tau_j^y) = (D_x^2 - \frac{h^4}{240} D_x^6) u(x_i, \tau_j^y) + O(h^6), \tag{2.38}$$

$$D_x^3 S(x_i, \tau_j^y) = (D_x^3 - \frac{h^2}{12} D_x^5 + \frac{h^4}{240} D_x^7) u(x_i, \tau_j^y) + O(h^6), \tag{2.39}$$

$$D_y S(x_i, \tau_j^y) = (D_y - \frac{7h^4}{5760} D_y^5) u(x_i, \tau_j^y) + O(h^6), \tag{2.40}$$

$$D_y^2 S(x_i, \tau_j^y) = (D_y^2 + \frac{7h^4}{1920} D_y^6) u(x_i, \tau_j^y) + O(h^6), \tag{2.41}$$

$$D_y^3 S(x_i, \tau_j^y) = (D_y^3 + \frac{h^2}{24} D_y^5 - \frac{7h^4}{1920} D_y^7) u(x_i, \tau_j^y) + O(h^6), \tag{2.42}$$

$$D_x D_y S(x_i, \tau_j^y) = (D_x D_y + \frac{h^4}{720} D_x^5 D_y - \frac{7h^4}{5760} D_x D_y^5) u(x_i, \tau_j^y) + O(h^6), \tag{2.43}$$

$$D_x D_y^2 S(x_i, \tau_j^y) = (D_x D_y^2 + \frac{h^4}{720} D_x^5 D_y^2 + \frac{7h^4}{1920} D_x D_y^6) u(x_i, \tau_j^y) + O(h^6), \tag{2.44}$$

$$D_x^2 D_y S(x_i, \tau_j^y) = (D_x^2 D_y - \frac{h^4}{240} D_x^6 D_y - \frac{7h^4}{5760} D_x^2 D_y^5) u(x_i, \tau_j^y) + O(h^6), \tag{2.45}$$

and for $i = 1, \cdots, n,\ j = 0, \cdots, n,$

$$S(\tau_i^x, y_j) = u(\tau_i^x, y_j) + \frac{h^6}{3072} D_y^6 u(\tau_i^x, y_j) + O(h^8), \tag{2.46}$$

$$D_x S(\tau_i^x, y_j) = (D_x - \frac{7h^4}{5760} D_x^5) u(\tau_i^x, y_j) + O(h^6), \tag{2.47}$$

$$D_x^2 S(\tau_i^x, y_j) = (D_x^2 + \frac{7h^4}{1920} D_x^6) u(\tau_i^x, y_j) + O(h^6), \tag{2.48}$$

$$D_x^3 S(\tau_i^x, y_j) = (D_x^3 + \frac{h^2}{24} D_x^5 - \frac{7h^4}{1920} D_x^7) u(\tau_i^x, y_j) + O(h^6), \tag{2.49}$$

$$D_yS(\tau_i^x, y_j) = (D_y + \frac{h^4}{720}D_y^5)u(\tau_i^x, y_j) + O(h^6), \quad (2.50)$$

$$D_y^2S(\tau_i^x, y_j) = (D_y^2 - \frac{h^4}{240}D_y^6)u(\tau_i^x, y_j) + O(h^6), \quad (2.51)$$

$$D_y^3S(\tau_i^x, y_j) = (D_y^3 - \frac{h^2}{12}D_y^5 + \frac{h^4}{240}D_y^7)u(\tau_i^x, y_j) + O(h^6), \quad (2.52)$$

$$D_xD_yS(\tau_i^x, y_j) = (D_xD_y - \frac{7h^4}{5760}D_x^5D_y + \frac{h^4}{720}D_xD_y^5)u(\tau_i^x, y_j) + O(h^6), \quad (2.53)$$

$$D_xD_y^2S(\tau_i^x, y_j) = (D_xD_y^2 - \frac{7h^4}{5760}D_x^5D_y^2 - \frac{h^4}{240}D_xD_y^6)u(\tau_i^x, y_j) + O(h^6), \quad (2.54)$$

$$D_x^2D_yS(\tau_i^x, y_j) = (D_x^2D_y + \frac{7h^4}{1920}D_x^6D_y + \frac{h^4}{720}D_x^2D_y^5)u(\tau_i^x, y_j) + O(h^6). \quad (2.55)$$

Proof. We first prove equation (2.36). Notice that $\mathbb{I}_{xy}u - u = \mathbb{I}_x(\mathbb{I}_yu - u) + \mathbb{I}_xu - u$. At the points (x_i, τ_j^y), we have that $(\mathbb{I}_yu - u)(x_i, \tau_j^y) = 0$ and $(\mathbb{I}_xu - u)(x_i, \tau_j^y) = \frac{h^6}{3072}D_x^6u(x_i, \tau_j^y) + O(h^8)$ (see [43], page 18). Equation (2.36) follows. Relations (2.37)-(2.39), (2.39) and (2.50)-(2.52) can be shown similarly.

To prove equation (2.43), we notice that

$$\begin{aligned}
&D_xD_yS(x_0, \tau_j^y) - D_xD_yu(x_0, \tau_j^y) \\
&= D_xD_y\mathbb{I}_{xy}u(x_0, \tau_j^y) - D_xD_yu(x_0, \tau_j^y) \\
&= D_x\mathbb{I}_xD_y\mathbb{I}_yu(x_0, \tau_j^y) - D_xD_yu(x_0, \tau_j^y) \\
&= D_x\mathbb{I}_x(D_y\mathbb{I}_yu(x_0, \tau_j^y) - D_yu(x_0, \tau_j^y)) + D_x\mathbb{I}_xD_yu(x_0, \tau_j^y) - D_xD_yu(x_0, \tau_j^y) \\
&= D_x\mathbb{I}_x(D_y\mathbb{I}_yu(x_0, \tau_j^y) - D_yu(x_0, \tau_j^y)) - D_x(D_y\mathbb{I}_yu(x_0, \tau_j^y) - D_yu(x_0, \tau_j^y)) \\
&\quad + D_x(D_y\mathbb{I}_yu(x_0, \tau_j^y) - D_yu(x_0, \tau_j^y)) + D_y(D_x\mathbb{I}_xu(x_0, \tau_j^y) - D_xu(x_0, \tau_j^y)) \\
&= D_x\mathbb{I}_xw(x_0, \tau_j^y) - D_xw(x_0, \tau_j^y) + D_xw(x_0, \tau_j^y) + D_y(D_x\mathbb{I}_xu(x_0, \tau_j^y) - D_xu(x_0, \tau_j^y)),
\end{aligned}$$

where $w(x_0, \tau_j^y) \equiv D_y\mathbb{I}_yu(x_0, \tau_j^y) - D_yu(x_0, \tau_j^y)$. From Theorem 1 in [43] (page 15) and Theorem 2 in [43] (page 18), we obtain

$$\begin{aligned}
&w(x_0, \tau_j^y) = -\frac{7h^4}{5760}D_y^5u(x_0, \tau_j^y) + O(h^6), \\
&D_x\mathbb{I}_xw(x_0, \tau_j^y) - D_xw(x_0, \tau_j^y) = \frac{h^4}{720}D_x^5w(x_0, \tau_j^y) + O(h^6) = O(h^6), \\
&D_x\mathbb{I}_xu(x_0, \tau_j^y) - D_xu(x_0, \tau_j^y) = \frac{h^4}{720}D_x^5u(x_0, \tau_j^y) + O(h^6).
\end{aligned}$$

Then we have

$$D_xD_yS(x_0,\tau_j^y) - D_xD_yu(x_0,\tau_j^y) = (-\frac{7h^4}{5760}D_xD_y^5 + \frac{h^4}{720}D_x^5D_y)u(x_0,\tau_j^y) + O(h^6).$$

The remaining relations can be proved in a similar manner. □

After discussing the errors of the interpolant S at collocation points and grid points, we present some results about the errors at other points.

Theorem 2.2.8. *If $u \in C^{10}(\bar{\Omega})$, then S satisfies the global error bounds*

$$\left\|D_x^kS - D_x^ku\right\|_\infty = O(h^{5-k}), \tag{2.56}$$

$$\left\|D_y^kS - D_y^ku\right\|_\infty = O(h^{5-k}), \tag{2.57}$$

for $k = 0, \cdots, 4$.

The theorem trivially extends Theorem 3 in [43] (page 19). The next theorem indicates the corresponding cross-derivative results.

Theorem 2.2.9. *If $u \in C^{10}(\bar{\Omega})$, then S satisfies the global error bounds*

$$\left\|D_x^kD_y^lS - D_x^kD_y^lu\right\|_\infty = O(h^{5-\max\{k,l\}}), \quad 0 \le k+l \le 4. \tag{2.58}$$

Proof. We give a proof for the special case when $k = l = 1$. Other cases can be shown similarly. Notice that

$$\begin{aligned}
&D_xD_yS(x,y) - D_xD_yu(x,y)\\
&= D_xD_y\mathbb{I}_{xy}u(x,y) - D_xD_yu(x,y)\\
&= D_x\mathbb{I}_xD_y\mathbb{I}_yu(x,y) - D_xD_yu(x,y)\\
&= D_x\mathbb{I}_x[D_y\mathbb{I}_yu(x,y) - D_yu(x,y)] + D_x\mathbb{I}_xD_yu(x,y) - D_xD_yu(x,y)\\
&= [D_x\mathbb{I}_xw(x,y) - D_xw(x,y)] + D_xw(x,y) + D_x\mathbb{I}_xD_yu(x,y) - D_xD_yu(x,y),
\end{aligned}$$

where $w(x,y) \equiv D_y \mathbb{I}_y u(x,y) - D_y u(x,y)$. By Theorem 3 in [43] (page 19), we have

$$\begin{aligned} w(x,y) &= O(h^4), \\ D_x \mathbb{I}_x w(x,y) - D_x w(x,y) &= O(h^4), \\ D_x \mathbb{I}_x D_y u(x,y) - D_x D_y u(x,y) &= O(h^4). \end{aligned}$$

Therefore, we have

$$D_x D_y S(x,y) - D_x D_y u(x,y) = O(h^4).$$

$\square$

After the error analysis of S, we derive some accurate approximations using S to the derivatives of u, which later help derive high order collocation approximations to u. For simplicity, we introduce the discrete difference operators at midpoints,

$$\begin{aligned} \delta_x w_{i,j} &\equiv w_{i-1,j} - 2w_{i,j} + w_{i+1,j}, \quad \text{for } i,j = 2,\cdots,n-1, \\ \delta_y w_{i,j} &\equiv w_{i,j-1} - 2w_{i,j} + w_{i,j+1}, \quad \text{for } i,j = 2,\cdots,n-1, \\ \delta_x^2 w_{i,j} &\equiv w_{i-2,j} - 4w_{i-1,j} + 6w_{i,j} - 4w_{i+1,j} + w_{i+2,j}, \quad \text{for } i,j = 3,\cdots,n-2, \\ \delta_y^2 w_{i,j} &\equiv w_{i,j-2} - 4w_{i,j-1} + 6w_{i,j} - 4w_{i,j+1} + w_{i,j+2}, \quad \text{for } i,j = 3,\cdots,n-2. \end{aligned}$$

Moreover, we define the notation in which $\delta_{x,y} D_x^k D_y^l S_{i,j}$ denotes $\delta_{x,y}(D_x^k D_y^l S)_{i,j}$, and $\delta_{x,y}^2 D_x^k D_y^l S_{i,j}$ denotes $\delta_{x,y}^2 (D_x^k D_y^l S)_{i,j}$, for $i,j \geq 0$.

Theorem 2.2.10. *If $u \in C^{10}(\bar{\Omega})$, then, for $i,j = 3,\cdots,n-2$, we have for $k = 5,\cdots,8$,*

$$D_x^k u_{i,j} = \frac{\delta_x^2 D_x^{k-4} S_{i,j}}{h^4} + O(h^2), \tag{2.59}$$

$$D_y^k u_{i,j} = \frac{\delta_y^2 D_y^{k-4} S_{i,j}}{h^4} + O(h^2), \tag{2.60}$$

for $k = 5,6,7$, $l = 1,2,3$,

$$D_x^k D_y^l u_{i,j} = \frac{\delta_x^2 D_x^{k-4} D_y^l S_{i,j}}{h^4} + O(h^2), \tag{2.61}$$

$$D_x^l D_y^k u_{i,j} = \frac{\delta_y^2 D_x^l D_y^{k-4} S_{i,j}}{h^4} + O(h^2), \tag{2.62}$$

and

$$\begin{aligned}
D_x^5 u_{i,j} &= \frac{8\delta_x D_x^3 S_{i,j} - \delta_x^2 D_x^3 S_{i,j}}{8h^2} + O(h^4), && (2.63)\\
D_x^6 u_{i,j} &= \frac{24\delta_x D_x^4 S_{i,j} - \delta_x^2 D_x^4 S_{i,j}}{24h^2} + O(h^4), && (2.64)\\
D_x^5 D_y u_{i,j} &= \frac{8\delta_x D_x^3 D_y S_{i,j} - \delta_x^2 D_x^3 D_y S_{i,j}}{8h^2} + O(h^4), && (2.65)\\
D_y^5 u_{i,j} &= \frac{8\delta_y D_y^3 S_{i,j} - \delta_y^2 D_y^3 S_{i,j}}{8h^2} + O(h^4), && (2.66)\\
D_y^6 u_{i,j} &= \frac{24\delta_y D_y^4 S_{i,j} - \delta_y^2 D_y^4 S_{i,j}}{24h^2} + O(h^4), && (2.67)\\
D_y^5 D_x u_{i,j} &= \frac{8\delta_y D_y^3 D_x S_{i,j} - \delta_y^2 D_y^3 D_x S_{i,j}}{8h^2} + O(h^4). && (2.68)
\end{aligned}$$

Proof. We first consider the proof of (2.59) when $k = 8$. Since $u \in C^{10}(\bar{\Omega})$, we have the finite difference (FD) relations

$$\begin{aligned}
D_x^8 u_{i,j} &= \frac{\delta_x^2 D_x^4 u_{i,j}}{h^4} + O(h^2), && (2.69)\\
D_x^{10} u_{i,j} &= \frac{\delta_x^2 D_x^6 u_{i,j}}{h^4} + O(h^2), && (2.70)\\
D_x^{10} u_{i,j} &= \frac{\delta_x D_x^8 u_{i,j}}{h^2} + O(h^2). && (2.71)
\end{aligned}$$

By equation (2.15), we have

$$D_x^4 u_{k,j} = D_x^4 S_{k,j} + \frac{h^2}{24} D_x^6 u_{k,j} - \frac{7h^4}{5760} D_x^8 u_{k,j} + O(h^6), \tag{2.72}$$

for $k = i-2, i-1, i, i+1, i+2$. Applying (2.72) to the right-hand side of (2.69), then (2.70) and (2.71), we obtain

$$\begin{aligned}
D_x^8 u_{i,j} &= \frac{\delta_x^2 D_x^4 S_{i,j}}{h^4} + \frac{h^2}{24}\frac{\delta_x^2 D_x^6 u_{i,j}}{h^4} - \frac{7}{5760}\delta_x^2 D_x^8 u_{i,j} + O(h^2),\\
&= \frac{\delta_x^2 D_x^4 S_{i,j}}{h^4} + \frac{h^2}{24} D_x^{10} u_{i,j} - \frac{7h^2}{5760}\delta_x D_x^{10} u_{i,j} + O(h^2),\\
&= \frac{\delta_x^2 D_x^4 S_{i,j}}{h^4} + O(h^2).
\end{aligned}$$

This verifies the asymptotic relation (2.59) when $k = 8$. The other cases of it and the relations (2.60)-(2.62) can be shown similarly. To prove (2.63), we use the FD relation

$$D_x^5 u_{i,j} = \frac{\delta_x D_x^3 u_{i,j}}{h^2} - \frac{h^2}{12} D_x^7 u_{i,j} + O(h^4).$$

By equation (2.13), we have

$$\begin{aligned} D_x^5 u_{i,j} &= \frac{\delta_x D_x^3 S_{i,j}}{h^2} - \frac{h^2}{24}\frac{\delta_x D_x^5 u_{i,j}}{h^2} + \frac{7h^4}{1920}\frac{\delta_x D_x^7 u_{i,j}}{h^2} - \frac{h^2}{12} D_x^7 u_{i,j} + O(h^4), \\ &= \frac{\delta_x D_x^3 S_{i,j}}{h^2} - \frac{h^2}{24} D_x^7 u_{i,j} + \frac{7h^4}{1920} D_x^9 u_{i,j} - \frac{h^2}{12} D_x^7 u_{i,j} + O(h^4), \\ &= \frac{\delta_x D_x^3 S_{i,j}}{h^2} - \frac{h^2}{8} D_x^7 u_{i,j} + O(h^4). \end{aligned} \tag{2.73}$$

Substituting the right-hand side of (2.59) with $k = 7$ for $D_x^7 u_{i,j}$ in (2.73), we obtain the relation (2.63). The rest of the relations can be proved similarly. □

The results in Theorems 2.2.3, 2.2.4, and 2.2.10 for the interior collocation points in T_i lead to the following corollary, which gives approximations to the derivatives of u at the midpoints using the interpolant S and its derivatives.

Corollary 2.2.11. *If $u \in C^{10}(\bar{\Omega})$, then, for $i, j = 3, \cdots, n-2$,*

$$D_x u_{i,j} = D_x S_{i,j} + \frac{7\delta_x^2 D_x S_{i,j}}{5760} + O(h^6), \tag{2.74}$$

$$D_x^2 u_{i,j} = D_x^2 S_{i,j} - \frac{7\delta_x^2 D_x^2 S_{i,j}}{1920} + O(h^6), \tag{2.75}$$

$$D_x^3 u_{i,j} = D_x^3 S_{i,j} - \frac{80\delta_x D_x^3 S_{i,j} - 17\delta_x^2 D_x^3 S_{i,j}}{1920} + O(h^6), \tag{2.76}$$

$$D_x^4 u_{i,j} = D_x^4 S_{i,j} + \frac{240\delta_x D_x^4 S_{i,j} - 17\delta_x^2 D_x^4 S_{i,j}}{5760} + O(h^6), \tag{2.77}$$

$$D_y u_{i,j} = D_y S_{i,j} + \frac{7\delta_y^2 D_y S_{i,j}}{5760} + O(h^6), \tag{2.78}$$

$$D_y^2 u_{i,j} = D_y^2 S_{i,j} - \frac{7\delta_y^2 D_y^2 S_{i,j}}{1920} + O(h^6), \tag{2.79}$$

$$D_y^3 u_{i,j} = D_y^3 S_{i,j} - \frac{80\delta_y D_y^3 S_{i,j} - 17\delta_y^2 D_y^3 S_{i,j}}{1920} + O(h^6), \tag{2.80}$$

$$D_y^4 u_{i,j} = D_y^4 S_{i,j} + \frac{240\delta_y D_y^4 S_{i,j} - 17\delta_y^2 D_y^4 S_{i,j}}{5760} + O(h^6), \tag{2.81}$$

$$D_xD_yu_{i,j} = D_xD_yS_{i,j} + \frac{7(\delta_x^2+\delta_y^2)D_xD_yS_{i,j}}{5760} + O(h^6), \tag{2.82}$$

$$D_xD_y^2u_{i,j} = D_xD_y^2S_{i,j} + \frac{7\delta_x^2D_xD_y^2S_{i,j}}{5760} - \frac{7\delta_y^2D_xD_y^2S_{i,j}}{1920} + O(h^6), \tag{2.83}$$

$$D_yD_x^2u_{i,j} = D_yD_x^2S_{i,j} + \frac{7\delta_y^2D_yD_x^2S_{i,j}}{5760} - \frac{7\delta_x^2D_yD_x^2S_{i,j}}{1920} + O(h^6), \tag{2.84}$$

$$D_x^2D_y^2u_{i,j} = D_x^2D_y^2S_{i,j} - \frac{7(\delta_x^2+\delta_y^2)D_x^2D_y^2S_{i,j}}{1920} + O(h^6), \tag{2.85}$$

$$D_xD_y^3u_{i,j} = D_xD_y^3S_{i,j} - \frac{(240\delta_yD_xD_y^3 - 17\delta_y^2D_xD_y^3 - 7\delta_x^2D_xD_y^3)S_{i,j}}{5760} + O(h^6), \tag{2.86}$$

$$D_yD_x^3u_{i,j} = D_yD_x^3S_{i,j} - \frac{(240\delta_xD_yD_x^3 - 17\delta_x^2D_yD_x^3 - 7\delta_y^2D_yD_x^3)S_{i,j}}{5760} + O(h^6). \tag{2.87}$$

Clearly, some of these relations are not applicable at the points in $\Pi_\partial \cup \Pi_{i\partial}$. To obtain high order approximations for the derivatives of u at these points, we derive some relations for the high derivatives of u.

Lemma 2.2.12. *If $u \in C^{10}(\bar{\Omega})$, then, for $k = 5, 6, 7, 8$, $l = 0, 1, 2, 3$, we have for $i = 1, 2$,*

$$\begin{aligned} D_x^kD_y^lu(\tau_i^x, y) &= 2D_x^kD_y^lu(\tau_{i+1}^x, y) - D_x^kD_y^lu(\tau_{i+2}^x, y) + O(h^2), \\ &= 4D_x^kD_y^lu(\tau_{i+1}^x, y) - 6D_x^kD_y^lu(\tau_{i+2}^x, y) + 4D_x^kD_y^lu(\tau_{i+3}^x, y) \\ &\quad -D_x^kD_y^lu(\tau_{i+4}^x, y) + O(h^4), \end{aligned} \tag{2.88}$$

for $i = n-1, n$,

$$\begin{aligned} D_x^kD_y^lu(\tau_i^x, y) &= 2D_x^kD_y^lu(\tau_{i-1}^x, y) - D_x^kD_y^lu(\tau_{i-2}^x, y) + O(h^2), \\ &= 4D_x^kD_y^lu(\tau_{i-1}^x, y) - 6D_x^kD_y^lu(\tau_{i-2}^x, y) + 4D_x^kD_y^lu(\tau_{i-3}^x, y) \\ &\quad -D_x^kD_y^lu(\tau_{i-4}^x, y) + O(h^4), \end{aligned} \tag{2.89}$$

for $i = 0$,

$$\begin{aligned} D_x^kD_y^lu(x_i, y) &= \frac{1}{2}(3D_x^kD_y^lu(\tau_{i+1}^x, y) - D_x^kD_y^lu(\tau_{i+2}^x), y) + O(h^2), \\ &= \frac{1}{16}(35D_x^kD_y^lu(\tau_{i+1}^x, y) - 35D_x^kD_y^lu(\tau_{i+2}^x, y) + 21D_x^kD_y^lu(\tau_{i+3}^x, y) \\ &\quad -5D_x^kD_y^lu(\tau_{i+4}^x, y)) + O(h^4), \end{aligned} \tag{2.90}$$

and for $i = n$,

$$
\begin{aligned}
D_x^k D_y^l u(x_i, y) &= \frac{1}{2}(3D_x^k D_y^l u(\tau_{i-1}^x, y) - D_x^k D_y^l u(\tau_{i-2}^x), y) + O(h^2), \\
&= \frac{1}{16}(35D_x^k D_y^l u(\tau_{i-1}^x, y) - 35D_x^k D_y^l u(\tau_{i-2}^x, y) + 21D_x^k D_y^l u(\tau_{i-3}^x, y) \\
&\quad -5D_x^k D_y^l u(\tau_{i-4}^x, y)) + O(h^4).
\end{aligned}
\tag{2.91}
$$

Similar relations hold for $D_x^l D_y^k u(x, \tau_i^y)$ and $D_x^l D_y^k u(x, y_i)$.

Proof. These relations can be trivially derived from linear combinations of Taylor's expansions. □

For convenience, we adopt the notation from [43], more specifically, γ_0, γ_1, γ_2, γ_3, γ_4, γ_5 and γ_6 which are either 0, 1, 2, 3, 4, 5, and 6, respectively, (in which case they are used to refer to collocation points close to the left or bottom boundary), or $n+1$, n, $n-1$, $n-2$, $n-3$, $n-4$, and $n-5$, respectively, (in which case they are used to refer to collocation points close to the right or top boundary). We also adopt the notation of some discrete operators from [43]

$$
\begin{aligned}
\eta_l^x z_{\gamma_i,j} \equiv\ & \frac{1}{l}[4(l \cdot \delta_x z_{\gamma_{i+1},j} - \delta_x^2 z_{\gamma_{i+1},j}) - 6(l \cdot \delta_x z_{\gamma_{i+2},j} - \delta_x^2 z_{\gamma_{i+2},j}) \\
& +4(l \cdot \delta_x z_{\gamma_{i+3},j} - \delta_x^2 z_{\gamma_{i+3},j}) - (l \cdot \delta_x z_{\gamma_{i+4},j} - \delta_x^2 z_{\gamma_{i+4},j})],
\end{aligned}
$$

$$
\begin{aligned}
\zeta_l^x z_{\gamma_i,j} \equiv\ & \frac{1}{l}[10(l \cdot \delta_x z_{\gamma_{i+1},j} - \delta_x^2 z_{\gamma_{i+1},j}) - 20(l \cdot \delta_x z_{\gamma_{i+2},j} - \delta_x^2 z_{\gamma_{i+2},j}) \\
& +15(l \cdot \delta_x z_{\gamma_{i+3},j} - \delta_x^2 z_{\gamma_{i+3},j}) - 4(l \cdot \delta_x z_{\gamma_{i+4},j} - \delta_x^2 z_{\gamma_{i+4},j})],
\end{aligned}
$$

$$
\begin{aligned}
\theta_l^x z_{\gamma_i,j} \equiv\ & \frac{1}{16l}[231(l \cdot \delta_x z_{\gamma_{i+1},j} - \delta_x^2 z_{\gamma_{i+1},j}) - 495(l \cdot \delta_x z_{\gamma_{i+2},j} - \delta_x^2 z_{\gamma_{i+2},j}) \\
& +385(l \cdot \delta_x z_{\gamma_{i+3},j} - \delta_x^2 z_{\gamma_{i+3},j}) - 105(l \cdot \delta_x z_{\gamma_{i+4},j} - \delta_x^2 z_{\gamma_{i+4},j})],
\end{aligned}
$$

$$
\begin{aligned}
\eta_l^y z_{j,\gamma_i} \equiv\ & \frac{1}{l}[4(l \cdot \delta_y z_{j,\gamma_{i+1}} - \delta_y^2 z_{j,\gamma_{i+1}}) - 6(l \cdot \delta_y z_{j,\gamma_{i+2}} - \delta_y^2 z_{j,\gamma_{i+2}}) \\
& +4(l \cdot \delta_y z_{j,\gamma_{i+3}} - \delta_y^2 z_{j,\gamma_{i+3}}) - (l \cdot \delta_y z_{j,\gamma_{i+4}} - \delta_y^2 z_{j,\gamma_{i+4}})],
\end{aligned}
$$

$$\zeta_l^y z_{j,\gamma_i} \equiv \frac{1}{l}[10(l\cdot\delta_y z_{j,\gamma_{i+1}} - \delta_y^2 z_{j,\gamma_{i+1}}) - 20(l\cdot\delta_y z_{j,\gamma_{i+2}} - \delta_y^2 z_{j,\gamma_{i+2}}) +15(l\cdot\delta_y z_{j,\gamma_{i+3}} - \delta_y^2 z_{j,\gamma_{i+3}}) - 4(l\cdot\delta_y z_{j,\gamma_{i+4}} - \delta_y^2 z_{j,\gamma_{i+4}})],$$

$$\theta_l^y z_{j,\gamma_i} \equiv \frac{1}{16l}[231(l\cdot\delta_y z_{j,\gamma_{i+1}} - \delta_y^2 z_{j,\gamma_{i+1}}) - 495(l\cdot\delta_y z_{j,\gamma_{i+2}} - \delta_y^2 z_{j,\gamma_{i+2}}) +385(l\cdot\delta_y z_{j,\gamma_{i+3}} - \delta_y^2 z_{j,\gamma_{i+3}}) - 105(l\cdot\delta_y z_{j,\gamma_{i+4}} - \delta_y^2 z_{j,\gamma_{i+4}})],$$

where $i = 0, 1, 2$, $j = 0, \cdots, n+1$, l is 6 or 24, and z is some function of two variables. We remind readers that $z_{i,j} \equiv z(\tau_i^x, \tau_j^y)$.

Lemma 2.2.13. *If $u \in C^{10}(\bar{\Omega})$, then at points in Π_{lr}, for $j = 3, \cdots, n-2$, the derivatives of S and u satisfy, for $k = 5, 6, 7, 8$,*

$$D_x^k u_{\gamma_1,j} = \frac{1}{h^4}(3\delta_x^2 D_x^{k-4} S_{\gamma_3,j} - 2\delta_x^2 D_x^{k-4} S_{\gamma_4,j}) + O(h^2), \tag{2.92}$$

$$D_x^k u_{\gamma_2,j} = \frac{1}{h^4}(2\delta_x^2 D_x^{k-4} S_{\gamma_3,j} - \delta_x^2 D_x^{k-4} S_{\gamma_4,j}) + O(h^2), \tag{2.93}$$

$$D_y^k u_{\{\gamma_1,\gamma_2\},j} = \frac{\delta_y^2 D_y^{k-4} S_{\{\gamma_1,\gamma_2\},j}}{h^4} + O(h^2), \tag{2.94}$$

$$D_y^5 u_{\{\gamma_1,\gamma_2\},j} = \frac{8\delta_y D_y^3 S_{\{\gamma_1,\gamma_2\},j} - \delta_y^2 D_y^3 S_{\{\gamma_1,\gamma_2\},j}}{8h^2} + O(h^4), \tag{2.95}$$

$$D_y^6 u_{\{\gamma_1,\gamma_2\},j} = \frac{24\delta_y D_y^4 S_{\{\gamma_1,\gamma_2\},j} - \delta_y^2 D_y^4 S_{\{\gamma_1,\gamma_2\},j}}{24h^2} + O(h^4). \tag{2.96}$$

and

$$D_x^5 u_{\gamma_1,j} = \frac{1}{h^2}\zeta_8^x D_x^3 S_{\gamma_2,j} + O(h^4), \tag{2.97}$$

$$D_x^5 u_{\gamma_2,j} = \frac{1}{h^2}\eta_8^x D_x^3 S_{\gamma_2,j} + O(h^4), \tag{2.98}$$

$$D_x^6 u_{\gamma_1,j} = \frac{1}{h^2}\zeta_{24}^x D_x^4 S_{\gamma_2,j} + O(h^4), \tag{2.99}$$

$$D_x^6 u_{\gamma_2,j} = \frac{1}{h^2}\eta_{24}^x D_x^4 S_{\gamma_2,j} + O(h^4), \tag{2.100}$$

and the cross-derivatives of S and u satisfy, for $k = 5, 6, 7,\ l = 1, 2, 3$,

$$
\begin{aligned}
D_x^k D_y^l u_{\gamma_1,j} &= \frac{1}{h^4}(3\delta_x^2 D_x^{k-4} D_y^l S_{\gamma_3,j} - 2\delta_x^2 D_x^{k-4} D_y^l S_{\gamma_4,j}) + O(h^2), && (2.101)\\
D_x^k D_y^l u_{\gamma_2,j} &= \frac{1}{h^4}(2\delta_x^2 D_x^{k-4} D_y^l S_{\gamma_3,j} - \delta_x^2 D_x^{k-4} D_y^l S_{\gamma_4,j}) + O(h^2), && (2.102)\\
D_x^l D_y^k u_{\{\gamma_1,\gamma_2\},j} &= \frac{\delta_y^2 D_x^l D_y^{k-4} S_{\{\gamma_1,\gamma_2\},j}}{h^4} + O(h^2), && (2.103)\\
&= \frac{8\delta_y D_x^l D_y^{k-2} S_{\{\gamma_1,\gamma_2\},j} - \delta_y^2 D_x^l D_y^{k-2} S_{\{\gamma_1,\gamma_2\},j}}{8h^2} + O(h^4), && (2.104)
\end{aligned}
$$

and

$$
\begin{aligned}
D_x^5 D_y u_{\gamma_1,j} &= \frac{1}{h^2}\zeta_8^x D_x^3 D_y S_{\gamma_2,j} + O(h^4), && (2.105)\\
D_x^5 D_y u_{\gamma_2,j} &= \frac{1}{h^2}\eta_8^x D_x^3 D_y S_{\gamma_2,j} + O(h^4). && (2.106)
\end{aligned}
$$

Similar relations hold for points in Π_{ud}.

At points in Π_c, the derivatives of S and u satisfy, for $k = 5, 6, 7, 8$,

$$
\begin{aligned}
D_x^k u_{\gamma_1,\{\gamma_1,\gamma_2\}} &= \frac{1}{h^4}(3\delta_x^2 D_x^{k-4} S_{\gamma_3,\{\gamma_1,\gamma_2\}} - 2\delta_x^2 D_x^{k-4} S_{\gamma_4,\{\gamma_1,\gamma_2\}}) + O(h^2), && (2.107)\\
D_x^k u_{\gamma_2,\{\gamma_1,\gamma_2\}} &= \frac{1}{h^4}(2\delta_x^2 D_x^{k-4} S_{\gamma_3,\{\gamma_1,\gamma_2\}} - \delta_x^2 D_x^{k-4} S_{\gamma_4,\{\gamma_1,\gamma_2\}}) + O(h^2), && (2.108)\\
D_y^k u_{\{\gamma_1,\gamma_2\},\gamma_1} &= \frac{1}{h^4}(3\delta_y^2 D_y^{k-4} S_{\{\gamma_1,\gamma_2\},\gamma_3} - 2\delta_y^2 D_y^{k-4} S_{\{\gamma_1,\gamma_2\},\gamma_4}) + O(h^2), && (2.109)\\
D_y^k u_{\{\gamma_1,\gamma_2\},\gamma_2} &= \frac{1}{h^4}(2\delta_y^2 D_y^{k-4} S_{\{\gamma_1,\gamma_2\},\gamma_3} - \delta_y^2 D_y^{k-4} S_{\{\gamma_1,\gamma_2\},\gamma_4}) + O(h^2), && (2.110)
\end{aligned}
$$

and

$$
\begin{aligned}
D_x^5 u_{\gamma_1,\{\gamma_1,\gamma_2\}} &= \frac{1}{h^2}\zeta_8^x D_x^3 S_{\gamma_2,\{\gamma_1,\gamma_2\}} + O(h^4), && (2.111)\\
D_x^5 u_{\gamma_2,\{\gamma_1,\gamma_2\}} &= \frac{1}{h^2}\eta_8^x D_x^3 S_{\gamma_2,\{\gamma_1,\gamma_2\}} + O(h^4), && (2.112)\\
D_y^5 u_{\{\gamma_1,\gamma_2\},\gamma_1} &= \frac{1}{h^2}\zeta_8^y D_y^3 S_{\{\gamma_1,\gamma_2\},\gamma_2} + O(h^4), && (2.113)\\
D_y^5 u_{\{\gamma_1,\gamma_2\},\gamma_2} &= \frac{1}{h^2}\eta_8^y D_y^3 S_{\{\gamma_1,\gamma_2\},\gamma_2} + O(h^4), && (2.114)\\
D_x^6 u_{\gamma_1,\{\gamma_1,\gamma_2\}} &= \frac{1}{h^2}\zeta_{24}^x D_x^4 S_{\gamma_2,\{\gamma_1,\gamma_2\}} + O(h^4), && (2.115)\\
D_x^6 u_{\gamma_2,\{\gamma_1,\gamma_2\}} &= \frac{1}{h^2}\eta_{24}^x D_x^4 S_{\gamma_2,\{\gamma_1,\gamma_2\}} + O(h^4), && (2.116)\\
D_y^6 u_{\{\gamma_1,\gamma_2\},\gamma_1} &= \frac{1}{h^2}\zeta_{24}^y D_y^4 S_{\{\gamma_1,\gamma_2\},\gamma_2} + O(h^4), && (2.117)\\
D_y^6 u_{\{\gamma_1,\gamma_2\},\gamma_2} &= \frac{1}{h^2}\eta_{24}^y D_y^4 S_{\{\gamma_1,\gamma_2\},\gamma_2} + O(h^4), && (2.118)
\end{aligned}
$$

and the cross-derivatives of S and u satisfy, for $k = 5, 6, 7$, $l = 1, 2, 3$,

$$D_x^k D_y^l u_{\gamma_1,\{\gamma_1,\gamma_2\}} = \frac{1}{h^4}(3\delta_x^2 D_x^{k-4} D_y^l S_{\gamma_3,\{\gamma_1,\gamma_2\}} - 2\delta_x^2 D_x^{k-4} D_y^l S_{\gamma_4,\{\gamma_1,\gamma_2\}}) + O(h^2), \quad (2.119)$$

$$D_x^k D_y^l u_{\gamma_2,\{\gamma_1,\gamma_2\}} = \frac{1}{h^4}(2\delta_x^2 D_x^{k-4} D_y^l S_{\gamma_3,\{\gamma_1,\gamma_2\}} - \delta_x^2 D_x^{k-4} D_y^l S_{\gamma_4,\{\gamma_1,\gamma_2\}}) + O(h^2), \quad (2.120)$$

$$D_x^l D_y^k u_{\{\gamma_1,\gamma_2\},\gamma_1} = \frac{1}{h^4}(3\delta_y^2 D_x^l D_y^{k-4} S_{\{\gamma_1,\gamma_2\},\gamma_3} - 2\delta_y^2 D_x^l D_y^{k-4} S_{\{\gamma_1,\gamma_2\},\gamma_4}) + O(h^2), \quad (2.121)$$

$$D_x^l D_y^k u_{\{\gamma_1,\gamma_2\},\gamma_2} = \frac{1}{h^4}(2\delta_y^2 D_x^l D_y^{k-4} S_{\{\gamma_1,\gamma_2\},\gamma_3} - \delta_y^2 D_x^l D_y^{k-4} S_{\{\gamma_1,\gamma_2\},\gamma_4}) + O(h^2), \quad (2.122)$$

and

$$D_x^5 D_y u_{\gamma_1,\{\gamma_1,\gamma_2\}} = \frac{1}{h^2}\zeta_8^x D_x^3 D_y S_{\gamma_2,\{\gamma_1,\gamma_2\}} + O(h^4), \quad (2.123)$$

$$D_x^5 D_y u_{\gamma_2,\{\gamma_1,\gamma_2\}} = \frac{1}{h^2}\eta_8^x D_x^3 D_y S_{\gamma_2,\{\gamma_1,\gamma_2\}} + O(h^4), \quad (2.124)$$

$$D_x D_y^5 u_{\{\gamma_1,\gamma_2\},\gamma_1} = \frac{1}{h^2}\zeta_8^y D_x D_y^3 S_{\{\gamma_1,\gamma_2\},\gamma_2} + O(h^4), \quad (2.125)$$

$$D_x D_y^5 u_{\{\gamma_1,\gamma_2\},\gamma_2} = \frac{1}{h^2}\eta_8^y D_x D_y^3 S_{\{\gamma_1,\gamma_2\},\gamma_2} + O(h^4). \quad (2.126)$$

Proof. We give the proof for the relation (2.93). The rest of relations can be shown similarly. When $k = 5$, from equations (2.88)-(2.89), we have

$$D_x^k u_{\gamma_2,j} = 2D_x^k u_{\gamma_3,j} - D_x^k u_{\gamma_4,j} + O(h^2).$$

Substituting (2.59) into the above relation for $D_x^k u_{\gamma_3,j}$ and $D_x^k u_{\gamma_4,j}$, we obtain relation (2.93). □

Combining Theorems 2.2.3, 2.2.4, and Lemma 2.2.13, we derive high order approximations for the derivatives of u at the interior-boundary collocation points using the interpolant S and its derivatives.

Corollary 2.2.14. *If $u \in C^{10}(\bar{\Omega})$, then* *at points in Π_{lr}, for $j = 3, \cdots, n-2$, the derivatives of S and u satisfy*

$$D_x u_{\gamma_1,j} = D_x S_{\gamma_1,j} + \frac{7}{5760}(3\delta_x^2 D_x S_{\gamma_3,j} - 2\delta_x^2 D_x S_{\gamma_4,j}) + O(h^6), \tag{2.127}$$

$$D_x^2 u_{\gamma_1,j} = D_x^2 S_{\gamma_1,j} - \frac{7}{1920}(3\delta_x^2 D_x^2 S_{\gamma_3,j} - 2\delta_x^2 D_x^2 S_{\gamma_4,j}) + O(h^6), \tag{2.128}$$

$$D_x^3 u_{\gamma_1,j} = D_x^3 S_{\gamma_1,j} - \frac{1}{24}\zeta_8^x D_x^3 S_{\gamma_2,j} + \frac{7}{1920}(3\delta_x^2 D_x^3 S_{\gamma_3,j} - 2\delta_x^2 D_x^3 S_{\gamma_4,j}) + O(h^6), \tag{2.129}$$

$$D_x^4 u_{\gamma_1,j} = D_x^4 S_{\gamma_1,j} + \frac{1}{24}\zeta_{24}^x D_x^4 S_{\gamma_2,j} - \frac{7}{5760}(3\delta_x^2 D_x^4 S_{\gamma_3,j} - 2\delta_x^2 D_x^4 S_{\gamma_4,j}) + O(h^6), \tag{2.130}$$

$$D_x u_{\gamma_2,j} = D_x S_{\gamma_2,j} + \frac{7}{5760}(2\delta_x^2 D_x S_{\gamma_3,j} - \delta_x^2 D_x S_{\gamma_4,j}) + O(h^6), \tag{2.131}$$

$$D_x^2 u_{\gamma_2,j} = D_x^2 S_{\gamma_2,j} - \frac{7}{1920}(2\delta_x^2 D_x^2 S_{\gamma_3,j} - \delta_x^2 D_x^2 S_{\gamma_4,j}) + O(h^6), \tag{2.132}$$

$$D_x^3 u_{\gamma_2,j} = D_x^3 S_{\gamma_2,j} - \frac{1}{24}\eta_8^x D_x^3 S_{\gamma_2,j} + \frac{7}{1920}(2\delta_x^2 D_x^3 S_{\gamma_3,j} - \delta_x^2 D_x^3 S_{\gamma_4,j}) + O(h^6), \tag{2.133}$$

$$D_x^4 u_{\gamma_2,j} = D_x^4 S_{\gamma_2,j} + \frac{1}{24}\eta_{24}^x D_x^4 S_{\gamma_2,j} - \frac{7}{5760}(2\delta_x^2 D_x^4 S_{\gamma_3,j} - \delta_x^2 D_x^4 S_{\gamma_4,j}) + O(h^6), \tag{2.134}$$

$$D_y u_{\{\gamma_1,\gamma_2\},j} = D_y S_{i,j} + \frac{7\delta_y^2 D_y S_{\{\gamma_1,\gamma_2\},j}}{5760} + O(h^6), \tag{2.135}$$

$$D_y^2 u_{\{\gamma_1,\gamma_2\},j} = D_y^2 S_{\{\gamma_1,\gamma_2\},j} - \frac{7\delta_y^2 D_y^2 S_{\{\gamma_1,\gamma_2\},j}}{1920} + O(h^6), \tag{2.136}$$

$$D_y^3 u_{\{\gamma_1,\gamma_2\},j} = D_y^3 S_{\{\gamma_1,\gamma_2\},j} - \frac{80\delta_y D_y^3 S_{\{\gamma_1,\gamma_2\},j} - 17\delta_y^2 D_y^3 S_{\{\gamma_1,\gamma_2\},j}}{1920} + O(h^6), \tag{2.137}$$

$$D_y^4 u_{\{\gamma_1,\gamma_2\},j} = D_y^4 S_{\{\gamma_1,\gamma_2\},j} + \frac{240\delta_y D_y^4 S_{\{\gamma_1,\gamma_2\},j} - 17\delta_y^2 D_y^4 S_{\{\gamma_1,\gamma_2\},j}}{5760} + O(h^6), \tag{2.138}$$

and the cross-derivatives of S and u satisfy

$$\begin{aligned} D_x D_y u_{\gamma_1,j} &= D_x D_y S_{\gamma_1,j} + \frac{7(3\delta_x^2 D_x D_y S_{\gamma_3,j} - 2\delta_x^2 D_x D_y S_{\gamma_4,j})}{5760} \\ &\quad + \frac{7\delta_y^2 D_x D_y S_{\gamma_1,j}}{5760} + O(h^6), \end{aligned} \tag{2.139}$$

$$\begin{aligned} D_x D_y^2 u_{\gamma_1,j} &= D_x D_y^2 S_{\gamma_1,j} + \frac{7(3\delta_x^2 D_x D_y S_{\gamma_3,j} - 2\delta_x^2 D_x D_y S_{\gamma_4,j})}{5760} \\ &\quad - \frac{7\delta_y^2 D_x D_y^2 S_{\gamma_1,j}}{1920} + O(h^6), \end{aligned} \tag{2.140}$$

$$\begin{aligned} D_x^2 D_y u_{\gamma_1,j} &= D_x^2 D_y S_{\gamma_1,j} - \frac{7(3\delta_x^2 D_x^2 D_y S_{\gamma_3,j} - 2\delta_x^2 D_x^2 D_y S_{\gamma_4,j})}{1920} \\ &\quad + \frac{7\delta_y^2 D_x^2 D_y S_{\gamma_1,j}}{5760} + O(h^6), \end{aligned} \tag{2.141}$$

$$
\begin{aligned}
D_x^2 D_y^2 u_{\gamma_1,j} &= D_x^2 D_y S_{\gamma_1,j} - \frac{7(3\delta_x^2 D_x^2 D_y^2 S_{\gamma_3,j} - 2\delta_x^2 D_x^2 D_y^2 S_{\gamma_4,j})}{1920} \\
&\quad - \frac{7\delta_y^2 D_x^2 D_y^2 S_{\gamma_1,j}}{1920} + O(h^6), && (2.142) \\
D_x D_y^3 u_{\gamma_1,j} &= D_x D_y^3 S_{\gamma_1,j} - \frac{80\delta_y D_x D_y^3 - 17\delta_y^2 D_x D_y^3}{1920} S_{\gamma_1,j} \\
&\quad + \frac{7(3\delta_x^2 D_x D_y^3 S_{\gamma_3,j} - 2\delta_x^2 D_x D_y^3 S_{\gamma_4,j})}{5760} + O(h^6), && (2.143) \\
D_x^3 D_y u_{\gamma_1,j} &= D_x^3 D_y S_{\gamma_1,j} - \frac{1}{24}\zeta_8^x D_x^3 D_y S_{\gamma_1,j} + \frac{7\delta_y^2 D_x D_y}{5760} S_{\gamma_1,j} \\
&\quad + \frac{7(3\delta_x^2 D_x^3 D_y S_{\gamma_3,j} - 2\delta_x^2 D_x^3 D_y S_{\gamma_4,j})}{1920} + O(h^6), && (2.144)
\end{aligned}
$$

$$
\begin{aligned}
D_x D_y u_{\gamma_2,j} &= D_x D_y S_{\gamma_2,j} + \frac{7(2\delta_x^2 D_x D_y S_{\gamma_3,j} - \delta_x^2 D_x D_y S_{\gamma_4,j})}{5760} \\
&\quad + \frac{7\delta_y^2 D_x D_y S_{\gamma_2,j}}{5760} + O(h^6), && (2.145) \\
D_x D_y^2 u_{\gamma_2,j} &= D_x D_y^2 S_{\gamma_2,j} + \frac{7(2\delta_x^2 D_x D_y S_{\gamma_3,j} - \delta_x^2 D_x D_y S_{\gamma_4,j})}{5760} \\
&\quad - \frac{7\delta_y^2 D_x D_y^2 S_{\gamma_2,j}}{1920} + O(h^6), && (2.146) \\
D_x^2 D_y u_{\gamma_2,j} &= D_x^2 D_y S_{\gamma_2,j} - \frac{7(2\delta_x^2 D_x^2 D_y S_{\gamma_3,j} - \delta_x^2 D_x^2 D_y S_{\gamma_4,j})}{1920} \\
&\quad + \frac{7\delta_y^2 D_x^2 D_y S_{\gamma_2,j}}{5760} + O(h^6), && (2.147)
\end{aligned}
$$

$$
\begin{aligned}
D_x^2 D_y^2 u_{\gamma_2,j} &= D_x^2 D_y^2 S_{\gamma_2,j} - \frac{7(2\delta_x^2 D_x^2 D_y^2 S_{\gamma_3,j} - \delta_x^2 D_x^2 D_y^2 S_{\gamma_4,j})}{1920} \\
&\quad - \frac{7\delta_y^2 D_x^2 D_y^2 S_{\gamma_2,j}}{1920} + O(h^6), && (2.148) \\
D_x D_y^3 u_{\gamma_2,j} &= D_x D_y^3 S_{\gamma_2,j} - \frac{80\delta_y D_x D_y^3 - 17\delta_y^2 D_x D_y^3}{1920} S_{\gamma_2,j} \\
&\quad + \frac{7(2\delta_x^2 D_x D_y^3 S_{\gamma_3,j} - \delta_x^2 D_x D_y^3 S_{\gamma_4,j})}{5760} + O(h^6), && (2.149) \\
D_x^3 D_y u_{\gamma_2,j} &= D_x^3 D_y S_{\gamma_2,j} - \frac{1}{24}\eta_8^x D_x^3 D_y S_{\gamma_2,j} + \frac{7\delta_y^2 D_x D_y}{5760} S_{\gamma_2,j} \\
&\quad + \frac{7(2\delta_x^2 D_x^3 D_y S_{\gamma_3,j} - \delta_x^2 D_x^3 D_y S_{\gamma_4,j})}{1920} + O(h^6). && (2.150)
\end{aligned}
$$

We have similar relations at points in Π_{ud}.

At points in Π_c, the derivatives of S and u satisfy

$$D_x u_{\gamma_1,\{\gamma_1,\gamma_2\}} = D_x S_{\gamma_1,\{\gamma_1,\gamma_2\}} + \frac{7}{5760}(3\delta_x^2 D_x S_{\gamma_3,\{\gamma_1,\gamma_2\}} - 2\delta_x^2 D_x S_{\gamma_4,\{\gamma_1,\gamma_2\}}) + O(h^6), \tag{2.151}$$

$$D_x^2 u_{\gamma_1,\{\gamma_1,\gamma_2\}} = D_x^2 S_{\gamma_1,\{\gamma_1,\gamma_2\}} - \frac{7}{1920}(3\delta_x^2 D_x^2 S_{\gamma_3,\{\gamma_1,\gamma_2\}} - 2\delta_x^2 D_x^2 S_{\gamma_4,\{\gamma_1,\gamma_2\}}) + O(h^6), \tag{2.152}$$

$$\begin{aligned} D_x^3 u_{\gamma_1,\{\gamma_1,\gamma_2\}} &= D_x^3 S_{\gamma_1,\{\gamma_1,\gamma_2\}} - \frac{1}{24}\zeta_8^x D_x^3 S_{\gamma_2,\{\gamma_1,\gamma_2\}} + \frac{7}{1920}(3\delta_x^2 D_x^3 S_{\gamma_3,\{\gamma_1,\gamma_2\}} \\ &\quad -2\delta_x^2 D_x^3 S_{\gamma_4,\{\gamma_1,\gamma_2\}}) + O(h^6), \end{aligned} \tag{2.153}$$

$$\begin{aligned} D_x^4 u_{\gamma_1,\{\gamma_1,\gamma_2\}} &= D_x^4 S_{\gamma_1,\{\gamma_1,\gamma_2\}} + \frac{1}{24}\zeta_{24}^x D_x^4 S_{\gamma_2,\{\gamma_1,\gamma_2\}} - \frac{7}{5760}(3\delta_x^2 D_x^4 S_{\gamma_3,\{\gamma_1,\gamma_2\}} \\ &\quad -2\delta_x^2 D_x^4 S_{\gamma_4,\{\gamma_1,\gamma_2\}}) + O(h^6), \end{aligned} \tag{2.154}$$

$$D_x u_{\gamma_2,\{\gamma_1,\gamma_2\}} = D_x S_{\gamma_2,\{\gamma_1,\gamma_2\}} + \frac{7}{5760}(2\delta_x^2 D_x S_{\gamma_3,\{\gamma_1,\gamma_2\}} - \delta_x^2 D_x S_{\gamma_4,\{\gamma_1,\gamma_2\}}) + O(h^6), \tag{2.155}$$

$$D_x^2 u_{\gamma_2,\{\gamma_1,\gamma_2\}} = D_x^2 S_{\gamma_2,\{\gamma_1,\gamma_2\}} - \frac{7}{1920}(2\delta_x^2 D_x^2 S_{\gamma_3,\{\gamma_1,\gamma_2\}} - \delta_x^2 D_x^2 S_{\gamma_4,\{\gamma_1,\gamma_2\}}) + O(h^6), \tag{2.156}$$

$$\begin{aligned} D_x^3 u_{\gamma_2,\{\gamma_1,\gamma_2\}} &= D_x^3 S_{\gamma_2,\{\gamma_1,\gamma_2\}} - \frac{1}{24}\eta_8^x D_x^3 S_{\gamma_2,\{\gamma_1,\gamma_2\}} + \frac{7}{1920}(2\delta_x^2 D_x^3 S_{\gamma_3,\{\gamma_1,\gamma_2\}} \\ &\quad -\delta_x^2 D_x^3 S_{\gamma_4,\{\gamma_1,\gamma_2\}}) + O(h^6), \end{aligned} \tag{2.157}$$

$$\begin{aligned} D_x^4 u_{\gamma_2,\{\gamma_1,\gamma_2\}} &= D_x^4 S_{\gamma_2,\{\gamma_1,\gamma_2\}} + \frac{1}{24}\eta_{24}^x D_x^4 S_{\gamma_2,\{\gamma_1,\gamma_2\}} - \frac{7}{5760}(2\delta_x^2 D_x^4 S_{\gamma_3,\{\gamma_1,\gamma_2\}} \\ &\quad -\delta_x^2 D_x^4 S_{\gamma_4,\{\gamma_1,\gamma_2\}}) + O(h^6), \end{aligned} \tag{2.158}$$

$$D_y u_{\{\gamma_1,\gamma_2\},\gamma_1} = D_y S_{\{\gamma_1,\gamma_2\},\gamma_1} + \frac{7}{5760}(3\delta_y^2 D_y S_{\{\gamma_1,\gamma_2\},\gamma_3} - 2\delta_y^2 D_y S_{\{\gamma_1,\gamma_2\},\gamma_4}) + O(h^6), \tag{2.159}$$

$$D_y^2 u_{\{\gamma_1,\gamma_2\},\gamma_1} = D_y^2 S_{\{\gamma_1,\gamma_2\},\gamma_1} - \frac{7}{1920}(3\delta_y^2 D_y^2 S_{\{\gamma_1,\gamma_2\},\gamma_3} - 2\delta_y^2 D_y^2 S_{\{\gamma_1,\gamma_2\},\gamma_4}) + O(h^6), \tag{2.160}$$

$$\begin{aligned} D_y^3 u_{\{\gamma_1,\gamma_2\},\gamma_1} &= D_y^3 S_{\{\gamma_1,\gamma_2\},\gamma_1} - \frac{1}{24}\zeta_8^y D_y^3 S_{\{\gamma_1,\gamma_2\},\gamma_2} + \frac{7}{1920}(3\delta_y^2 D_y^3 S_{\{\gamma_1,\gamma_2\},\gamma_3} \\ &\quad -2\delta_y^2 D_y^3 S_{\{\gamma_1,\gamma_2\},\gamma_4}) + O(h^6), \end{aligned} \tag{2.161}$$

$$\begin{aligned} D_y^4 u_{\{\gamma_1,\gamma_2\},\gamma_1} &= D_y^4 S_{\{\gamma_1,\gamma_2\},\gamma_1} + \frac{1}{24}\zeta_{24}^y D_y^4 S_{\{\gamma_1,\gamma_2\},\gamma_2} - \frac{7}{5760}(3\delta_y^2 D_y^4 S_{\{\gamma_1,\gamma_2\},\gamma_3} \\ &\quad -2\delta_y^2 D_y^4 S_{\{\gamma_1,\gamma_2\},\gamma_4}) + O(h^6), \end{aligned} \tag{2.162}$$

$$D_y u_{\{\gamma_1,\gamma_2\},\gamma_2} = D_y S_{\{\gamma_1,\gamma_2\},\gamma_2} + \frac{7}{5760}(2\delta_y^2 D_y S_{\{\gamma_1,\gamma_2\},\gamma_3} - \delta_y^2 D_y S_{\{\gamma_1,\gamma_2\},\gamma_4}) + O(h^6), \quad (2.163)$$

$$D_y^2 u_{\{\gamma_1,\gamma_2\},\gamma_2} = D_y^2 S_{\{\gamma_1,\gamma_2\},\gamma_2} - \frac{7}{1920}(2\delta_y^2 D_y^2 S_{\{\gamma_1,\gamma_2\},\gamma_3} - \delta_y^2 D_y^2 S_{\{\gamma_1,\gamma_2\},\gamma_4}) + O(h^6), \quad (2.164)$$

$$\begin{aligned} D_y^3 u_{\{\gamma_1,\gamma_2\},\gamma_2} &= D_y^3 S_{\{\gamma_1,\gamma_2\},\gamma_2} - \frac{1}{24}\eta_8^y D_y^3 S_{\{\gamma_1,\gamma_2\},\gamma_2} + \frac{7}{1920}(2\delta_y^2 D_y^3 S_{\{\gamma_1,\gamma_2\},\gamma_3} \\ &\quad -\delta_y^2 D_y^3 S_{\{\gamma_1,\gamma_2\},\gamma_4}) + O(h^6), \quad (2.165) \\ D_y^4 u_{\{\gamma_1,\gamma_2\},\gamma_2} &= D_y^4 S_{\{\gamma_1,\gamma_2\},\gamma_2} + \frac{1}{24}\eta_{24}^y D_y^4 S_{\{\gamma_1,\gamma_2\},\gamma_2} - \frac{7}{5760}(2\delta_y^2 D_y^4 S_{\{\gamma_1,\gamma_2\},\gamma_3} \\ &\quad -\delta_y^2 D_y^4 S_{\{\gamma_1,\gamma_2\},\gamma_4}) + O(h^6), \quad (2.166) \end{aligned}$$

and the cross-derivatives of S and u satisfy

$$\begin{aligned} D_x D_y u_{\gamma_1,\gamma_1} &= D_x D_y S_{\gamma_1,\gamma_1} + \frac{7(3\delta_x^2 D_x D_y S_{\gamma_3,\gamma_1} - 2\delta_x^2 D_x D_y S_{\gamma_4,\gamma_1})}{5760} \\ &\quad + \frac{7(3\delta_y^2 D_x D_y S_{\gamma_1,\gamma_3} - 2\delta_y^2 D_x D_y S_{\gamma_1,\gamma_4})}{5760} + O(h^6), \quad (2.167) \\ D_x D_y^2 u_{\gamma_1,\gamma_1} &= D_x D_y^2 S_{\gamma_1,\gamma_1} + \frac{7(3\delta_x^2 D_x D_y S_{\gamma_3,\gamma_1} - 2\delta_x^2 D_x D_y S_{\gamma_4,\gamma_1})}{5760} \\ &\quad - \frac{7(3\delta_y^2 D_x D_y^2 S_{\gamma_1,\gamma_3} - 2\delta_y^2 D_x D_y^2 S_{\gamma_1,\gamma_4})}{1920} + O(h^6), \quad (2.168) \\ D_x^2 D_y u_{\gamma_1,\gamma_1} &= D_x^2 D_y S_{\gamma_1,\gamma_1} - \frac{7(3\delta_x^2 D_x^2 D_y S_{\gamma_3,\gamma_1} - 2\delta_x^2 D_x^2 D_y S_{\gamma_4,\gamma_1})}{1920} \\ &\quad + \frac{7(3\delta_y^2 D_x D_y S_{\gamma_1,\gamma_3} - 2\delta_y^2 D_x D_y S_{\gamma_1,\gamma_4})}{5760} + O(h^6), \quad (2.169) \end{aligned}$$

$$\begin{aligned} D_x^2 D_y^2 u_{\gamma_1,\gamma_1} &= D_x^2 D_y^2 S_{\gamma_1,\gamma_1} - \frac{7(3\delta_x^2 D_x^2 D_y^2 S_{\gamma_3,\gamma_1} - 2\delta_x^2 D_x^2 D_y^2 S_{\gamma_4,\gamma_1})}{1920} \\ &\quad - \frac{7(3\delta_y^2 D_x^2 D_y^2 S_{\gamma_1,\gamma_3} - 2\delta_y^2 D_x^2 D_y^2 S_{\gamma_1,\gamma_4})}{1920} + O(h^6), \quad (2.170) \\ D_x D_y^3 u_{\gamma_1,\gamma_1} &= D_x D_y^3 S_{\gamma_1,\gamma_1} - \frac{1}{24}\zeta_8^y D_x D_y^3 S_{\gamma_1,\gamma_2} + \frac{7(3\delta_y^2 D_x D_y^3 S_{\gamma_1,\gamma_3} - 2\delta_y^2 D_x D_y^3 S_{\gamma_1,\gamma_4})}{1920} \\ &\quad + \frac{7(3\delta_x^2 D_x D_y^3 S_{\gamma_3,\gamma_1} - 2\delta_x^2 D_x D_y^3 S_{\gamma_4,\gamma_1})}{5760} + O(h^6), \quad (2.171) \\ D_x^3 D_y u_{\gamma_1,\gamma_1} &= D_x^3 D_y S_{\gamma_1,\gamma_1} - \frac{1}{24}\zeta_8^x D_x^3 D_y S_{\gamma_2,\gamma_1} + \frac{7(3\delta_y^2 D_x^3 D_y S_{\gamma_1,\gamma_3} - 2\delta_y^2 D_x^3 D_y S_{\gamma_1,\gamma_4})}{5760} \\ &\quad + \frac{7(3\delta_x^2 D_x^3 D_y S_{\gamma_3,\gamma_1} - 2\delta_x^2 D_x^3 D_y S_{\gamma_4,\gamma_1})}{1920} + O(h^6), \quad (2.172) \end{aligned}$$

$$
\begin{aligned}
D_xD_yu_{\gamma_1,\gamma_2} &= D_xD_yS_{\gamma_1,\gamma_2} + \frac{7(3\delta_x^2D_xD_yS_{\gamma_3,\gamma_2} - 2\delta_x^2D_xD_yS_{\gamma_4,\gamma_2})}{5760} \\
&\quad + \frac{7(2\delta_y^2D_xD_yS_{\gamma_1,\gamma_3} - \delta_y^2D_xD_yS_{\gamma_1,\gamma_4})}{5760} + O(h^6), \qquad (2.173) \\
D_xD_y^2u_{\gamma_1,\gamma_2} &= D_xD_y^2S_{\gamma_1,\gamma_2} + \frac{7(3\delta_x^2D_xD_yS_{\gamma_3,\gamma_2} - 2\delta_x^2D_xD_yS_{\gamma_4,\gamma_2})}{5760} \\
&\quad - \frac{7(2\delta_y^2D_xD_y^2S_{\gamma_1,\gamma_3} - \delta_y^2D_xD_y^2S_{\gamma_1,\gamma_4})}{1920} + O(h^6), \qquad (2.174) \\
D_x^2D_yu_{\gamma_1,\gamma_2} &= D_x^2D_yS_{\gamma_1,\gamma_2} - \frac{7(3\delta_x^2D_x^2D_yS_{\gamma_3,\gamma_2} - 2\delta_x^2D_x^2D_yS_{\gamma_4,\gamma_2})}{1920} \\
&\quad + \frac{7(2\delta_y^2D_xD_yS_{\gamma_1,\gamma_3} - \delta_y^2D_xD_yS_{\gamma_1,\gamma_4})}{5760} + O(h^6), \qquad (2.175)
\end{aligned}
$$

$$
\begin{aligned}
D_x^2D_y^2u_{\gamma_1,\gamma_2} &= D_x^2D_y^2S_{\gamma_1,\gamma_2} - \frac{7(3\delta_x^2D_x^2D_y^2S_{\gamma_3,\gamma_2} - 2\delta_x^2D_x^2D_y^2S_{\gamma_4,\gamma_2})}{1920} \\
&\quad - \frac{7(2\delta_y^2D_x^2D_y^2S_{\gamma_1,\gamma_3} - \delta_y^2D_x^2D_y^2S_{\gamma_1,\gamma_4})}{1920} + O(h^6), \qquad (2.176) \\
D_xD_y^3u_{\gamma_1,\gamma_2} &= D_xD_y^3S_{\gamma_1,\gamma_2} - \frac{1}{24}\eta_8^yD_xD_y^3S_{\gamma_1,\gamma_2} + \frac{7(2\delta_y^2D_xD_y^3S_{\gamma_1,\gamma_3} - \delta_y^2D_xD_y^3S_{\gamma_1,\gamma_4})}{1920} \\
&\quad + \frac{7(3\delta_x^2D_xD_y^3S_{\gamma_3,\gamma_2} - 2\delta_x^2D_xD_y^3S_{\gamma_4,\gamma_2})}{5760} + O(h^6), \qquad (2.177) \\
D_x^3D_yu_{\gamma_1,\gamma_2} &= D_x^3D_yS_{\gamma_1,\gamma_2} - \frac{1}{24}\zeta_8^xD_x^3D_yS_{\gamma_2,\gamma_2} + \frac{7(2\delta_y^2D_x^3D_yS_{\gamma_1,\gamma_3} - \delta_y^2D_x^3D_yS_{\gamma_1,\gamma_4})}{5760} \\
&\quad + \frac{7(3\delta_x^2D_x^3D_yS_{\gamma_3,\gamma_2} - 2\delta_x^2D_x^3D_yS_{\gamma_4,\gamma_2})}{1920} + O(h^6), \qquad (2.178)
\end{aligned}
$$

$$
\begin{aligned}
D_xD_yu_{\gamma_2,\gamma_1} &= D_xD_yS_{\gamma_2,\gamma_1} + \frac{7(2\delta_x^2D_xD_yS_{\gamma_3,\gamma_1} - \delta_x^2D_xD_yS_{\gamma_4,\gamma_1})}{5760} \\
&\quad + \frac{7(3\delta_y^2D_xD_yS_{\gamma_2,\gamma_3} - 2\delta_y^2D_xD_yS_{\gamma_2,\gamma_4})}{5760} + O(h^6), \qquad (2.179) \\
D_xD_y^2u_{\gamma_2,\gamma_1} &= D_xD_y^2S_{\gamma_2,\gamma_1} + \frac{7(2\delta_x^2D_xD_yS_{\gamma_3,\gamma_1} - \delta_x^2D_xD_yS_{\gamma_4,\gamma_1})}{5760} \\
&\quad - \frac{7(3\delta_y^2D_xD_y^2S_{\gamma_2,\gamma_3} - 2\delta_y^2D_xD_y^2S_{\gamma_2,\gamma_4})}{1920} + O(h^6), \qquad (2.180) \\
D_x^2D_yu_{\gamma_2,\gamma_1} &= D_xD_yS_{\gamma_2,\gamma_1} - \frac{7(2\delta_x^2D_x^2D_yS_{\gamma_3,\gamma_1} - \delta_x^2D_x^2D_yS_{\gamma_4,\gamma_1})}{1920} \\
&\quad + \frac{7(3\delta_y^2D_xD_yS_{\gamma_2,\gamma_3} - 2\delta_y^2D_xD_yS_{\gamma_2,\gamma_4})}{5760} + O(h^6), \qquad (2.181)
\end{aligned}
$$

$$\begin{aligned}
D_x^2D_y^2u_{\gamma_2,\gamma_1} &= D_x^2D_y^2S_{\gamma_2,\gamma_1} - \frac{7(2\delta_x^2D_x^2D_y^2S_{\gamma_3,\gamma_1} - \delta_x^2D_x^2D_y^2S_{\gamma_4,\gamma_1})}{1920} \\
&\quad - \frac{7(3\delta_y^2D_x^2D_y^2S_{\gamma_2,\gamma_3} - 2\delta_y^2D_x^2D_y^2S_{\gamma_2,\gamma_4})}{1920} + O(h^6), && (2.182) \\
D_xD_y^3u_{\gamma_2,\gamma_1} &= D_xD_y^3S_{\gamma_2,\gamma_1} - \frac{1}{24}\zeta_8^y D_xD_y^3S_{\gamma_2,\gamma_2} + \frac{7(3\delta_y^2D_xD_y^3S_{\gamma_2,\gamma_3} - 2\delta_y^2D_xD_y^3S_{\gamma_2,\gamma_4})}{1920} \\
&\quad + \frac{7(2\delta_x^2D_xD_y^3S_{\gamma_3,\gamma_1} - \delta_x^2D_xD_y^3S_{\gamma_4,\gamma_1})}{5760} + O(h^6), && (2.183) \\
D_x^3D_yu_{\gamma_2,\gamma_1} &= D_x^3D_yS_{\gamma_2,\gamma_1} - \frac{1}{24}\eta_8^x D_x^3D_yS_{\gamma_2,\gamma_1} + \frac{7(3\delta_y^2D_x^3D_yS_{\gamma_2,\gamma_3} - 2\delta_y^2D_x^3D_yS_{\gamma_2,\gamma_4})}{5760} \\
&\quad + \frac{7(2\delta_x^2D_x^3D_yS_{\gamma_3,\gamma_1} - \delta_x^2D_x^3D_yS_{\gamma_4,\gamma_1})}{1920} + O(h^6), && (2.184)
\end{aligned}$$

$$\begin{aligned}
D_xD_yu_{\gamma_2,\gamma_2} &= D_xD_yS_{\gamma_2,\gamma_2} + \frac{7(2\delta_x^2D_xD_yS_{\gamma_3,\gamma_2} - \delta_x^2D_xD_yS_{\gamma_4,\gamma_2})}{5760} \\
&\quad + \frac{7(2\delta_y^2D_xD_yS_{\gamma_2,\gamma_3} - \delta_y^2D_xD_yS_{\gamma_2,\gamma_4})}{5760} + O(h^6), && (2.185) \\
D_xD_y^2u_{\gamma_2,\gamma_2} &= D_xD_y^2S_{\gamma_2,\gamma_2} + \frac{7(2\delta_x^2D_xD_yS_{\gamma_3,\gamma_2} - \delta_x^2D_xD_yS_{\gamma_4,\gamma_2})}{5760} \\
&\quad - \frac{7(2\delta_y^2D_xD_y^2S_{\gamma_2,\gamma_3} - \delta_y^2D_xD_y^2S_{\gamma_2,\gamma_4})}{1920} + O(h^6), && (2.186) \\
D_x^2D_yu_{\gamma_2,\gamma_2} &= D_x^2D_yS_{\gamma_2,\gamma_2} - \frac{7(2\delta_x^2D_x^2D_yS_{\gamma_3,\gamma_2} - \delta_x^2D_x^2D_yS_{\gamma_4,\gamma_2})}{1920} \\
&\quad + \frac{7(2\delta_y^2D_xD_yS_{\gamma_2,\gamma_3} - \delta_y^2D_xD_yS_{\gamma_2,\gamma_4})}{5760} + O(h^6), && (2.187)
\end{aligned}$$

$$\begin{aligned}
D_x^2D_y^2u_{\gamma_2,\gamma_2} &= D_x^2D_y^2S_{\gamma_2,\gamma_2} - \frac{7(2\delta_x^2D_x^2D_y^2S_{\gamma_3,\gamma_2} - \delta_x^2D_x^2D_y^2S_{\gamma_4,\gamma_2})}{1920} \\
&\quad - \frac{7(2\delta_y^2D_x^2D_y^2S_{\gamma_2,\gamma_3} - \delta_y^2D_x^2D_y^2S_{\gamma_2,\gamma_4})}{1920} + O(h^6), && (2.188) \\
D_xD_y^3u_{\gamma_2,\gamma_2} &= D_xD_y^3S_{\gamma_2,\gamma_2} - \frac{1}{24}\eta_8^y D_xD_y^3S_{\gamma_2,\gamma_2} + \frac{7(2\delta_y^2D_xD_y^3S_{\gamma_2,\gamma_3} - \delta_y^2D_xD_y^3S_{\gamma_2,\gamma_4})}{1920} \\
&\quad + \frac{7(2\delta_x^2D_xD_y^3S_{\gamma_3,\gamma_2} - \delta_x^2D_xD_y^3S_{\gamma_4,\gamma_2})}{5760} + O(h^6), && (2.189) \\
D_x^3D_yu_{\gamma_2,\gamma_2} &= D_x^3D_yS_{\gamma_2,\gamma_2} - \frac{1}{24}\eta_8^x D_x^3D_yS_{\gamma_2,\gamma_2} + \frac{7(2\delta_y^2D_x^3D_yS_{\gamma_2,\gamma_3} - \delta_y^2D_x^3D_yS_{\gamma_2,\gamma_4})}{5760} \\
&\quad + \frac{7(2\delta_x^2D_x^3D_yS_{\gamma_3,\gamma_2} - \delta_x^2D_x^3D_yS_{\gamma_4,\gamma_2})}{1920} + O(h^6). && (2.190)
\end{aligned}$$

We now derive high order approximations for the derivatives of u at the boundary collocation points, i.e. points in Π_∂.

Lemma 2.2.15. *If $u \in C^{10}(\bar{\Omega})$, then,*

at points in Π^{∂}_{lr}, for $j = 3, \cdots, n-2$, the derivatives of S and u satisfy, for $k = 5, 6, 7$,

$$D_x^k u_{\gamma_0,j} = \frac{1}{2h^4}(7\delta_x^2 D_x^{k-4} S_{\gamma_3,j} - 5\delta_x^2 D_x^{k-4} S_{\gamma_4,j}) + O(h^2), \quad (2.191)$$

$$D_y^k u_{\gamma_0,j} = \frac{\delta_y^2 D_y^{k-4} S_{\gamma_0,j}}{h^4} + O(h^2), \quad (2.192)$$

$$= \frac{8\delta_y D_y^{k-2} S_{\gamma_0,j} - \delta_y^2 D_y^{k-2} S_{\gamma_0,j}}{8h^2} + O(h^4), \quad (2.193)$$

and

$$D_x^5 u_{\gamma_0,j} = \frac{1}{h^2}\theta_8^x D_x^3 S_{\gamma_2,j} + O(h^4), \quad (2.194)$$

and the cross-derivatives of S and u satisfy, for $k = 5, 6$, $l = 1, 2$,

$$D_x^k D_y^l u_{\gamma_0,j} = \frac{1}{2h^4}(7\delta_x^2 D_x^{k-4} D_y^l S_{\gamma_3,j} - 5\delta_x^2 D_x^{k-4} D_y^l S_{\gamma_4,j}) + O(h^2), \quad (2.195)$$

$$D_x^l D_y^k u_{\gamma_0,j} = \frac{\delta_y^2 D_x^l D_y^{k-4} S_{\gamma_0,j}}{h^4} + O(h^2). \quad (2.196)$$

Similar relations hold at points in Π^{∂}_{ud}.

At points in Π^{∂}_{cx}, the derivatives of S and u satisfy, for $k = 5, 6, 7$,

$$D_x^k u_{\gamma_0,\{\gamma_1,\gamma_2\}} = \frac{1}{2h^4}(7\delta_x^2 D_x^{k-4} S_{\gamma_3,\{\gamma_1,\gamma_2\}} - 5\delta_x^2 D_x^{k-4} S_{\gamma_4,\{\gamma_1,\gamma_2\}}) + O(h^2), \quad (2.197)$$

$$D_y^k u_{\gamma_0,\gamma_1} = \frac{1}{h^4}(3\delta_y^2 D_y^{k-4} S_{\gamma_0,\gamma_3} - 2\delta_x^2 D_y^{k-4} S_{\gamma_0,\gamma_4}) + O(h^2), \quad (2.198)$$

$$D_y^k u_{\gamma_0,\gamma_2} = \frac{1}{h^4}(2\delta_y^2 D_y^{k-4} S_{\gamma_0,\gamma_3} - \delta_x^2 D_y^{k-4} S_{\gamma_0,\gamma_4}) + O(h^2), \quad (2.199)$$

and

$$D_x^5 u_{\gamma_0,\{\gamma_1,\gamma_2\}} = \frac{1}{h^2}\theta_8^x D_x^3 S_{\gamma_2,\{\gamma_1,\gamma_2\}} + O(h^4), \quad (2.200)$$

$$D_y^5 u_{\gamma_0,\gamma_1} = \frac{1}{h^2}\zeta_8^y D_y^3 S_{\gamma_0,\gamma_3} + O(h^4), \quad (2.201)$$

$$D_y^5 u_{\gamma_0,\gamma_2} = \frac{1}{h^2}\eta_8^y D_y^3 S_{\gamma_0,\gamma_3} + O(h^4), \quad (2.202)$$

and the cross-derivatives of S and u satisfy, for $k = 5, 6$, $l = 1, 2$,

$$D_x^k D_y^l u_{\gamma_0,\{\gamma_1,\gamma_2\}} = \frac{1}{2h^4}(7\delta_x^2 D_x^{k-4} D_y^l S_{\gamma_3,\{\gamma_1,\gamma_2\}} - 5\delta_x^2 D_x^{k-4} D_y^l S_{\gamma_4,\{\gamma_1,\gamma_2\}}) + O(h^2), \quad (2.203)$$

$$D_x^l D_y^k u_{\gamma_0,\gamma_1} = \frac{1}{h^4}(3\delta_y^2 D_x^l D_y^{k-4} S_{\gamma_0,\gamma_3} - 2\delta_y^2 D_x^l D_y^{k-4} S_{\gamma_0,\gamma_4}) + O(h^2), \quad (2.204)$$

$$D_x^l D_y^k u_{\gamma_0,\gamma_2} = \frac{1}{h^4}(2\delta_y^2 D_x^l D_y^{k-4} S_{\gamma_0,\gamma_3} - \delta_y^2 D_x^l D_y^{k-4} S_{\gamma_0,\gamma_4}) + O(h^2). \quad (2.205)$$

Similar relations hold for $(x, y) \in \Pi_{cy}^{\partial}$.

At points in Π_c^{∂}, *the derivatives of* S *and* u *satisfy, for* $k = 5, 6, 7$,

$$D_x^k u_{\gamma_0,\gamma_0} = \frac{1}{2h^4}(7\delta_x^2 D_x^{k-4} S_{\gamma_3,\gamma_0} - 5\delta_x^2 D_x^{k-4} S_{\gamma_4,\gamma_0}) + O(h^2), \tag{2.206}$$

$$D_y^k u_{\gamma_0,\gamma_0} = \frac{1}{2h^4}(7\delta_y^2 D_y^{k-4} S_{\gamma_0,\gamma_3} - 5\delta_y^2 D_y^{k-4} S_{\gamma_0,\gamma_4}) + O(h^2), \tag{2.207}$$

and

$$D_x^5 u_{\gamma_0,\gamma_0} = \frac{1}{h^2}\theta_8^x D_x^3 S_{\gamma_2,\gamma_0} + O(h^4), \tag{2.208}$$

$$D_y^5 u_{\gamma_0,\gamma_0} = \frac{1}{h^2}\theta_8^y D_y^3 S_{\gamma_2,\gamma_0} + O(h^4), \tag{2.209}$$

and the cross-derivatives of S *and* u *satisfy, for* $k = 5, 6,\ l = 1, 2$,

$$D_x^k D_y^l u_{\gamma_0,\gamma_0} = \frac{1}{2h^4}(7\delta_x^2 D_x^{k-4} D_y^l S_{\gamma_3,\gamma_0} - 5\delta_x^2 D_x^{k-4} D_y^l S_{\gamma_4,\gamma_0}) + O(h^2), \tag{2.210}$$

$$D_x^l D_y^k u_{\gamma_0,\gamma_0} = \frac{1}{2h^4}(7\delta_y^2 D_x^l D_y^{k-4} S_{\gamma_0,\gamma_3} - 5\delta_y^2 D_x^l D_y^{k-4} S_{\gamma_0,\gamma_4}) + O(h^2). \tag{2.211}$$

We summarize the results in Theorem 2.2.7 and Lemma 2.2.15 by the following corollary.

Corollary 2.2.16. *If* $u \in C^{10}(\bar{\Omega})$, *we have* *at points* $(\tau^x_{\gamma_0}, \tau^y_j)$, $j = 3, \cdots, n-2$,

$$D_x u_{\gamma_0,j} = D_x S_{\gamma_0,j} - \frac{1}{1440}(7\delta_x^2 D_x S_{\gamma_3,j} - 5\delta_x^2 D_x S_{\gamma_4,j}) + O(h^6), \tag{2.212}$$

$$D_x^2 u_{\gamma_0,j} = D_x^2 S_{\gamma_0,j} + \frac{1}{480}(7\delta_x^2 D_x^2 S_{\gamma_3,j} - 5\delta_x^2 D_x^2 S_{\gamma_4,j}) + O(h^6), \tag{2.213}$$

$$D_x^3 u_{\gamma_0,j} = D_x^3 S_{\gamma_0,j} + \frac{1}{12}\theta_8^x D_x^3 S_{\gamma_2,j} - \frac{1}{480}(7\delta_x^2 D_x^3 S_{\gamma_3,j} - 5\delta_x^2 D_x^3 S_{\gamma_4,j}) + O(h^6), \tag{2.214}$$

$$D_y u_{\gamma_0,j} = D_y S_{\gamma_0,j} + \frac{7}{5760}\delta_y^2 D_y S_{\gamma_0,j} + O(h^6), \tag{2.215}$$

$$D_y^2 u_{\gamma_0,j} = D_y^2 S_{\gamma_0,j} - \frac{7}{1920}\delta_y^2 D_y^2 S_{\gamma_0,j} + O(h^6), \tag{2.216}$$

$$D_y^3 u_{\gamma_0,j} = D_y^3 S_{\gamma_0,j} - \frac{8\delta_y D_y^3 S_{\gamma_0,j} - \delta_y^2 D_y^3 S_{\gamma_0,j}}{192} + \frac{7}{1920}\delta_y^2 D_y^3 S_{\gamma_0,j} + O(h^6), \tag{2.217}$$

$$D_x D_y u_{\gamma_0,j} = D_x D_y S_{\gamma_0,j} - \frac{7\delta_x^2 D_x D_y S_{\gamma_3,j} - 5\delta_x^2 D_x D_y S_{\gamma_4,j}}{1440} + \frac{7\delta_y^2 D_x D_y S_{\gamma_0,j}}{5760} + O(h^6), \tag{2.218}$$

$$D_x^2 D_y u_{\gamma_0,j} = D_x^2 D_y S_{\gamma_0,j} + \frac{7\delta_x^2 D_x^2 D_y S_{\gamma_3,j} - 5\delta_x^2 D_x^2 D_y S_{\gamma_4,j}}{480} + \frac{7\delta_y^2 D_x^2 D_y S_{\gamma_0,j}}{5760} + O(h^6), \tag{2.219}$$

$$D_x D_y^2 u_{\gamma_0,j} = D_x D_y^2 S_{\gamma_0,j} - \frac{7\delta_x^2 D_x D_y^2 S_{\gamma_3,j} - 5\delta_x^2 D_x D_y^2 S_{\gamma_4,j}}{1440} - \frac{7\delta_y^2 D_x D_y^2 S_{\gamma_0,j}}{1920} + O(h^6), \tag{2.220}$$

and similar relations at the points $(\tau^x_j, \tau^y_{\gamma_0})$.

At points $(\tau^x_{\gamma_0}, \tau^y_{\{\gamma_1,\gamma_2\}})$,

$$D_x u_{\gamma_0,\{\gamma_1,\gamma_2\}} = D_x S_{\gamma_0,\{\gamma_1,\gamma_2\}} - \frac{1}{1440}(7\delta_x^2 D_x S_{\gamma_3,\{\gamma_1,\gamma_2\}} - 5\delta_x^2 D_x S_{\gamma_4,\{\gamma_1,\gamma_2\}}) + O(h^6), \tag{2.221}$$

$$D_x^2 u_{\gamma_0,\{\gamma_1,\gamma_2\}} = D_x^2 S_{\gamma_0,\{\gamma_1,\gamma_2\}} + \frac{1}{480}(7\delta_x^2 D_x^2 S_{\gamma_3,\{\gamma_1,\gamma_2\}} - 5\delta_x^2 D_x^2 S_{\gamma_4,\{\gamma_1,\gamma_2\}}) + O(h^6), \tag{2.222}$$

$$\begin{aligned} D_x^3 u_{\gamma_0,\{\gamma_1,\gamma_2\}} &= D_x^3 S_{\gamma_0,\{\gamma_1,\gamma_2\}} + \frac{1}{12}\theta_8^x D_x^3 S_{\gamma_2,\{\gamma_1,\gamma_2\}} - \frac{1}{480}(7\delta_x^2 D_x^3 S_{\gamma_3,\{\gamma_1,\gamma_2\}} \\ &\quad -5\delta_x^2 D_x^3 S_{\gamma_4,\{\gamma_1,\gamma_2\}}) + O(h^6), \end{aligned} \tag{2.223}$$

$$D_y u_{\gamma_0,\gamma_1} = D_y S_{\gamma_0,\gamma_1} + \frac{7}{5760}(3\delta_y^2 D_y S_{\gamma_0,\gamma_3} - 2\delta_y^2 D_y S_{\gamma_0,\gamma_4}) + O(h^6), \tag{2.224}$$

$$D_y u_{\gamma_0,\gamma_2} = D_y S_{\gamma_0,\gamma_2} + \frac{7}{5760}(2\delta_y^2 D_y S_{\gamma_0,\gamma_3} - \delta_y^2 D_y S_{\gamma_0,\gamma_4}) + O(h^6), \tag{2.225}$$

$$D_y^2 u_{\gamma_0,\gamma_1} = D_y^2 S_{\gamma_0,\gamma_1} - \frac{7}{1920}(3\delta_y^2 D_y^2 S_{\gamma_0,\gamma_3} - 2\delta_y^2 D_y^2 S_{\gamma_0,\gamma_4}) + O(h^6), \tag{2.226}$$

$$D_y^2 u_{\gamma_0,\gamma_2} = D_y^2 S_{\gamma_0,\gamma_2} - \frac{7}{1920}(2\delta_y^2 D_y^2 S_{\gamma_0,\gamma_3} - \delta_y^2 D_y^2 S_{\gamma_0,\gamma_4}) + O(h^6), \tag{2.227}$$

$$D_y^3 u_{\gamma_0,\gamma_1} = D_y^3 S_{\gamma_0,\gamma_1} - \frac{\zeta_8^y D_y^3 S_{\gamma_0,\gamma_3}}{24} + \frac{7(3\delta_y^2 D_y^3 S_{\gamma_0,\gamma_3} - 2\delta_y^2 D_y^3 S_{\gamma_0,\gamma_4})}{1920} + O(h^6), \tag{2.228}$$

$$D_y^3 u_{\gamma_0,\gamma_2} = D_y^3 S_{\gamma_0,\gamma_2} - \frac{\eta_8^y D_y^3 S_{\gamma_0,\gamma_3}}{24} + \frac{7(2\delta_y^2 D_y^3 S_{\gamma_0,\gamma_3} - \delta_y^2 D_y^3 S_{\gamma_0,\gamma_4})}{1920} + O(h^6), \tag{2.229}$$

$$\begin{aligned}
D_x D_y u_{\gamma_0,\gamma_1} &= D_x D_y S_{\gamma_0,\gamma_1} - \frac{7\delta_x^2 D_x D_y S_{\gamma_3,\gamma_1} - 5\delta_x^2 D_x D_y S_{\gamma_4,\gamma_1}}{1440} \\
&\quad + \frac{7(3\delta_y^2 D_x D_y S_{\gamma_0,\gamma_3} - 2\delta_y^2 D_x D_y S_{\gamma_0,\gamma_4})}{5760} + O(h^6), &(2.230)\\
D_x D_y u_{\gamma_0,\gamma_2} &= D_x D_y S_{\gamma_0,\gamma_2} - \frac{7\delta_x^2 D_x D_y S_{\gamma_3,\gamma_2} - 5\delta_x^2 D_x D_y S_{\gamma_4,\gamma_2}}{1440} \\
&\quad + \frac{7(2\delta_y^2 D_x D_y S_{\gamma_0,\gamma_3} - \delta_y^2 D_x D_y S_{\gamma_0,\gamma_4})}{5760} + O(h^6), &(2.231)\\
D_x^2 D_y u_{\gamma_0,\gamma_1} &= D_x^2 D_y S_{\gamma_0,\gamma_1} + \frac{7\delta_x^2 D_x^2 D_y S_{\gamma_3,\gamma_1} - 5\delta_x^2 D_x^2 D_y S_{\gamma_4,\gamma_1}}{480} \\
&\quad + \frac{7(3\delta_y^2 D_x^2 D_y S_{\gamma_0,\gamma_3} - 2\delta_y^2 D_x^2 D_y S_{\gamma_0,\gamma_4})}{5760} + O(h^6), &(2.232)
\end{aligned}$$

$$\begin{aligned}
D_x^2 D_y u_{\gamma_0,\gamma_2} &= D_x^2 D_y S_{\gamma_0,\gamma_2} + \frac{7\delta_x^2 D_x^2 D_y S_{\gamma_3,\gamma_2} - 5\delta_x^2 D_x^2 D_y S_{\gamma_4,\gamma_2}}{480} \\
&\quad + \frac{7(2\delta_y^2 D_x^2 D_y S_{\gamma_0,\gamma_3} - \delta_y^2 D_x^2 D_y S_{\gamma_0,\gamma_4})}{5760} + O(h^6), &(2.233)\\
D_x D_y^2 u_{\gamma_0,\gamma_1} &= D_x D_y^2 S_{\gamma_0,\gamma_1} - \frac{7\delta_x^2 D_x D_y^2 S_{\gamma_3,\gamma_1} - 5\delta_x^2 D_x D_y^2 S_{\gamma_4,\gamma_1}}{1440} \\
&\quad - \frac{7(3\delta_y^2 D_x D_y^2 S_{\gamma_0,\gamma_3} - 2\delta_y^2 D_x D_y^2 S_{\gamma_0,\gamma_4})}{1920} + O(h^6), &(2.234)\\
D_x D_y^2 u_{\gamma_0,\gamma_2} &= D_x D_y^2 S_{\gamma_0,\gamma_2} - \frac{7\delta_x^2 D_x D_y^2 S_{\gamma_3,\gamma_2} - 5\delta_x^2 D_x D_y^2 S_{\gamma_4,\gamma_2}}{1440} \\
&\quad - \frac{7(2\delta_y^2 D_x D_y^2 S_{\gamma_0,\gamma_3} - \delta_y^2 D_x D_y^2 S_{\gamma_0,\gamma_4})}{1920} + O(h^6), &(2.235)
\end{aligned}$$

and similar relations at points $(\tau^x_{\{\gamma_1,\gamma_2\}}, \tau^y_{\gamma_0})$.

At the four corner points (x_0, y_0), (x_0, y_n), (x_n, y_0), *and* (x_n, y_n), *we have*

$$D_x u_{\gamma_0,\gamma_0} = D_x S_{\gamma_0,\gamma_0} - \frac{7\delta_x^2 D_x S_{\gamma_3,\gamma_0} - 5\delta_x^2 D_x S_{\gamma_4,\gamma_0}}{1440} + O(h^6), \tag{2.236}$$

$$D_x^2 u_{\gamma_0,\gamma_0} = D_x^2 S_{\gamma_0,\gamma_0} + \frac{7\delta_x^2 D_x^2 S_{\gamma_3,\gamma_0} - 5\delta_x^2 D_x^2 S_{\gamma_4,\gamma_0}}{480} + O(h^6), \tag{2.237}$$

$$D_x^3 u_{\gamma_0,\gamma_0} = D_x^3 S_{\gamma_0,\gamma_0} + \frac{\theta_8^x D_x^3 S_{\gamma_2,\gamma_0}}{12} - \frac{7\delta_x^2 D_x^3 S_{\gamma_3,\gamma_0} - 5\delta_x^2 D_x^3 S_{\gamma_4,\gamma_0}}{480} + O(h^6), \tag{2.238}$$

$$D_y u_{\gamma_0,\gamma_0} = D_y S_{\gamma_0,\gamma_0} - \frac{7\delta_y^2 D_y S_{\gamma_3,\gamma_0} - 5\delta_y^2 D_y S_{\gamma_4,\gamma_0}}{1440} + O(h^6), \tag{2.239}$$

$$D_y^2 u_{\gamma_0,\gamma_0} = D_y^2 S_{\gamma_0,\gamma_0} + \frac{7\delta_y^2 D_y^2 S_{\gamma_3,\gamma_0} - 5\delta_y^2 D_y^2 S_{\gamma_4,\gamma_0}}{480} + O(h^6), \tag{2.240}$$

$$D_y^3 u_{\gamma_0,\gamma_0} = D_y^3 S_{\gamma_0,\gamma_0} + \frac{\theta_8^y D_y^3 S_{\gamma_2,\gamma_0}}{12} - \frac{7\delta_y^2 D_y^3 S_{\gamma_3,\gamma_0} - 5\delta_y^2 D_y^3 S_{\gamma_4,\gamma_0}}{480} + O(h^6), \tag{2.241}$$

$$\begin{aligned}
D_x D_y u_{\gamma_0,\gamma_0} &= D_x D_y S_{\gamma_0,\gamma_0} - \frac{7\delta_x^2 D_x D_y S_{\gamma_3,\gamma_0} - 5\delta_x^2 D_x D_y S_{\gamma_4,\gamma_0}}{1440} \\
&\quad - \frac{7\delta_y^2 D_x D_y S_{\gamma_0,\gamma_3} - 5\delta_y^2 D_x D_y S_{\gamma_0,\gamma_4}}{1440} + O(h^6), && (2.242) \\
D_x^2 D_y u_{\gamma_0,\gamma_0} &= D_x^2 D_y S_{\gamma_0,\gamma_0} + \frac{7\delta_x^2 D_x^2 D_y S_{\gamma_3,\gamma_0} - 5\delta_x^2 D_x^2 D_y S_{\gamma_4,\gamma_0}}{480} \\
&\quad - \frac{7\delta_y^2 D_x^2 D_y S_{\gamma_0,\gamma_3} - 5\delta_y^2 D_x^2 D_y S_{\gamma_0,\gamma_4}}{1440} + O(h^6), && (2.243) \\
D_x D_y^2 u_{\gamma_0,\gamma_0} &= D_x D_y^2 S_{\gamma_0,\gamma_0} - \frac{7\delta_x^2 D_x D_y^2 S_{\gamma_3,\gamma_0} - 5\delta_x^2 D_x D_y^2 S_{\gamma_4,\gamma_0}}{1440} \\
&\quad + \frac{7\delta_y^2 D_x D_y^2 S_{\gamma_0,\gamma_3} - 5\delta_y^2 D_x D_y^2 S_{\gamma_0,\gamma_4}}{480} + O(h^6). && (2.244)
\end{aligned}$$

Chapter 3

Bi-Quartic Spline Collocation

This chapter develops optimal bi-quartic spline collocation methods for the numerical solution of general two-dimensional fourth-order linear boundary value problems (BVPs) on rectangular domains.

3.1 Two-Dimensional Fourth Order BVP

Consider the elliptic linear partial differential equation

$$\begin{aligned} Lu \equiv\; & p^{(1)}(x,y)D_x^4u + p^{(2)}(x,y)D_x^3D_yu + p^{(3)}(x,y)D_x^2D_y^2u + p^{(4)}(x,y)D_xD_y^3u + p^{(5)}(x,y)D_y^4u \\ & +q^{(1)}(x,y)D_x^3u + q^{(2)}(x,y)D_x^2D_yu + q^{(3)}(x,y)D_xD_y^2(x,y)u + q^{(4)}(x,y)D_y^3u \\ & +r^{(1)}(x,y)D_x^2u + r^{(2)}(x,y)D_xD_yu + r^{(3)}D_y^2u + s^{(1)}(x,y)D_xu + s^{(2)}(x,y)D_yu + t(x,y)u \\ =\; & f(x,y), \quad \text{on } \Omega \equiv [0,1]\times[0,1], \end{aligned} \tag{3.1}$$

subject to boundary conditions

$$\begin{aligned} B_ku \equiv\; & \alpha^{(k,1)}(x,y)D_x^3u + \alpha^{(k,2)}(x,y)D_x^2D_yu + \alpha^{(k,3)}(x,y)D_xD_y^2u + \alpha^{(k,4)}(x,y)D_y^3u \\ & +\beta^{(k,1)}(x,y)D_x^2u + \beta^{(k,2)}(x,y)D_xD_yu + \beta^{(k,3)}(x,y)D_y^2u + \phi^{(k,1)}(x,y)D_xu \\ & +\phi^{(k,2)}(x,y)D_yu + \psi^{(k)}(x,y)u \\ =\; & g_k(x,y) \quad \text{on } \partial\Omega,\ k=1,2, \end{aligned} \tag{3.2}$$

where the coefficients and the right-hand sides are given functions, and $u(x, y)$ is the unknown function, which is assumed to be in $C^{10}(\bar{\Omega})$ in this thesis.

3.2 Bi-Quartic Spline Collocation Method

When we apply a bi-quartic spline collocation method Problem (3.1)-(3.2), naturally we will find $w_\Delta \in \widehat{S}_\Delta$ by forcing it to satisfy the equations

$$Lw_\Delta(\tau_i^x, \tau_j^y) = f(\tau_i^x, \tau_j^y), \quad (\tau_i^x, \tau_j^y) \in \Pi - \Pi_\partial, \tag{3.3}$$

$$B_k w_\Delta(\tau_i^x, \tau_j^y) = g_k(\tau_i^x, \tau_j^y), \quad (\tau_i^x, \tau_j^y) \in \Pi_\partial, \ k = 1, 2. \tag{3.4}$$

From the introduction in Chapter 2, we know that to uniquely determine a bi-quartic spline, we need $(n + 4)^2$ independent linear equations when we have a gridsize n. Since we have n^2 interior collocation points, (3.3) includes n^2 linear equations. Thus we need another $8(n + 2)$ equations to obtain a total of $n^2 + 8(n + 2) = (n + 4)^2$ equations.

There are $n+2$ collocation points on each side of $\partial\Omega$, but each corner point is shared by two sides. Therefore, depending on the definition of B_1 and B_2, we may or may not be able to obtain $4(n+2)$ equations from each of B_1 and B_2. For example, if B_1 involves the normal derivative of u, $\frac{\partial u}{\partial \mathrm{n}}$, we could obtain two equations on a corner point, namely one involving $\frac{\partial u}{\partial x}$ and one $\frac{\partial u}{\partial y}$. Thus for some choices of B_1 and B_2 we are able to obtain $8(n + 2)$ equations.

However, it is very common that the boundary condition equations B_1 (or B_2) on $\partial\Omega$ is the Dirichlet condition $u = 0$ on $\partial\Omega$. This boundary condition $u = 0$ can only provide $4(n + 1)$ (instead of $4(n + 2)$) independent equations at the boundary collocation points, since the condition $u = 0$ at each corner is shared by two sides of $\partial\Omega$. Therefore we need to make up four additional boundary conditions for this case of B_1. While there are a variety of ways to achieve this, we choose to formulate equations involving the cross-derivative u_{xy} at the four corners $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$. In the next chapter, when we discuss the bi-quartic spline collocation methods for the biharmonic problems, we will discuss in detail how the additional equations are generated. Throughout this chapter, we assume that if these extra boundary conditions are needed, there is a way to formulate them, and we refer to them as *extra corner conditions*, without explicitly mentioning how they are formulated.

The standard bi-quartic spline collocation method for the solution of problem (3.1)-(3.2) determines a spline $w_\Delta \in \widehat{S}_\Delta$ by forcing it to satisfy the equations (3.3)-(3.4) and the extra corner conditions, if needed. Numerical experiments show that this method has second order convergence, which means that it is non-optimal, in the sense that the spline interpolation in the same approximation space $\widehat{S}_\Delta$ can achieve sixth-order convergence at the grid points and midpoints and fifth-order convergence at other points. Before proceeding to introduce optimal (sixth-order) bi-quartic spline collocation methods, we consider the residual relations that the interpolant S of u, defined by (2.5)-(2.8), satisfies.

3.3 Bi-Quartic Spline Interpolant Residual

Consider the interpolant S of u which is defined by (2.5)-(2.8). From the results in Theorems 2.2.3, 2.2.4, and 2.2.7, we have, for $i, j = 1, \cdots, n$,

$$
\begin{aligned}
(LS)_{i,j} \;=\;& (Lu)_{i,j} - p^{(1)}_{i,j}\frac{h^2}{24}D_x^6 u_{i,j} + p^{(1)}_{i,j}\frac{7h^4}{5760}D_x^8 u_{i,j} + p^{(2)}_{i,j}\frac{h^2}{24}D_x^5 D_y u_{i,j} - p^{(2)}_{i,j}\frac{7h^4}{1920}D_x^7 D_y u_{i,j} \\
& - p^{(2)}_{i,j}\frac{7h^4}{5760}D_x^3 D_y^5 u_{i,j} + p^{(3)}_{i,j}\frac{7h^4}{1920}D_x^2 D_y^6 u_{i,j} + p^{(3)}_{i,j}\frac{7h^4}{1920}D_x^6 D_y^2 u_{i,j} + p^{(4)}_{i,j}\frac{h^2}{24}D_x D_y^5 u_{i,j} \\
& - p^{(4)}_{i,j}\frac{7h^4}{1920}D_x D_y^7 u_{i,j} - p^{(4)}_{i,j}\frac{7h^4}{5760}D_x^5 D_y^3 u_{i,j} - p^{(5)}_{i,j}\frac{h^2}{24}D_y^6 u_{i,j} + p^{(5)}_{i,j}\frac{7h^4}{5760}D_y^8 u_{i,j} \\
& + q^{(1)}_{i,j}\frac{h^2}{24}D_x^5 u_{i,j} - q^{(1)}_{i,j}\frac{7h^4}{1920}D_x^7 u_{i,j} - q^{(2)}_{i,j}\frac{7h^4}{5760}D_x^2 D_y^5 u_{i,j} + q^{(2)}_{i,j}\frac{7h^4}{1920}D_x^6 D_y u_{i,j} \\
& - q^{(3)}_{i,j}\frac{7h^4}{5760}D_x^5 D_y^2 u_{i,j} + q^{(3)}_{i,j}\frac{7h^4}{1920}D_x D_y^6 u_{i,j} + q^{(4)}_{i,j}\frac{h^2}{24}D_y^5 u_{i,j} - q^{(4)}_{i,j}\frac{7h^4}{1920}D_y^7 u_{i,j} \\
& + r^{(1)}_{i,j}\frac{7h^4}{1920}D_x^6 u_{i,j} - r^{(2)}_{i,j}\frac{7h^4}{5760}D_x^5 D_y u_{i,j} - r^{(2)}_{i,j}\frac{7h^4}{5760}D_x D_y^5 u_{i,j} + r^{(3)}_{i,j}\frac{7h^4}{1920}D_y^6 u_{i,j} \\
& - s^{(1)}_{i,j}\frac{7h^4}{5760}D_x^5 u_{i,j} - s^{(2)}_{i,j}\frac{7h^4}{5760}D_y^5 u_{i,j} + O(h^6),
\end{aligned}
\tag{3.5}
$$

and for $k = 1, 2$ and $j = 1, \cdots, n$,

$$
\begin{aligned}
&(B_k S)_{\gamma_0,j} \\
&= (B_k u)_{\gamma_0,j} - \alpha^{(k,1)}_{\gamma_0,j}\frac{h^2}{12}D_x^5 u_{\gamma_0,j} + \alpha^{(k,1)}_{\gamma_0,j}\frac{h^4}{240}D_x^7 u_{\gamma_0,j} - \alpha^{(k,2)}_{\gamma_0,j}\frac{h^4}{240}D_x^6 D_y u_{\gamma_0,j} \\
&\quad - \alpha^{(k,2)}_{\gamma_0,j}\frac{7h^4}{5760}D_x^2 D_y^5 u_{\gamma_0,j} + \alpha^{(k,3)}_{\gamma_0,j}\frac{h^4}{720}D_x^5 D_y^2 u_{\gamma_0,j} + \alpha^{(k,3)}_{\gamma_0,j}\frac{7h^4}{1920}D_x D_y^6 u_{\gamma_0,j} + \alpha^{(k,4)}_{\gamma_0,j}\frac{h^2}{24}D_y^5 u_{\gamma_0,j} \\
&\quad - \alpha^{(k,4)}_{\gamma_0,j}\frac{7h^4}{1920}D_y^7 u_{\gamma_0,j} - \beta^{(k,1)}_{\gamma_0,j}\frac{h^4}{240}D_x^6 u_{\gamma_0,j} + \beta^{(k,2)}_{\gamma_0,j}\frac{h^4}{720}D_x^5 D_y u_{\gamma_0,j} - \beta^{(k,2)}_{\gamma_0,j}\frac{7h^4}{5760}D_x D_y^5 u_{\gamma_0,j} \\
&\quad + \beta^{(k,3)}_{\gamma_0,j}\frac{7h^4}{1920}D_y^6 u_{\gamma_0,j} + \phi^{(k,1)}_{\gamma_0,j}\frac{h^4}{720}D_x^5 u_{\gamma_0,j} - \phi^{(k,2)}_{\gamma_0,j}\frac{7h^4}{5760}D_y^5 u_{\gamma_0,j} + O(h^6). \qquad (3.6)
\end{aligned}
$$

Similar relations hold at the other boundary collocation points.

3.4 Optimal Bi-Quartic Spline Collocation

This section presents two bi-quartic spline collocation methods for the numerical solution of the problem (3.1)-(3.2): the *one-step* or *extrapolated* method and the *three-step* or *deferred-correction* method. They are the extensions of the optimal quartic spline collocation methods for one-dimensional BVPs in [43], which employ the same perturbation technique used in the optimal spline collocation methods of [17], [27] and [28]. The main idea in the above perturbation technique is to develop appropriate perturbations $P_L = P_{L_1} + P_{L_2}$, $P_{B_1} = P_{B_{11}} + P_{B_{12}}$, and $P_{B_2} = P_{B_{21}} + P_{B_{22}}$ of L, B_1, and B_2, respectively, using the residual relations of the interpolant S. The exact expressions of P_L, P_{B_1} and P_{B_2} will be derived later in this section. In the one-step or extrapolated method, a sixth-order bi-quartic spline collocation approximation $u_\Delta(x)$ is determined by the equations

$$
\begin{aligned}
(L + P_{L_1} + P_{L_2})u_\Delta(x)(\tau_i^x, \tau_j^y) &= f(\tau_i^x, \tau_j^y), \quad (\tau_i^x, \tau_j^y) \in \Pi - \Pi_\partial, \qquad (3.7) \\
(B_k + P_{B_{k1}} + P_{B_{k2}})u_\Delta(x)(\tau_i^x, \tau_j^y) &= g_k(\tau_i^x, \tau_j^y), \quad (\tau_i^x, \tau_j^y) \in \Pi_\partial, k = 1, 2, \qquad (3.8)
\end{aligned}
$$

and the extra corner conditions, if needed. The three-step or deferred-correction method consists of three steps. In the first step, the standard second-order bi-quartic spline collocation approximation w_Δ is determined by (3.3)-(3.4) and the extra corner conditions, if needed. In the second step, a fourth-order bi-quartic spline collocation approximation v_Δ is determined by

$$\begin{aligned} Lv_\Delta(\tau_i^x, \tau_j^y) &= f(\tau_i^x, \tau_j^y) - P_{L_1} w_\Delta(\tau_i^x, \tau_j^y), \quad (\tau_i^x, \tau_j^y) \in \Pi - \Pi_\partial, &(3.9)\\ B_k v_\Delta(\tau_i^x, \tau_j^y) &= g_k(\tau_i^x, \tau_j^y) - P_{B_{k1}} w_\Delta(\tau_i^x, \tau_j^y), \quad (\tau_i^x, \tau_j^y) \in \Pi_\partial, k = 1, 2, &(3.10)\end{aligned}$$

and the extra corner conditions, if needed. Finally, in the third step, a sixth-order bi-quartic spline collocation approximation u_Δ is determined by

$$Lu_\Delta(\tau_i^x, \tau_j^y) = f(\tau_i^x, \tau_j^y) - P_{L_1} v_\Delta(\tau_i^x, \tau_j^y) - P_{L_2} v_\Delta(\tau_i^x, \tau_j^y), \quad (\tau_i^x, \tau_j^y) \in \Pi - \Pi_\partial, \quad (3.11)$$

$$B_k u_\Delta(\tau_i^x, \tau_j^y) = g_k(\tau_i^x, \tau_j^y) - P_{B_{k1}} v_\Delta(\tau_i^x, \tau_j^y) - P_{B_{k2}} v_\Delta(\tau_i^x, \tau_j^y), (\tau_i^x, \tau_j^y) \in \Pi_\partial, k = 1, 2, \quad (3.12)$$

and the extra corner conditions, if needed. In the following, we explain how the perturbations P_L, P_{B_1} and P_{B_2}, as well as their components P_{L_1}, P_{L_2}, $P_{B_{11}}$, $P_{B_{12}}$, $P_{B_{21}}$ and $P_{B_{22}}$ are defined, and we give more details of how the two optimal bi-quartic spline collocation methods are derived.

Please note that in this chapter, we only present the derivation of the three-step method and the one-step method. The theoretical analysis of the order of convergence of these methods will be discussed in Chapter 7.

3.4.1 Three-Step Method

Step 1:

By (3.5)-(3.6), it is easy to see that the interpolant S satisfies

$$\begin{aligned} LS(\tau_i^x, \tau_j^y) &= f(\tau_i^x, \tau_j^y) + O(h^2), \quad (\tau_i^x, \tau_j^y) \in \Pi - \Pi_\partial, &(3.13)\\ B_k S(\tau_i^x, \tau_j^y) &= g_k(\tau_i^x, \tau_j^y) + O(h^2), \quad (\tau_i^x, \tau_j^y) \in \Pi_\partial, k = 1, 2. &(3.14)\end{aligned}$$

As mentioned before, in the *first step of the three-step method*, the spline $w_\Delta \in \widehat{S}_\Delta$

satisfies the collocation equations

$$\begin{aligned} Lw_\Delta(\tau_i^x, \tau_j^y) &= f(\tau_i^x, \tau_j^y), \quad (\tau_i^x, \tau_j^y) \in \Pi - \Pi_\partial, \\ B_k w_\Delta(\tau_i^x, \tau_j^y) &= g_k(\tau_i^x, \tau_j^y), \quad (\tau_i^x, \tau_j^y) \in \Pi_\partial, k = 1, 2. \end{aligned}$$

It is worth noticing that the collocation equations of the first step resemble the residual equations (3.13)-(3.14), except that w_Δ is substituted in place of S, and the $O(h^2)$ (and higher order) terms of the residual are dropped.

Step 2:

By (3.5)-(3.6), it is also easy to see that the interpolant S satisfies for $i, j = 1, \cdots, n$,

$$\begin{aligned} LS_{i,j} &= f_{i,j} - p_{i,j}^{(1)} \frac{h^2}{24} D_x^6 u_{i,j} + p_{i,j}^{(2)} \frac{h^2}{24} D_x^5 D_y u_{i,j} + p_{i,j}^{(4)} \frac{h^2}{24} D_x D_y^5 u_{i,j} - p_{i,j}^{(5)} \frac{h^2}{24} D_y^6 u_{i,j} \\ &\quad + q_{i,j}^{(1)} \frac{h^2}{24} D_x^5 u_{i,j} + q_{i,j}^{(4)} \frac{h^2}{24} D_y^5 u_{i,j} + O(h^4), \end{aligned} \tag{3.15}$$

and for $k = 1, 2$ and $j = 1, \cdots, n$,

$$B_k S_{\gamma_0,j} = g_k(\tau_{\gamma_0}^x, \tau_j^y) - \alpha_{\gamma_0,j}^{(k,1)} \frac{h^2}{12} D_x^5 u_{\gamma_0,j} + \alpha_{\gamma_0,j}^{(k,4)} \frac{h^2}{24} D_y^5 u_{\gamma_0,j} + O(h^4), \quad k = 1, 2, \tag{3.16}$$

and similar relations at other boundary collocation points. Notice that relations (3.15)-(3.16) are derived from (3.5)-(3.6), by not showing explicitly the $O(h^4)$ terms of the residual.

Consider the derivatives of u in the right-hand sides of equations (3.15)-(3.16), and the fourth-order approximations to these derivatives given by relations (2.63)-(2.68), (2.95)-(2.100), (2.104)-(2.106), (2.111)-(2.118), (2.123)-(2.126), (2.193)-(2.194), (2.200)-(2.202) and (2.208)-(2.209), which use values of derivatives of S.

We compute approximations to the derivatives of u in the right-hand sides of equations (3.15)-(3.16), by relations similar to (2.63)-(2.68), (2.95)-(2.100), (2.104)-(2.106), (2.111) - (2.118), (2.123)-(2.126), (2.193)-(2.194), (2.200)-(2.202), and (2.208)-(2.209), except that w_Δ is substituted in place of S. For example, since from equation (2.63) we have

$$D_x^5 u_{i,j} = \frac{8\delta_x D_x^3 S_{i,j} - \delta_x^2 D_x^3 S_{i,j}}{8h^2} + O(h^4), i, j = 3, \cdots, n-2,$$

we approximate $D_x^5 u_{i,j}$ by

$$D_x^5 u_{i,j} \approx \frac{8\delta_x D_x^3 w_\Delta(\tau_i^x, \tau_j^y) - \delta_x^2 D_x^3 w_\Delta(\tau_i^x, \tau_j^y)}{8h^2}. \tag{3.17}$$

We expect this approximation to be of second order. While we cannot prove it theoretically at the moment, some numerical experiments support our belief that it is indeed of second order. In Chapter 8, we apply the three-step method to Problem 8.1.2 and obtain w_Δ in the first step. Table (3.1) shows the order of convergence of $D_x^5 w_\Delta$ and $D_y^5 w_\Delta$.

Table 3.1: Errors and orders of convergence for the approximations of $D_x^5 w_\Delta$ and $D_y^5 w_\Delta$ for Problem 8.1.2

$u(x,y) = [1-\cos(2\pi x)][1-\cos(2\pi y)]$, $\Omega = (0,1)\times(0,1)$				
	$D_x^5 u_{i,j}$		$D_y^5 u_{i,j}$	
n	error	order	error	order
8	1.5507e+04		1.5507e+04	
16	2.0522e+03	2.9177	2.0522e+03	2.9177
32	9.5848e+01	4.4203	9.5848e+01	4.4203

By substituting these approximations into the right-hand sides of equations (3.15)-(3.16) and dropping the $O(h^4)$ remainders, we obtain perturbed right-hand sides $\bar{f}$ and $\bar{g}_k$. For simplicity, we only give the expression of $\bar{f}$ at the interior collocation points (τ_i^x, τ_j^y), for $i, j = 3, \cdots, n-2$:

$$\begin{aligned}
\bar{f}_{i,j} &= f_{i,j} - P_{L_1} w_\Delta(\tau_i^x, \tau_j^y) \\
&\equiv f_{i,j} - p_{i,j}^{(1)} \frac{(24\delta_x D_x^4 - \delta_x^2 D_x^4) w_\Delta(\tau_i^x, \tau_j^y)}{576} + p_{i,j}^{(2)} \frac{(8\delta_x D_x^3 D_y - \delta_x^2 D_x^3 D_y) w_\Delta(\tau_i^x, \tau_j^y)}{192} \\
&\quad + p_{i,j}^{(4)} \frac{(8\delta_y D_x D_y^3 - \delta_y^2 D_x D_y^3) w_\Delta(\tau_i^x, \tau_j^y)}{192} - p_{i,j}^{(5)} \frac{(24\delta_y D_y^4 - \delta_y^2 D_y^4) w_\Delta(\tau_i^x, \tau_j^y)}{576} \\
&\quad + q_{i,j}^{(1)} \frac{(8\delta_x D_x^3 - \delta_x^2 D_x^3) w_\Delta(\tau_i^x, \tau_j^y)}{192} + q_{i,j}^{(4)} \frac{(8\delta_y D_y^3 - \delta_y^2 D_y^3) w_\Delta(\tau_i^x, \tau_j^y)}{192}.
\end{aligned} \tag{3.18}$$

Relation (3.18) defines the perturbation operator P_{L_1} at interior collocation points. The perturbation P_{L_1} at interior-boundary collocation points, as well as the perturbations

$P_{B_{k1}}$, $k = 1, 2$, are defined similarly. Having defined the perturbations P_{L_1} and $P_{B_{k1}}$, $k = 1, 2$, we derive the collocation equations of the second step.

In the *second step of the three-step method*, the spline $v_\Delta \in \widehat{S}_\Delta$ satisfies the collocation equations

$$\begin{aligned} Lv_\Delta(\tau_i^x, \tau_j^y) &= \bar{f}(\tau_i^x, \tau_j^y), \quad (\tau_i^x, \tau_j^y) \in \Pi - \Pi_\partial, \\ B_k v_\Delta(\tau_i^x, \tau_j^y) &= \bar{g}_k(\tau_i^x, \tau_j^y), \quad (\tau_i^x, \tau_j^y) \in \Pi_\partial, k = 1, 2, \end{aligned}$$

which are the same equations as (3.9)-(3.10).

Step 3:

Consider the derivatives of u in the right-hand sides of equations (3.5)-(3.6) and the approximations to these derivatives given by the relations in Theorem 2.2.10, Lemma 2.2.13, and Lemma 2.2.15, which use values of derivatives of S. Notice that some of these approximations are fourth-order, namely (2.63)-(2.68), (2.95)-(2.100), (2.104)-(2.106), (2.111)-(2.118), (2.123)-(2.126), (2.193)-(2.194), (2.200)-(2.202) and (2.208)-(2.209), while the rest are second-order. Recall also that the fourth-order approximations were used in Step 2 to approximate derivatives in the right-hand sides of equations (3.15)-(3.16), which, as mentioned above, show explicitly only the $O(h^2)$ terms of the residual in relations (3.5)-(3.6).

We compute approximations to the derivatives of u in the right-hand sides of equations (3.5)-(3.6) associated with $O(h^2)$ terms, by relations similar to (2.63)-(2.68), (2.95)-(2.100), (2.104)-(2.106), (2.111)-(2.118), (2.123)-(2.126), (2.193)-(2.194), (2.200)-(2.202), and (2.208)-(2.209), except that v_Δ is substituted in place of S. For example, since from equation (2.63) we have

$$D_x^5 u_{i,j} = \frac{8\delta_x D_x^3 S_{i,j} - \delta_x^2 D_x^3 S_{i,j}}{8h^2} + O(h^4), i, j = 3, \cdots, n-2,$$

we approximate $D_x^5 u_{i,j}$ in the term $q_{i,j}^{(1)} \frac{h^2}{24} D_x^5 u_{i,j}$ of (3.5) by

$$D_x^5 u_{i,j} \approx \frac{8\delta_x D_x^3 v_\Delta(\tau_i^x, \tau_j^y) - \delta_x^2 D_x^3 v_\Delta(\tau_i^x, \tau_j^y)}{8h^2}.$$

We also compute approximations to the derivatives of u in the right-hand sides of equations (3.5)-(3.6) associated with $O(h^4)$ terms, by relations similar to the rest of the relations in Theorem 2.2.10, Lemma 2.2.13, and Lemma 2.2.15, except that v_Δ is substituted in place of S. For example, since from equation (2.59) we have

$$D_x^5 u_{i,j} = \frac{\delta_x^2 D_x^{k-4} S_{i,j}}{h^4} + O(h^2), \quad i,j = 3, \cdots, n-2,$$

we approximate $D_x^5 u_{i,j}$ in the term $-s_{i,j}^{(1)} \frac{7h^4}{5760} D_x^5 u_{i,j}$ of (3.5) by

$$D_x^5 u_{i,j} \approx \frac{\delta_x^2 D_x^{k-4} v_\Delta(\tau_i^x, \tau_j^y)}{h^4}.$$

By substituting these approximations into the right-hand sides of equations (3.5)-(3.6) and dropping the $O(h^6)$ remainders, we obtain new perturbed right-hand sides $\bar{\bar{f}}$ and $\bar{\bar{g}}_k$. For simplicity, we only give the expression of $\bar{\bar{f}}$ at the interior collocation points (τ_i^x, τ_j^y), for $i,j = 3, \cdots, n-2$:

$$\begin{aligned}
\bar{\bar{f}}_{i,j} &= f_{i,j} - P_L v_\Delta(\tau_i^x, \tau_j^y) \\
&\equiv f_{i,j} - p_{i,j}^{(1)} \frac{(240\delta_x D_x^4 - 17\delta_x^2 D_x^4) v_\Delta(\tau_i^x, \tau_j^y)}{5760} \\
&\quad + p_{i,j}^{(2)} \frac{240\delta_x D_x^3 D_y - 17\delta_x^2 D_x^3 D_y - 7\delta_y^2 D_x^3 D_y) v_\Delta(\tau_i^x, \tau_j^y)}{5760} \\
&\quad + p_{i,j}^{(3)} \frac{7(\delta_x^2 + \delta_y^2) D_x^2 D_y^2 v_\Delta(\tau_i^x, \tau_j^y)}{1920} + p_{i,j}^{(4)} \frac{(240\delta_y D_x D_y^3 - 17\delta_y^2 D_x D_y^3 - 7\delta_x^2 D_x D_y^3) v_\Delta(\tau_i^x, \tau_j^y)}{5760} \\
&\quad - p_{i,j}^{(5)} \frac{(240\delta_y D_y^4 - 17\delta_y^2 D_y^4) v_\Delta(\tau_i^x, \tau_j^y)}{5760} + q_{i,j}^{(1)} \frac{(80\delta_x D_x^3 - 17\delta_x^2 D_x^3) v_\Delta(\tau_i^x, \tau_j^y)}{1920} \\
&\quad + q_{i,j}^{(2)} \frac{7(3\delta_x^2 D_x^2 D_y - \delta_y^2 D_x^2 D_y) v_\Delta(\tau_i^x, \tau_j^y)}{5760} + q_{i,j}^{(3)} \frac{7(3\delta_y^2 D_x D_y^2 - \delta_x^2 D_x D_y^2) v_\Delta(\tau_i^x, \tau_j^y)}{5760} \\
&\quad + q_{i,j}^{(4)} \frac{(80\delta_y D_y^3 - 17\delta_y^2 D_y^3) v_\Delta(\tau_i^x, \tau_j^y)}{1920} + r_{i,j}^{(1)} \frac{7\delta_x^2 D_x^2 v_\Delta(\tau_i^x, \tau_j^y)}{1920} \\
&\quad - r_{i,j}^{(2)} \frac{7(\delta_x^2 + \delta_y^2) D_x D_y v_\Delta(\tau_i^x, \tau_j^y)}{5760} + r_{i,j}^{(3)} \frac{7\delta_y^2 D_y^2 v_\Delta(\tau_i^x, \tau_j^y)}{1920} - s_{i,j}^{(1)} \frac{7\delta_x^2 D_x v_\Delta(\tau_i^x, \tau_j^y)}{5760} \\
&\quad - s_{i,j}^{(2)} \frac{7\delta_y^2 D_y v_\Delta(\tau_i^x, \tau_j^y)}{5760} \qquad (3.19) \\
&\equiv f_{i,j} - P_{L_1} v_\Delta(\tau_i^x, \tau_j^y) - P_{L_2} v_\Delta(\tau_i^x, \tau_j^y). \qquad (3.20)
\end{aligned}$$

Relations (3.19) and (3.20) define the perturbation operators P_L, and P_{L_2}, respectively, at interior collocation points, taking into account that P_{L_1} is already defined. The perturbations P_L and P_{L_2} at interior-boundary collocation points, as well as the perturbations P_{B_k} and $P_{B_{k2}}$, $k = 1, 2$, are defined similarly. Having defined the perturbations P_{L_2} and $P_{B_{k2}}$, $k = 1, 2$, we derive the collocation equations of the third step. In the *third step of the three-step method* the spline $u_\Delta \in \widehat{S}_\Delta$ satisfies the collocation equations

$$\begin{aligned} Lu_\Delta(\tau_i^x, \tau_j^y) &= \bar{\bar{f}}(\tau_i^x, \tau_j^y), \quad (\tau_i^x, \tau_j^y) \in \Pi - \Pi_\partial, \\ B_k u_\Delta(\tau_i^x, \tau_j^y) &= \bar{\bar{g}}_k(\tau_i^x, \tau_j^y), \quad (\tau_i^x, \tau_j^y) \in \Pi_\partial, k = 1, 2, \end{aligned}$$

which are the same equations as (3.11)-(3.12).

3.4.2 One-Step Method

Having defined the perturbations P_L and P_{B_k}, $k = 1, 2$, the collocation equations of the one-step method (3.7)-(3.8) are also defined. Here, we give more explanation how these equations are derived.

Consider the equations (3.1)-(3.2) at the collocation points, that is,

$$\begin{aligned} & p_{i,j}^{(1)} D_x^4 u_{i,j} + p_{i,j}^{(2)} D_x^3 D_y u_{i,j} + p_{i,j}^{(3)} D_x^2 D_y^2 u_{i,j} + p_{i,j}^{(4)} D_x D_y^3 u_{i,j} + p_{i,j}^{(5)} D_y^4 u_{i,j} \\ & + q_{i,j}^{(1)} D_x^3 u_{i,j} + q_{i,j}^{(2)} D_x^2 D_y u_{i,j} + q_{i,j}^{(3)} D_x D_y^2 u_{i,j} + q_{i,j}^{(4)} D_y^3 u_{i,j} \\ & + r_{i,j}^{(1)} D_x^2 u_{i,j} + r_{i,j}^{(2)} D_x D_y u_{i,j} + r^{(3)} D_y^2 u_{i,j} + s_{i,j}^{(1)} D_x u_{i,j} + s_{i,j}^{(2)} D_y u_{i,j} + t_{i,j} u_{i,j} \\ = \; & f(\tau_i^x, \tau_j^y), \quad (\tau_i^x, \tau_j^y) \in \Pi - \Pi_\partial, \end{aligned} \tag{3.21}$$

$$\begin{aligned} & \alpha_{i,j}^{(k,1)} D_x^3 u_{i,j} + \alpha_{i,j}^{(k,2)} D_x^2 D_y u_{i,j} + \alpha_{i,j}^{(k,3)} D_x D_y^2 u_{i,j} + \alpha_{i,j}^{(k,4)} D_y^3 u_{i,j} \\ & + \beta_{i,j}^{(k,1)} D_x^2 u_{i,j} + \beta_{i,j}^{(k,2)} D_x D_y u_{i,j} + \beta_{i,j}^{(k,3)} D_y^2 u_{i,j} + \phi_{i,j}^{(k,1)} D_x u_{i,j} \\ & + \phi_{i,j}^{(k,2)} D_y u_{i,j} + \psi_{i,j}^{(k)} u_{i,j} \\ = \; & g_k(\tau_i^x, \tau_j^y), \quad (\tau_i^x, \tau_j^y) \in \Pi_\partial, k = 1, 2. \end{aligned} \tag{3.22}$$

In Corollaries 2.2.11, 2.2.14 and 2.2.16, sixth-order approximations of the derivatives of u at the collocation points are derived using values of the derivatives of S. We substitute these approximations into (3.21)-(3.22) and obtain new equations involving values of the derivatives of S. For simplicity, we present the equations only at interior collocation points (τ_i^x, τ_j^y), $i, j = 3, \cdots, n-2$:

$$
\begin{aligned}
& p_{i,j}^{(1)} D_x^4 S_{i,j} + p_{i,j}^{(2)} D_x^3 D_y S_{i,j} + p_{i,j}^{(3)} D_x^2 D_y^2 S_{i,j} + p_{i,j}^{(4)} D_x D_y^3 S_{i,j} + p_{i,j}^{(5)} D_y^4 S_{i,j} \\
& + q_{i,j}^{(1)} D_x^3 S_{i,j} + q_{i,j}^{(2)} D_x^2 D_y S_{i,j} + q_{i,j}^{(3)} D_x D_y^2 S_{i,j} + q_{i,j}^{(4)} D_y^3 S_{i,j} \\
& + r_{i,j}^{(1)} D_x^2 S_{i,j} + r_{i,j}^{(2)} D_x D_y S_{i,j} + r^{(3)} D_y^2 S_{i,j} + s_{i,j}^{(1)} D_x S_{i,j} + s_{i,j}^{(2)} D_y S_{i,j} + t_{i,j} S_{i,j} \\
& + p_{i,j}^{(1)} \frac{(240\delta_x D_x^4 - 17\delta_x^2 D_x^4) S_{i,j}}{5760} - p_{i,j}^{(2)} \frac{(240\delta_x D_x^3 D_y - 17\delta_x^2 D_x^3 D_y - 7\delta_y^2 D_x^3 D_y) S_{i,j}}{5760} \\
& - p_{i,j}^{(3)} \frac{7(\delta_x^2 + \delta_y^2) D_x^2 D_y^2 S_{i,j}}{1920} - p_{i,j}^{(4)} \frac{(240\delta_y D_x D_y^3 - 17\delta_y^2 D_x D_y^3 - 7\delta_x^2 D_x D_y^3) S_{i,j}}{5760} \\
& + p_{i,j}^{(5)} \frac{(240\delta_y D_y^4 - 17\delta_y^2 D_y^4) S_{i,j}}{5760} - q_{i,j}^{(1)} \frac{(80\delta_x D_x^3 - 17\delta_x^2 D_x^3) S_{i,j}}{1920} \\
& + q_{i,j}^{(2)} \frac{7(\delta_y^2 D_x^2 D_y - 3\delta_x^2 D_x^2 D_y) S_{i,j}}{5760} + q_{i,j}^{(3)} \frac{7(\delta_x^2 D_x D_y^2 - 3\delta_y^2 D_x D_y^2) S_{i,j}}{5760} \\
& - q_{i,j}^{(4)} \frac{(80\delta_y D_y^3 - 17\delta_y^2 D_y^3) S_{i,j}}{1920} - r_{i,j}^{(1)} \frac{7\delta_x^2 D_x^2 S_{i,j}}{1920} + r_{i,j}^{(2)} \frac{7(\delta_x^2 + \delta_y^2) D_x D_y S_{i,j}}{5760} \\
& - r_{i,j}^{(3)} \frac{7\delta_y^2 D_y^2 S_{i,j}}{1920} + s_{i,j}^{(1)} \frac{7\delta_x^2 D_x S_{i,j}}{5760} + s_{i,j}^{(2)} \frac{7\delta_y^2 D_y S_{i,j}}{5760} \\
= \; & f_{i,j} + O(h^6).
\end{aligned}
$$

Notice that the left side of (3.23) is $LS_{i,j} + P_L S_{i,j}$ with $P_L(\cdot)_{i,j}$ defined in (3.19). That is, we have $(L + P_L)S(\tau_i^x, \tau_j^y) = f(\tau_i^x, \tau_j^y) + O(h^6)$. Similarly, we get the respective relations at the interior-boundary and boundary collocation points. Thus we have,

$$
(L + P_L)S(\tau_i^x, \tau_j^y) = f(\tau_i^x, \tau_j^y) + O(h^6), \quad (\tau_i^x, \tau_j^y) \in \Pi - \Pi_\partial, \tag{3.23}
$$

$$
(B_k + P_{B_k})S(\tau_i^x, \tau_j^y) = g_k(\tau_i^x, \tau_j^y) + O(h^6), \quad (\tau_i^x, \tau_j^y) \in \Pi_\partial, k = 1, 2. \tag{3.24}
$$

We are now ready to derive the collocation equations of the one-step method. In the one-step method, the spline $u_\Delta \in \widehat{S}_\Delta$ satisfies the collocation equations

$$
\begin{aligned}
(L + P_L)u_\Delta(\tau_i^x, \tau_j^y) &= f(\tau_i^x, \tau_j^y), \quad (\tau_i^x, \tau_j^y) \in \Pi - \Pi_\partial, \\
(B_k + P_{B_k})u_\Delta(\tau_i^x, \tau_j^y) &= g_k(\tau_i^x, \tau_j^y), \quad (\tau_i^x, \tau_j^y) \in \Pi_\partial, k = 1, 2,
\end{aligned}
$$

which are the same equations as (3.7)-(3.8). It is worth noticing that the collocation equations of the one-step method resemble the residual equations (3.23)-(3.24), except that u_Δ is substituted in place of S, and the $O(h^6)$ terms of the residual are dropped.

Chapter 4

Biharmonic Problems

4.1 Introduction

In this chapter, we introduce several types of biharmonic problems. In particular, we consider the numerical solution of the biharmonic Dirichlet problem, a two-dimensional fourth-order boundary value problem, which is defined by

$$\begin{aligned} Lu \equiv D_x^4 u + 2D_x^2 D_y^2 u + D_y^4 u &= f(x,y) \quad \text{in} \quad \Omega \equiv (0,1)\times(0,1), & (4.1)\\ B_1 u \equiv u &= g_1(x,y) \quad \text{on} \quad \partial\Omega, & (4.2)\\ B_2 u \equiv \frac{\partial u}{\partial \mathrm{n}} &= g_2(x,y) \quad \text{on} \quad \partial\Omega, & (4.3)\end{aligned}$$

where $\frac{\partial u}{\partial \mathrm{n}}$ is the outer normal derivative of u on $\partial\Omega$. In this special case of (3.1)-(3.2), the coefficient functions satisfy $p^{(1)}(x,y) = p^{(5)}(x,y) = 1$, $p^{(3)}(x,y) = 2$, $\psi^{(1)}(x,y) = 1$, $\phi^{(2,1)}(0,y) = \phi^{(2,1)}(1,y) = 1$, $\phi^{(2,2)}(x,0) = \phi^{(2,1)}(x,1) = 1$, and the rest of the coefficient functions are all zero.

In general, there is no known analytical solution to this problem. In some special case, however, the solution can be expressed as the sum of an infinite series [42], which will be discussed in Chapter 8. In this chapter, we only discuss the numerical solution to this problem.

The two optimal Bi-Quartic SC methods developed in the preceding chapter, namely, the one-step and the three-step methods, can both be applied to the biharmonic Dirichlet problem. Though both methods achieve the same order of convergence, for some problems the one-step method may obtain more accurate numerical results than the three-step method. On the other hand, as we will see in later chapters, the three-step Bi-Quartic SC method runs faster than the one-step method, since the linear systems arising from the three-step method can be solved efficiently by a fast solver based on Fast Fourier Transforms and preconditioned GMRES as will be described in the next chapter.

4.2 Three-step Bi-Quartic SC

Since we have already given the detailed formulation of the three-step bi-quartic spline collocation method for a general two-dimensional fourth-order boundary value problem in Section 3.4.1, we only sketch the derivation of the three-step spline collocation method for the biharmonic Dirichlet problem, but we give the perturbed right-hand sides $\bar{f}$, $\bar{g}$, $\bar{\bar{f}}$, and $\bar{\bar{g}}$ in more details.

Step 1: Determine a spline $w_\Delta \in \widehat{S}_\Delta$ by the equations

$$\begin{aligned}
Lw_\Delta(\tau_i^x, \tau_j^y) &= f(\tau_i^x, \tau_j^y), \quad i, j = 1, \cdots, n, &(4.4)\\
w_\Delta(\tau_{\gamma_0}^x, \tau_j^y) &= g_1(\tau_{\gamma_0}^x, \tau_j^y), \quad j = 0, \cdots, n+1, &(4.5)\\
w_\Delta(\tau_i^x, \tau_{\gamma_0}^y) &= g_1(\tau_i^x, \tau_{\gamma_0}^y), \quad i = 1, \cdots, n, &(4.6)\\
D_x w_\Delta(\tau_{\gamma_0}^x, \tau_j^y) &= g_2(\tau_{\gamma_0}^x, \tau_j^y), \quad j = 0, \cdots, n+1, &(4.7)\\
D_y w_\Delta(\tau_i^x, \tau_{\gamma_0}^y) &= g_2(\tau_i^x, \tau_{\gamma_0}^y), \quad i = 0, \cdots, n+1. &(4.8)
\end{aligned}$$

To uniquely determine the spline $w_\Delta \in \widehat{S}_\Delta$, $(n+4)^2$ independent equations are needed. However, the system (4.4)-(4.8) has only $(n+4)^2 - 4$ equations. Therefore, four additional equations are needed, which can be made up by approximating some (cross-)derivatives of w_Δ at some collocation points. For the biharmonic Dirichlet problem in this thesis, we introduce four additional equations by approximating the cross-derivative $D_x D_y w_\Delta$ at the four corners. From (2.30), we have

$$D_x D_y S(\tau^x_{\gamma_0}, \tau^y_{\gamma_0}) = D_x D_y u(\tau^x_{\gamma_0}, \tau^y_{\gamma_0}) + O(h^4). \tag{4.9}$$

Notice that the values of $D_x D_y u_{\gamma_0,\gamma_0}$ are not given. However, by the Taylor's expansions, it is easy to see that

$$D_x D_y u(\tau^x_{\gamma_0}, \tau^y_{\gamma_0}) = \frac{1}{h} \sqcap D_y u(\tau^x_{\gamma_0}, \tau^y_{\gamma_0}) + O(h^7),$$

where $h = \frac{1}{n}$ is the stepsize, and $\sqcap$ is a discrete operator defined by

$$\begin{aligned}
\sqcap z(x_0, y) &= \frac{49}{20} z(x_0, y) + 6z(x_1, y) - \frac{15}{2} z(x_2, y) + \frac{20}{3} z(x_3, y) - \frac{15}{4} z(x_4, y) \\
&\quad + \frac{6}{5} z(x_5, y) - \frac{50}{3} z(x_6, y), \\
\sqcap z(x_n, y) &= \frac{49}{20} z(x_n, y) - 6z(x_{n-1}, y) + \frac{15}{2} z(x_{n-2}, y) - \frac{20}{3} z(x_{n-3}, y) + \frac{15}{4} z(x_{n-4}, y) \\
&\quad - \frac{6}{5} z(x_{n-5}, y) + \frac{50}{3} z(x_{n-6}, y),
\end{aligned}$$

for any function $z(x, y)$. From the boundary condition (4.3), we have

$$\begin{aligned}
D_y u(x, y_0) &= g_2(x, y_0), \\
D_y u(x, y_n) &= g_2(x, y_n).
\end{aligned}$$

Therefore,

$$D_x D_y u(\tau^x_{\gamma_0}, \tau^y_{\gamma_0}) = \frac{1}{h} \sqcap g_2(\tau^x_{\gamma_0}, \tau^y_{\gamma_0}) + O(h^7), \tag{4.10}$$

To keep the notation consistent, we define $g_c(\tau^x_{\gamma_0}, \tau^y_{\gamma_0})$ by

$$g_c(\tau^x_{\gamma_0}, \tau^y_{\gamma_0}) \equiv \frac{1}{h} \sqcap g_2(\tau^x_{\gamma_0}, \tau^y_{\gamma_0}), \tag{4.11}$$

and we remind readers that $(\tau^x_{\gamma_0}, \tau^y_{\gamma_0})$ is any of the points (x_0, y_0), (x_0, y_n), (x_n, y_0), and (x_n, y_n). Then the equation (4.10) can be written as

$$D_x D_y u(\tau^x_{\gamma_0}, \tau^y_{\gamma_0}) = g_c(\tau^x_{\gamma_0}, \tau^y_{\gamma_0}) + O(h^7), \tag{4.12}$$

Now we make up four additional equations for the system (4.4)-(4.8) by

$$
\begin{aligned}
D_x D_y w_\Delta(x_0, y_0) &= g_c(x_0, y_0), && (4.13)\\
D_x D_y w_\Delta(x_0, y_n) &= g_c(x_0, y_n), && (4.14)\\
D_x D_y w_\Delta(x_n, y_0) &= g_c(x_n, y_0), && (4.15)\\
D_x D_y w_\Delta(x_n, y_n) &= g_c(x_n, y_n). && (4.16)
\end{aligned}
$$

Step 2:

(i) From (3.5)-(3.6) and (4.1)-(4.3), we have

$$
\begin{aligned}
LS_{i,j} &= f_{i,j} - \frac{h^2}{24} D_x^6 u_{i,j} - \frac{h^2}{24} D_y^6 u_{i,j} + O(h^4), \quad i,j = 1, \cdots, n, && (4.17)\\
S_{\gamma_0,j} &= g_1(\gamma_0, \tau_j^y) + O(h^6), \quad j = 0, \cdots, n+1, && (4.18)\\
S_{i,\gamma_0} &= g_1(\tau_i^x, \gamma_0) + O(h^6), \quad i = 1, \cdots, n && (4.19)\\
D_x S_{\gamma_0,j} &= g_2(\gamma_0, \tau_j^y) + O(h^4), \quad j = 0, \cdots, n+1, && (4.20)\\
D_y S_{i,\gamma_0} &= g_2(\tau_i^x, \gamma_0) + O(h^4), \quad i = 1, \cdots, n. && (4.21)
\end{aligned}
$$

From (4.9), we have

$$
D_x D_y S(\tau_{\gamma_0}^x, \tau_{\gamma_0}^y) = g_c(\tau_{\gamma_0}^x, \tau_{\gamma_0}^y) + O(h^4), \tag{4.22}
$$

(ii) Compute approximations of $D_x^6 u_{i,j}$ and $D_y^6 u_{i,j}$ by the relations (2.64), (2.67), (2.96), (2.99), (2.100), and (2.115)-(2.118), except that w_Δ is substituted in place of S. Substituting these approximations of $D_x^6 u_{i,j}$ and $D_y^6 u_{i,j}$ into the right-hand side of (4.17), we obtain the perturbed right-hand side $\bar{f}$:

$$
\begin{aligned}
\bar{f}_{i,j} &= f_{i,j} - P_{L_1} w_\Delta(\tau_i^x, \tau_j^y) && (4.23)\\
&\equiv f_{i,j} - \frac{(24\delta_x D_x^4 - \delta_x^2 D_x^4) w_\Delta(\tau_i^x, \tau_j^y)}{576} - \frac{(24\delta_y D_y^4 - \delta_y^2 D_y^4) w_\Delta(\tau_i^x, \tau_j^y)}{576},\\
&\quad i,j = 3, \cdots, n-2,\\
\bar{f}_{\gamma_1,j} &= f_{\gamma_1,j} - P_{L_1} w_\Delta(\tau_{\gamma_1}^x, \tau_j^y) && (4.24)\\
&\equiv f_{\gamma_1,j} - \frac{\zeta_{24}^x D_x^4 w_\Delta(\tau_{\gamma_2}^x, \tau_j^y)}{24} - \frac{(24\delta_y D_y^4 - \delta_y^2 D_y^4) w_\Delta(\tau_{\gamma_1}^x, \tau_j^y)}{576}, \quad j = 3, \cdots, n-2,
\end{aligned}
$$

$$\begin{aligned}\bar{f}_{\gamma_2,j} &= f_{\gamma_2,j} - P_{L_1} w_\Delta(\tau^x_{\gamma_2}, \tau^y_j) \\ &\equiv f_{\gamma_2,j} - \frac{\eta^x_{24} D^4_x w_\Delta(\tau^x_{\gamma_2}, \tau^y_j)}{24} - \frac{(24\delta_y D^4_y - \delta^2_y D^4_y) w_\Delta(\tau^x_{\gamma_2}, \tau^y_j)}{576}, \quad j = 3, \cdots, n-2,\end{aligned} \tag{4.25}$$

$$\begin{aligned}\bar{f}_{i,\gamma_1} &= f_{i,\gamma_1} - P_{L_1} w_\Delta(\tau^x_i, \tau^y_{\gamma_1}) \\ &\equiv f_{i,\gamma_1} - \frac{(24\delta_x D^4_x - \delta^2_x D^4_x) w_\Delta(\tau^x_i, \tau^y_{\gamma_1})}{576} - \frac{\zeta^y_{24} D^4_y w_\Delta(\tau^x_i, \tau^y_{\gamma_2})}{24}, \quad i = 3, \cdots, n-2,\end{aligned} \tag{4.26}$$

$$\begin{aligned}\bar{f}_{i,\gamma_2} &= f_{i,\gamma_2} - P_{L_1} w_\Delta(\tau^x_i, \tau^y_{\gamma_2}) \\ &\equiv f_{i,\gamma_2} - \frac{(24\delta_x D^4_x - \delta^2_x D^4_x) w_\Delta(\tau^x_i, \tau^y_{\gamma_1})}{576} - \frac{\eta^y_{24} D^4_y w_\Delta(\tau^x_i, \tau^y_{\gamma_2})}{24}, \quad i = 3, \cdots, n-2,\end{aligned} \tag{4.27}$$

$$\begin{aligned}\bar{f}_{\gamma_1,\gamma_1} &= f_{\gamma_1,\gamma_1} - P_{L_1} w_\Delta(\tau^x_{\gamma_1}, \tau^y_{\gamma_1}) \\ &\equiv f_{\gamma_1,\gamma_1} - \frac{\zeta^x_{24} D^4_x w_\Delta(\tau^x_{\gamma_2}, \tau^y_{\gamma_1})}{24} - \frac{\zeta^y_{24} D^4_y w_\Delta(\tau^x_{\gamma_1}, \tau^y_{\gamma_2})}{24},\end{aligned} \tag{4.28}$$

$$\begin{aligned}\bar{f}_{\gamma_2,\gamma_1} &= f_{\gamma_2,\gamma_1} - P_{L_1} w_\Delta(\tau^x_{\gamma_2}, \tau^y_{\gamma_1}) \\ &\equiv f_{\gamma_2,\gamma_1} - \frac{\eta^x_{24} D^4_x w_\Delta(\tau^x_{\gamma_2}, \tau^y_{\gamma_1})}{24} - \frac{\zeta^y_{24} D^4_y w_\Delta(\tau^x_{\gamma_2}, \tau^y_{\gamma_2})}{24},\end{aligned} \tag{4.29}$$

$$\begin{aligned}\bar{f}_{\gamma_1,\gamma_2} &= f_{\gamma_1,\gamma_2} - P_{L_1} w_\Delta(\tau^x_{\gamma_1}, \tau^y_{\gamma_2}) \\ &\equiv f_{\gamma_1,\gamma_2} - \frac{\zeta^x_{24} D^4_x w_\Delta(\tau^x_{\gamma_2}, \tau^y_{\gamma_2})}{24} - \frac{\eta^y_{24} D^4_y w_\Delta(\tau^x_{\gamma_1}, \tau^y_{\gamma_2})}{24},\end{aligned} \tag{4.30}$$

$$\begin{aligned}\bar{f}_{\gamma_2,\gamma_2} &= f_{\gamma_2,\gamma_2} - P_{L_1} w_\Delta(\tau^x_{\gamma_2}, \tau^y_{\gamma_2}) \\ &\equiv f_{\gamma_2,\gamma_2} - \frac{\eta^x_{24} D^4_x w_\Delta(\tau^x_{\gamma_2}, \tau^y_{\gamma_2})}{24} - \frac{\eta^y_{24} D^4_y w_\Delta(\tau^x_{\gamma_2}, \tau^y_{\gamma_2})}{24}.\end{aligned} \tag{4.31}$$

(iii) Determine a spline $v_\Delta \in \widehat{S}_\Delta$ by the following equations

$$L v_\Delta(\tau^x_i, \tau^y_j) = \bar{f}(\tau^x_i, \tau^y_j), \quad i, j = 1, \cdots, n, \tag{4.32}$$

$$v_\Delta(\tau^x_{\gamma_0}, \tau^y_j) = g_1(\gamma_0, \tau^y_j), \quad j = 0, \cdots, n+1, \tag{4.33}$$

$$v_\Delta(\tau^x_i, \tau^y_{\gamma_0}) = g_1(\tau^x_i, \gamma_0), \quad i = 1, \cdots, n \tag{4.34}$$

$$D_x v_\Delta(\tau^x_{\gamma_0}, \tau^y_j) = g_2(\gamma_0, \tau^y_j), \quad j = 0, \cdots, n+1, \tag{4.35}$$

$$D_y v_\Delta(\tau^x_i, \tau^y_{\gamma_0}) = g_2(\tau^x_i, \gamma_0), \quad i = 0, \cdots, n+1, \tag{4.36}$$

$$D_x D_y v_\Delta(\tau^x_{\gamma_0}, \tau^y_{\gamma_0}) = g_c(\tau^x_{\gamma_0}, \tau^y_{\gamma_0}), \tag{4.37}$$

It is worth noticing that, in this step, the right-hand sides g_1, g_2, and g_c of the boundary condition equations do not need any perturbation and remain the same as in the first step.

Step 3:

(i) From (3.5)-(3.6) and (4.1)-(4.3), we have

$$\begin{aligned}
LS_{i,j} &= Lu_{i,j} - \frac{h^2}{24}D_x^6 u_{i,j} + \frac{7h^4}{5760}D_x^8 u_{i,j} + \frac{7h^4}{960}D_x^2 D_y^6 u_{i,j} + \frac{7h^4}{960}D_x^6 D_y^2 u_{i,j} \\
&\quad -\frac{h^2}{24}D_y^6 u_{i,j} + \frac{7h^4}{5760}D_y^8 u_{i,j} + O(h^6), \quad i,j = 1,\cdots,n, \qquad (4.38) \\
S_{\gamma_0,j} &= g_1(\gamma_0, \tau_j^y) + O(h^6), \quad j = 0,\cdots,n+1, \qquad (4.39) \\
S_{i,\gamma_0} &= g_1(\tau_i^x, \gamma_0) + O(h^6), \quad i = 1,\cdots,n \qquad (4.40) \\
D_x S_{\gamma_0,j} &= g_2(\gamma_0, \tau_j^y) + \frac{h^4}{720}D_x^5 u_{\gamma_0,j} + O(h^6), \quad j = 0,\cdots,n+1, \qquad (4.41) \\
D_y S_{i,\gamma_0} &= g_2(\tau_i^x, \gamma_0) + \frac{h^4}{720}D_y^5 u_{i,\gamma_0} + O(h^6), \quad i = 0,\cdots,n+1. \qquad (4.42)
\end{aligned}$$

From (2.30) and (4.10), we have

$$D_x D_y S(\tau_{\gamma_0}^x, \tau_{\gamma_0}^y) = g_c(\tau_{\gamma_0}^x, \tau_{\gamma_0}^y) + (\frac{h^4}{720}D_x^5 D_y + \frac{h^4}{720}D_x D_y^5)u(\tau_{\gamma_0}^x, \tau_{\gamma_0}^y) + O(h^6). \qquad (4.43)$$

(ii) Compute approximations of the derivatives of u in the right-hand sides of equations (4.38)-(4.43). Specifically, the derivatives of u associated with $O(h^2)$ terms are approximated by relations (2.64), (2.67), (2.96), (2.99), (2.100), and (2.115)-(2.118), and the derivatives associated with $O(h^4)$ terms are approximated by relations (2.59)-(2.62), (2.92)-(2.94), (2.101)-(2.104), (2.191), (2.197), (2.206)-(2.207), (2.210)-(2.211), except that v_Δ is substituted in place of S. Substituting these approximations into the right-hand sides of equations (4.38)-(4.43) and ignoring the $O(h^6)$ remainders, we obtain new perturbed right-hand sides $\bar{\bar{f}}$, $\bar{\bar{g}}_2$, and $\bar{\bar{g}}_c$:

$$\begin{aligned}
\bar{\bar{f}}_{i,j} &= f_{i,j} - P_L v_\Delta(\tau_i^x, \tau_j^y) \qquad (4.44) \\
&\equiv f_{i,j} - \frac{(240\delta_x D_x^4 - 17\delta_x^2 D_x^4)v_\Delta(\tau_i^x, \tau_j^y)}{5760} + \frac{7(\delta_x^2 + \delta_y^2)D_x^2 D_y^2 v_\Delta(\tau_i^x, \tau_j^y)}{960} \\
&\quad -\frac{(240\delta_y D_y^4 - 17\delta_y^2 D_y^4)v_\Delta(\tau_i^x, \tau_j^y)}{5760}, \quad i,j = 3,\cdots,n-2,
\end{aligned}$$

$$
\begin{aligned}
\bar{\bar{f}}_{\gamma_1,j} &= f_{\gamma_1,j} - P_L v_\Delta(\tau^x_{\gamma_1}, \tau^y_j) \qquad (4.45)\\
&\equiv f_{\gamma_1,j} - \frac{\zeta^x_{24} D^4_x v_\Delta(\tau^x_{\gamma_2}, \tau^y_j)}{24} + \frac{7(3\delta^2_x D^4_x v_\Delta(\tau^x_{\gamma_3}, \tau^y_j) - 2\delta^2_x D^4_x v_\Delta(\tau^x_{\gamma_4}, \tau^y_j))}{5760}\\
&+ \frac{7(3\delta^2_x D^2_x D^2_y v_\Delta(\tau^x_{\gamma_3}, \tau^y_j) - 2\delta^2_x D^2_x D^2_y v_\Delta(\tau^x_{\gamma_4}, \tau^y_j))}{960} + \frac{7\delta^2_y D^2_x D^2_y v_\Delta(\tau^x_{\gamma_1}, \tau^y_j)}{960}\\
&- \frac{(240\delta_y D^4_y - 17\delta^2_y D^4_y) v_\Delta(\tau^x_{\gamma_1}, \tau^y_j)}{5760}, \quad j = 3, \cdots, n-2,
\end{aligned}
$$

$$
\begin{aligned}
\bar{\bar{f}}_{\gamma_2,j} &= f_{\gamma_2,j} - P_L v_\Delta(\tau^x_{\gamma_2}, \tau^y_j) \qquad (4.46)\\
&\equiv f_{\gamma_2,j} - \frac{\eta^x_{24} D^4_x v_\Delta(\tau^x_{\gamma_2}, \tau^y_j)}{24} + \frac{7(2\delta^2_x D^4_x v_\Delta(\tau^x_{\gamma_3}, \tau^y_j) - \delta^2_x D^4_x v_\Delta(\tau^x_{\gamma_4}, \tau^y_j))}{5760}\\
&+ \frac{7(2\delta^2_x D^2_x D^2_y v_\Delta(\tau^x_{\gamma_3}, \tau^y_j) - \delta^2_x D^2_x D^2_y v_\Delta(\tau^x_{\gamma_4}, \tau^y_j))}{960} + \frac{7\delta^2_y D^2_x D^2_y v_\Delta(\tau^x_{\gamma_2}, \tau^y_j)}{960}\\
&- \frac{(240\delta_y D^4_y - 17\delta^2_y D^4_y) v_\Delta(\tau^x_{\gamma_2}, \tau^y_j)}{5760}, \quad j = 3, \cdots, n-2,
\end{aligned}
$$

$$
\begin{aligned}
\bar{\bar{f}}_{i,\gamma_1} &= f_{i,\gamma_1} - P_L v_\Delta(\tau^x_i, \tau^y_{\gamma_1}) \qquad (4.47)\\
&\equiv f_{i,\gamma_1} - \frac{(240\delta_x D^4_x - 17\delta^2_x D^4_x) v_\Delta(\tau^x_i, \tau^y_{\gamma_1})}{5760} + \frac{7\delta^2_x D^2_x D^2_y v_\Delta(\tau^x_i, \tau^y_{\gamma_1})}{960}\\
&+ \frac{7(3\delta^2_y D^2_x D^2_y v_\Delta(\tau^x_i, \tau^y_{\gamma_3}) - 2\delta^2_y D^2_x D^2_y v_\Delta(\tau^x_i, \tau^y_{\gamma_4}))}{960} - \frac{\zeta^y_{24} D^4_y v_\Delta(\tau^x_i, \tau^y_{\gamma_2})}{24}\\
&+ \frac{7(3\delta^2_y D^4_y v_\Delta(\tau^x_i, \tau^y_{\gamma_3}) - 2\delta^2_y D^4_y v_\Delta(\tau^x_i, \tau^y_{\gamma_4}))}{5760}, \quad i = 3, \cdots, n-2,
\end{aligned}
$$

$$
\begin{aligned}
\bar{\bar{f}}_{i,\gamma_2} &= f_{i,\gamma_2} - P_L v_\Delta(\tau^x_i, \tau^y_{\gamma_2}) \qquad (4.48)\\
&\equiv f_{i,\gamma_2} - \frac{(240\delta_x D^4_x - 17\delta^2_x D^4_x) v_\Delta(\tau^x_i, \tau^y_{\gamma_2})}{5760} + \frac{7\delta^2_x D^2_x D^2_y v_\Delta(\tau^x_i, \tau^y_{\gamma_2})}{960}\\
&+ \frac{7(2\delta^2_y D^2_x D^2_y v_\Delta(\tau^x_i, \tau^y_{\gamma_3}) - \delta^2_y D^2_x D^2_y v_\Delta(\tau^x_i, \tau^y_{\gamma_4}))}{960} - \frac{\eta^y_{24} D^4_y v_\Delta(\tau^x_i, \tau^y_{\gamma_2})}{24}\\
&+ \frac{7(2\delta^2_y D^4_y v_\Delta(\tau^x_i, \tau^y_{\gamma_3}) - \delta^2_y D^4_y v_\Delta(\tau^x_i, \tau^y_{\gamma_4}))}{5760}, \quad i = 3, \cdots, n-2,
\end{aligned}
$$

$$
\begin{aligned}
\bar{\bar{f}}_{\gamma_1,\gamma_1} &= f_{\gamma_1,\gamma_1} - P_L v_\Delta(\tau^x_{\gamma_1}, \tau^y_{\gamma_1}) \qquad (4.49)\\
&\equiv f_{\gamma_1,\gamma_1} - \frac{\zeta^x_{24} D^4_x v_\Delta(\tau^x_{\gamma_2}, \tau^y_{\gamma_1})}{24} + \frac{7(3\delta^2_x D^4_x v_\Delta(\tau^x_{\gamma_3}, \tau^y_{\gamma_1}) - 2\delta^2_x D^4_x v_\Delta(\tau^x_{\gamma_4}, \tau^y_{\gamma_1}))}{5760}\\
&+ \frac{7(3\delta^2_x D^2_x D^2_y v_\Delta(\tau^x_{\gamma_3}, \tau^y_{\gamma_1}) - 2\delta^2_x D^2_x D^2_y v_\Delta(\tau^x_{\gamma_4}, \tau^y_{\gamma_1}))}{960}\\
&+ \frac{7(3\delta^2_y D^2_x D^2_y v_\Delta(\tau^x_{\gamma_1}, \tau^y_{\gamma_3}) - 2\delta^2_y D^2_x D^2_y v_\Delta(\tau^x_{\gamma_1}, \tau^y_{\gamma_4}))}{960}\\
&- \frac{\zeta^y_{24} D^4_y v_\Delta(\tau^x_{\gamma_1}, \tau^y_{\gamma_2})}{24} + \frac{7(3\delta^2_y D^4_y v_\Delta(\tau^x_{\gamma_1}, \tau^y_{\gamma_3}) - 2\delta^2_y D^4_y v_\Delta(\tau^x_{\gamma_1}, \tau^y_{\gamma_4}))}{5760},
\end{aligned}
$$

$$
\begin{aligned}
\bar{\bar{f}}_{\gamma_1,\gamma_2} &= f_{\gamma_1,\gamma_2} - P_L v_\Delta(\tau^x_{\gamma_1}, \tau^y_{\gamma_2}) \qquad (4.50)\\
&\equiv f_{\gamma_1,\gamma_2} - \frac{\zeta^x_{24} D^4_x v_\Delta(\tau^x_{\gamma_2}, \tau^y_{\gamma_2})}{24} + \frac{7(3\delta^2_x D^4_x v_\Delta(\tau^x_{\gamma_3}, \tau^y_{\gamma_2}) - 2\delta^2_x D^4_x v_\Delta(\tau^x_{\gamma_4}, \tau^y_{\gamma_2}))}{5760}\\
&+ \frac{7(3\delta^2_x D^2_x D^2_y v_\Delta(\tau^x_{\gamma_3}, \tau^y_{\gamma_2}) - 2\delta^2_x D^2_x D^2_y v_\Delta(\tau^x_{\gamma_4}, \tau^y_{\gamma_2}))}{960}\\
&+ \frac{7(2\delta^2_y D^2_x D^2_y v_\Delta(\tau^x_{\gamma_1}, \tau^y_{\gamma_3}) - \delta^2_y D^2_x D^2_y v_\Delta(\tau^x_{\gamma_1}, \tau^y_{\gamma_4}))}{960}\\
&- \frac{\eta^y_{24} D^4_y v_\Delta(\tau^x_{\gamma_1}, \tau^y_{\gamma_2})}{24} + \frac{7(2\delta^2_y D^4_y v_\Delta(\tau^x_{\gamma_1}, \tau^y_{\gamma_3}) - \delta^2_y D^4_y v_\Delta(\tau^x_{\gamma_1}, \tau^y_{\gamma_4}))}{5760},
\end{aligned}
$$

$$
\begin{aligned}
\bar{\bar{f}}_{\gamma_2,\gamma_1} &= f_{\gamma_2,\gamma_1} - P_L v_\Delta(\tau^x_{\gamma_2}, \tau^y_{\gamma_1}) \qquad (4.51)\\
&\equiv f_{\gamma_2,\gamma_1} - \frac{\eta^x_{24} D^4_x v_\Delta(\tau^x_{\gamma_2}, \tau^y_{\gamma_1})}{24} + \frac{7(2\delta^2_x D^4_x v_\Delta(\tau^x_{\gamma_3}, \tau^y_{\gamma_1}) - \delta^2_x D^4_x v_\Delta(\tau^x_{\gamma_4}, \tau^y_{\gamma_1}))}{5760}\\
&+ \frac{7(2\delta^2_x D^2_x D^2_y v_\Delta(\tau^x_{\gamma_3}, \tau^y_{\gamma_1}) - \delta^2_x D^2_x D^2_y v_\Delta(\tau^x_{\gamma_4}, \tau^y_{\gamma_1}))}{960}\\
&+ \frac{7(3\delta^2_y D^2_x D^2_y v_\Delta(\tau^x_{\gamma_1}, \tau^y_{\gamma_3}) - 2\delta^2_y D^2_x D^2_y v_\Delta(\tau^x_{\gamma_1}, \tau^y_{\gamma_4}))}{960}\\
&- \frac{\zeta^y_{24} D^4_y v_\Delta(\tau^x_{\gamma_1}, \tau^y_{\gamma_2})}{24} + \frac{7(3\delta^2_y D^4_y v_\Delta(\tau^x_{\gamma_1}, \tau^y_{\gamma_3}) - 2\delta^2_y D^4_y v_\Delta(\tau^x_{\gamma_1}, \tau^y_{\gamma_4}))}{5760},
\end{aligned}
$$

$$
\begin{aligned}
\bar{\bar{f}}_{\gamma_2,\gamma_2} &= f_{\gamma_2,\gamma_2} - P_L v_\Delta(\tau^x_{\gamma_2}, \tau^y_{\gamma_2}) \qquad (4.52)\\
&\equiv f_{\gamma_2,\gamma_2} - \frac{\eta^x_{24} D^4_x v_\Delta(\tau^x_{\gamma_2}, \tau^y_{\gamma_2})}{24} + \frac{7(2\delta^2_x D^4_x v_\Delta(\tau^x_{\gamma_3}, \tau^y_{\gamma_2}) - \delta^2_x D^4_x v_\Delta(\tau^x_{\gamma_4}, \tau^y_{\gamma_2}))}{5760}\\
&+ \frac{7(2\delta^2_x D^2_x D^2_y v_\Delta(\tau^x_{\gamma_3}, \tau^y_{\gamma_2}) - \delta^2_x D^2_x D^2_y v_\Delta(\tau^x_{\gamma_4}, \tau^y_{\gamma_2}))}{960}\\
&+ \frac{7(2\delta^2_y D^2_x D^2_y v_\Delta(\tau^x_{\gamma_1}, \tau^y_{\gamma_3}) - \delta^2_y D^2_x D^2_y v_\Delta(\tau^x_{\gamma_1}, \tau^y_{\gamma_4}))}{960}\\
&- \frac{\eta^y_{24} D^4_y v_\Delta(\tau^x_{\gamma_1}, \tau^y_{\gamma_2})}{24} + \frac{7(2\delta^2_y D^4_y v_\Delta(\tau^x_{\gamma_1}, \tau^y_{\gamma_3}) - \delta^2_y D^4_y v_\Delta(\tau^x_{\gamma_1}, \tau^y_{\gamma_4}))}{5760},
\end{aligned}
$$

$$
\begin{aligned}
\bar{\bar{g}}_2(\tau^x_{\gamma_0}, \tau^y_j) &= g_2(\tau^x_{\gamma_0}, \tau^y_j) - P_{B_2} v_\Delta(\tau^x_{\gamma_0}, \tau^y_j) \qquad (4.53)\\
&\equiv g_2(\tau^x_{\gamma_0}, \tau^y_j) + \frac{1}{1440}(7\delta^2_x D_x v_\Delta(\tau^x_{\gamma_3}, \tau^y_j) - 5\delta^2_x D_x v_\Delta(\tau^x_{\gamma_4}, \tau^y_j)), \quad j = 0, \cdots, n+1,\\
\bar{\bar{g}}_2(\tau^x_i, \tau^y_{\gamma_0}) &= g_2(\tau^x_i, \tau^y_{\gamma_0}) - P_{B_2} v_\Delta(\tau^x_i, \tau^y_{\gamma_0}) \qquad (4.54)\\
&\equiv g_2(\tau^x_i, \tau^y_{\gamma_0}) + \frac{1}{1440}(7\delta^2_y D_y v_\Delta(\tau^x_i, \tau^y_{\gamma_3}) - 5\delta^2_y D_y v_\Delta(\tau^x_i, \tau^y_{\gamma_4})), \quad i = 0, \cdots, n+1,\\
\bar{\bar{g}}_c(\tau^x_{\gamma_0}, \tau^y_{\gamma_0}) &= g_c(\tau^x_{\gamma_0}, \tau^y_{\gamma_0}) - P_{B_c} v_\Delta(\tau^x_{\gamma_0}, \tau^y_{\gamma_0}) \qquad (4.55)\\
&\equiv g_c(\tau^x_{\gamma_0}, \tau^y_{\gamma_0}) + \frac{7\delta^2_x D_x D_y v_\Delta(\tau^x_{\gamma_3}, \tau^y_{\gamma_0}) - 5\delta^2_x D_x D_y v_\Delta(\tau^x_{\gamma_4}, \tau^y_{\gamma_0})}{1440}\\
&+ \frac{7\delta^2_y D_x D_y v_\Delta(\tau^x_{\gamma_0}, \tau^y_{\gamma_3}) - 5\delta^2_y D_x D_y v_\Delta(\tau^x_{\gamma_0}, \tau^y_{\gamma_4})}{1440}.
\end{aligned}
$$

(iii) Determine a spline $u_\Delta \in \widehat{S}_\Delta$ by the following equations

$$\begin{aligned}
Lu_\Delta(\tau_i^x, \tau_j^y) &= \bar{\bar{f}}(\tau_i^x, \tau_j^y), \quad i, j = 1, \cdots, n, & (4.56)\\
u_\Delta(\tau_{\gamma_0}^x, \tau_j^y) &= g_1(\tau_{\gamma_0}^x, \tau_j^y), \quad j = 0, \cdots, n+1, & (4.57)\\
u_\Delta(\tau_i^x, \tau_{\gamma_0}^y) &= g_1(\tau_i^x, \tau_{\gamma_0}^y), \quad i = 1, \cdots, n & (4.58)\\
D_x u_\Delta(\tau_{\gamma_0}^x, \tau_j^y) &= \bar{\bar{g}}_2(\tau_{\gamma_0}^x, \tau_j^y), \quad j = 0, \cdots, n+1, & (4.59)\\
D_y u_\Delta(\tau_i^x, \tau_{\gamma_0}^y) &= \bar{\bar{g}}_2(\tau_i^x, \tau_{\gamma_0}^y), \quad i = 0, \cdots, n+1, & (4.60)\\
D_x D_y u_\Delta(\tau_{\gamma_0}^x, \tau_{\gamma_0}^y) &= \bar{\bar{g}}_c(\tau_{\gamma_0}^x, \tau_{\gamma_0}^y). & (4.61)
\end{aligned}$$

It is worth noticing that the right-hand side g_1 does not change and remains the same.

4.3 Linear Systems from the Three-Step Method

In this section, we discuss the linear systems arising from the three-step Bi-Quartic SC method. Noticing that the linear systems generated in the three steps have the same coefficient matrix and different right-hand side vectors, we derive the linear system arising from Step 3 as an example. The respective linear systems from Step 1 and Step 2 can be obtained in a similar manner.

Suppose the bi-quartic spline approximation $u_\Delta \in \widehat{S}_\Delta$ satisfying equations (4.56)-(4.61) is written as

$$u_\Delta(x, y) = \sum_{i=-1}^{n+2} \sum_{j=-1}^{n+2} c_{i,j}^u \widehat{\phi}_i^x(x) \widehat{\phi}_j^y(y).$$

Relations (4.56)-(4.61) generate a linear system which consists of $(n+4)^2$ equations. In forming this linear system, we adopt a bottom-up then left-to-right ordering of the collocation points, and present the collocation equations for each vertical line of collocation points. The coefficients $c_{i,j}^u$ are placed in a vector $(\bar{c}_1^u, \bar{c}_2^u, \cdots, \bar{c}_{(n+4)^2}^u)^T$, where $\bar{c}_k^u = c_{i,j}^u$ with $k = (i+1)(n+4)+j+2$. The resulting coefficient matrix is an $(n+4)^2 \times (n+4)^2$ almost block-pentadiagonal matrix with almost pentadiagonal blocks of size $(n+4) \times (n+4)$.

The equations of the first block (on the points $\{(\tau_0^x, \tau_j^y), j = 0, \cdots, n+1\}$) are:

$u_{\Delta 0,0} = (g_1)_{0,0},\ D_y u_{\Delta 0,0} = (\bar{\bar{g}}_2)_{0,0},\ u_{\Delta 0,1} = (g_1)_{0,1},\ \cdots,$

$u_{\Delta 0,n} = (g_1)_{0,n},\ D_y u_{\Delta 0,n+1} = (\bar{\bar{g}}_2)_{0,n+1},\ u_{\Delta 0,n+1} = (g_1)_{0,n+1}.$

The equations of the second block (again on the points $\{(\tau_0^x, \tau_j^y), j = 0, \cdots, n+1\}$) are:

$D_x u_{\Delta 0,0} = (\bar{\bar{g}}_2)_{0,0},\ D_x D_y u_{\Delta 0,0} = \bar{\bar{g}}_c(0,0),\ D_x u_{\Delta 0,1} = (\bar{\bar{g}}_2)_{0,1},\ \cdots,$

$D_x u_{\Delta 0,n} = (\bar{\bar{g}}_2)_{0,n},\ D_x D_y u_{\Delta 0,n+1} = \bar{\bar{g}}_c(0,1),\ D_x u_{\Delta 0,n+1} = (\bar{\bar{g}}_2)_{0,n+1}.$

The equations of the next n blocks (on the points $\{(\tau_i^x, \tau_j^y), j = 0, \cdots, n+1\}$, for each $i = 1, \cdots, n$) are:

$u_{\Delta i,0} = (g_1)_{i,0},\ D_y u_{\Delta i,0} = (\bar{\bar{g}}_2)_{i,0},\ L u_{\Delta i,1} = \bar{\bar{f}}_{i,1},\ \cdots,$

$L u_{\Delta i,n} = \bar{\bar{f}}_{i,n},\ D_y u_{\Delta i,n+1} = (\bar{\bar{g}}_2)_{i,n+1},\ u_{\Delta i,n+1} = (g_1)_{i,n+1}.$

The equations of the $(n+3)$rd block (on the points $\{(\tau_{n+1}^x, \tau_j^y), j = 0, \cdots, n+1\}$) are:

$D_x u_{\Delta n+1,0} = (\bar{\bar{g}}_2)_{n+1,0},\ D_x D_y u_{\Delta n+1,0} = \bar{\bar{g}}_c(1,0),\ D_x u_{\Delta n+1,1} = (\bar{\bar{g}}_2)_{n+1,1},\ \cdots,$

$D_x u_{\Delta n+1,n} = (\bar{\bar{g}}_2)_{n+1,n},\ D_x D_y u_{\Delta n+1,n+1} = \bar{\bar{g}}_c(1,1),\ D_x u_{\Delta n+1,n+1} = (\bar{\bar{g}}_2)_{n+1,n+1}.$

The equations of the last ($(n+4)$th) block of equations (again on the points $\{(\tau_{n+1}^x, \tau_j^y), j = 0, \cdots, n+1\}$) are:

$u_{\Delta n+1,0} = (g_1)_{n+1,1},\ D_y u_{\Delta n+1,0} = (\bar{\bar{g}}_2)_{n+1,0},\ u_{\Delta n+1,1} = (g_1)_{n+1,1},\ \cdots,$

$u_{\Delta n+1,n} = (g_1)_{n+1,n},\ D_y u_{\Delta n+1,n+1} = (\bar{\bar{g}}_2)_{n+1,n+1},\ u_{\Delta n+1,n+1} = (g_1)_{n+1,n+1}.$

For later convenience, we introduce some notation that presents the collocation equations in a more concise form. The notation consists of a tableau, where the entries show the left-hand side of the equations, and each row of the tableau corresponds to one block of equations. Here is the tableau of the collocation equations for the biharmonic Dirichlet problem with non-homogeneous boundary conditions:

$$\left\{\begin{array}{ccccccc}
u_{\Delta 0,0} & D_y u_{\Delta 0,0} & u_{\Delta 0,1} & \cdots & u_{\Delta 0,n} & D_y u_{\Delta 0,n+1} & u_{\Delta 0,n+1} \\
D_x u_{\Delta 0,0} & D_x D_y u_{\Delta 0,0} & D_x u_{\Delta 0,1} & \cdots & D_x u_{\Delta 0,n} & D_x D_y u_{\Delta 0,n+1} & D_x u_{\Delta 0,n+1} \\
u_{\Delta 1,0} & D_y u_{\Delta 1,0} & L u_{\Delta 1,1} & \cdots & L u_{\Delta 1,n} & D_y u_{\Delta 1,n+1} & u_{\Delta 1,n+1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
u_{\Delta n,0} & D_y u_{\Delta n,0} & L u_{\Delta n,1} & \cdots & L u_{\Delta n,n} & D_y u_{\Delta n,n+1} & u_{\Delta n,n+1} \\
D_x u_{\Delta n+1,0} & D_x D_y u_{\Delta n+1,0} & D_x u_{\Delta n+1,1} & \cdots & D_x u_{\Delta n+1,n} & D_x D_y u_{\Delta n+1,n+1} & D_x u_{\Delta n+1,n+1} \\
u_{\Delta n+1,0} & D_y u_{\Delta n+1,0} & u_{\Delta n+1,1} & \cdots & u_{\Delta n+1,n} & D_y u_{\Delta n+1,n+1} & u_{\Delta n+1,n+1}.
\end{array}\right\}$$

The resulting linear system in a block-wise form is

$$\widehat{A}\boldsymbol{\varsigma} \equiv \left|\begin{array}{cccccccccc}
\frac{1}{24}W & \frac{11}{24}W & \frac{11}{24}W & \frac{1}{24}W & & & & & & \\
\frac{-1}{6h}W & \frac{-1}{2h}W & \frac{1}{2h}W & \frac{1}{6h}W & & & & & & \\
P_{1,-1} & P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} & & & & & \\
 & P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} & & & & \\
 & & . & . & . & . & . & & & \\
 & & & P_{n-1,n-3} & P_{n-1,n-2} & P_{n-1,n-1} & P_{n-1,n} & P_{n-1,n+1} & & \\
 & & & & P_{n,n-2} & P_{n,n-1} & P_{n,n} & P_{n,n+1} & P_{n,n+2} & \\
 & & & & & \frac{-1}{6h}W & \frac{-1}{2h}W & \frac{1}{2h}W & \frac{1}{6h}W & \\
 & & & & & \frac{1}{24}W & \frac{11}{24}W & \frac{11}{24}W & \frac{1}{24}W &
\end{array}\right\|\begin{array}{c}
\varsigma_{-1} \\ \varsigma_0 \\ \varsigma_1 \\ \varsigma_2 \\ . \\ \varsigma_{n-1} \\ \varsigma_n \\ \varsigma_{n+2} \\ \varsigma_{n+1}
\end{array}\right| = \left|\begin{array}{c}
\varepsilon_{-1} \\ \varepsilon_0 \\ \varepsilon_1 \\ \varepsilon_2 \\ . \\ \varepsilon_{n-1} \\ \varepsilon_n \\ \varepsilon_{n+2} \\ \varepsilon_{n+1}
\end{array}\right| \tag{4.62}$$

where W is an $(n+4)\times(n+4)$ matrix of the form

$$W = \frac{1}{384}\begin{bmatrix}
16 & 176 & 176 & 16 & & & \\
\frac{-64}{h} & \frac{-192}{h} & \frac{192}{h} & \frac{64}{h} & & & \\
1 & 76 & 230 & 76 & 1 & & \\
 & . & . & . & . & . & \\
 & & 1 & 76 & 230 & 76 & 1 \\
 & & & \frac{-64}{h} & \frac{-192}{h} & \frac{192}{h} & \frac{64}{h} \\
 & & & 16 & 176 & 176 & 16
\end{bmatrix},$$

P_{ij}, $i = 1, \cdots, n$, $j = i-2, i-1, i, i+1, i+2$, are $(n+4) \times (n+4)$ matrices of the forms

$$P_{i,i-2} = P_{i,i+2} = \frac{1}{384h^4} \begin{bmatrix} \frac{h^4}{24} & \frac{11h^4}{24} & \frac{11h^4}{24} & \frac{h^4}{24} & & & \\ \frac{-h^3}{6} & \frac{-h^3}{2} & \frac{h^3}{2} & \frac{h^3}{6} & & & \\ 14 & 120 & 116 & 120 & 14 & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & 14 & 120 & 116 & 120 & 14 \\ & & & \frac{-h^3}{6} & \frac{-h^3}{2} & \frac{h^3}{2} & \frac{h^3}{6} \\ & & & \frac{h^4}{24} & \frac{11h^4}{24} & \frac{11h^4}{24} & \frac{h^4}{24} \end{bmatrix},$$

$$P_{i,i-1} = P_{i,i+1} = \frac{1}{384h^4} \begin{bmatrix} \frac{19h^4}{6} & \frac{209h^4}{6} & \frac{209h^4}{6} & \frac{19h^4}{6} & & & \\ \frac{-38h^3}{3} & -38h^3 & 38h^3 & \frac{38h^3}{3} & & & \\ 120 & -416 & -944 & -416 & 120 & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & 120 & -416 & -944 & -416 & 120 \\ & & & \frac{-38h^3}{3} & -38h^3 & 38h^3 & \frac{38h^3}{3} \\ & & & \frac{19h^4}{6} & \frac{209h^4}{6} & \frac{209h^4}{6} & \frac{19h^4}{6} \end{bmatrix},$$

$$P_{i,i} = \frac{1}{384h^4} \begin{bmatrix} \frac{115h^4}{12} & \frac{1265h^4}{12} & \frac{1265h^4}{12} & \frac{115h^4}{12} & & & \\ \frac{-115h^3}{3} & -115h^3 & 115h^3 & \frac{115h^3}{3} & & & \\ 116 & -944 & 3960 & -944 & 116 & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & 116 & -944 & 3960 & -944 & 116 \\ & & & \frac{-115h^3}{3} & -115h^3 & 115h^3 & \frac{115h^3}{3} \\ & & & \frac{115h^4}{12} & \frac{1265h^4}{12} & \frac{1265h^4}{12} & \frac{115h^4}{12} \end{bmatrix},$$

and ς_k and ε_k, $k = -1, \cdots, n+2$, are vectors of size $n+4$.

Notice that the coefficient matrix $\widehat{A}$ in (4.62) is a $(4(n+4), 4(n+4))$-banded sparse matrix. A straightforward application of the Gaussian-elimination method to (4.62) would need $O(n^4)$ flops. In the next chapter, we will present a fast solver for (4.62), whose complexity is only $O(n^2 \log n)$.

4.4 One-step Bi-Quartic SC

Recall that the one-step method determines a bi-quartic spline collocation approximation $u_\Delta(x)$ by the equations

$$
\begin{aligned}
(L+P_L)u_\Delta(x)(\tau_i^x,\tau_j^y) &= f(\tau_i^x,\tau_j^y), \quad (\tau_i^x,\tau_j^y)\in\Pi-\Pi_\partial,\\
(B_k+P_{B_k})u_\Delta(x)(\tau_i^x,\tau_j^y) &= g_k(\tau_i^x,\tau_j^y), \quad (\tau_i^x,\tau_j^y)\in\Pi_\partial, k=1,2,\\
(D_xD_y+P_{B_c})u_\Delta(x)(\tau_i^x,\tau_j^y) &= g_c(\tau_0^x,\tau_0^y).
\end{aligned}
$$

Since we have already defined the operators P_L, P_{B_k}, and P_{B_c} in Section 4.2, the formulation of the one-step method is straightforward.

We now describe the linear system arising from the one-step method for the biharmonic Dirichlet problem. We number the equations of the linear system as in the three-step method. Let $\tilde{\tilde{A}}$ be the coefficient matrix of the resulting linear system. Figure 4.1 shows the sparsity pattern of $\tilde{\tilde{A}}$. The order of $\tilde{\tilde{A}}$ is $(n+4)^2\times(n+4)^2$, and each non-zero block in $\tilde{\tilde{A}}$ is of $(n+4)\times(n+4)$. Let B_{ij} denote the block at ith row and jth column, i.e.,

$$\tilde{\tilde{A}} = \{B_{ij}\}, \quad ,i,j=1,\cdots,n+4,$$

where the B_{ij}'s are defined as follows:

$$
\begin{aligned}
B_{11} &= \frac{1}{9216}\begin{bmatrix}
16 & 176 & 176 & 16 & & & & & & & & & \\
\frac{-11513}{180h} & \frac{-34439}{180h} & \frac{8474}{45h} & \frac{352}{5h} & \frac{-53}{18h} & \frac{-371}{90h} & \frac{92}{15h} & \frac{-134}{45h} & \frac{83}{180h} & \frac{1}{36h} & & & \\
1 & 76 & 230 & 76 & 1 & & & & & & & & \\
 & & & & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\
 & & & & & & & & 1 & 76 & 230 & 76 & 1 \\
 & & & \frac{-1}{36h} & \frac{-83}{180h} & \frac{134}{45h} & \frac{-92}{15h} & \frac{371}{90h} & \frac{53}{18h} & \frac{-352}{5h} & \frac{-8474}{45h} & \frac{34439}{180h} & \frac{11513}{180h} \\
 & & & & & & & & & 16 & 176 & 176 & 16
\end{bmatrix}, \\
&= B_{14} \\
B_{12} &= B_{13} = 11B_{11},
\end{aligned}
$$

$$B_{21} = \frac{11513}{26542080h} \times$$
$$\left[\begin{array}{ccccccccccccc}
-16 & -176 & -176 & -16 & & & & & & & & & \\
\frac{736384}{11513h} & \frac{2202752}{11513h} & \frac{-2168000}{11513h} & \frac{-810560}{11513h} & \frac{33920}{11513h} & \frac{47488}{11513h} & \frac{-70656}{11513h} & \frac{34304}{11513h} & \frac{-5312}{11513h} & \frac{-320}{11513h} & & & \\
-1 & -76 & -230 & -76 & -1 & & & & & & & & \\
 & & & & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\
 & & & & & & & & -1 & -76 & -230 & -76 & -1 \\
 & & & \frac{320}{11513h} & \frac{5312}{11513h} & \frac{-34304}{11513h} & \frac{70656}{11513h} & \frac{-47488}{11513h} & \frac{-33920}{11513h} & \frac{810560}{11513h} & \frac{2168000}{11513h} & \frac{-2202752}{11513h} & \frac{-736384}{11513h} \\
 & & & & & & & & & & & & \\
 & & & & & & & & & -16 & -176 & -176 & -16
\end{array}\right],$$

$$B_{22} = \frac{34439}{26542080h} \times$$
$$\left[\begin{array}{ccccccccccccc}
-16 & -176 & -176 & -16 & & & & & & & & & \\
\frac{2202752}{34439h} & \frac{6589056}{34439h} & \frac{-6484800}{34439h} & \frac{-2425280}{34439h} & \frac{101760}{34439h} & \frac{142464}{34439h} & \frac{-211968}{34439h} & \frac{102912}{34439h} & \frac{-15936}{34439h} & \frac{-960}{34439h} & & & \\
-1 & -76 & -230 & -76 & -1 & & & & & & & & \\
 & & & & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\
 & & & & & & & & -1 & -76 & -230 & -76 & -1 \\
 & & & \frac{960}{34439h} & \frac{15936}{34439h} & \frac{-102912}{34439h} & \frac{211968}{34439h} & \frac{-142464}{34439h} & \frac{-101760}{34439h} & \frac{2425280}{34439h} & \frac{6484800}{34439h} & \frac{-6589056}{34439h} & \frac{-2202752}{34439h} \\
 & & & & & & & & & & & & \\
 & & & & & & & & & -16 & -176 & -176 & -16
\end{array}\right],$$

$$B_{23} = \frac{4237}{3317760h} \times$$
$$\begin{bmatrix}
16 & 176 & 176 & 16 & & & & & & & & & \\
\frac{-271000}{4237h} & \frac{-810600}{4237h} & \frac{797568}{4237h} & \frac{76496896}{4237h} & \frac{-3256320}{4237h} & \frac{-4558848}{4237h} & \frac{6782976}{4237h} & \frac{-3293184}{4237h} & \frac{509952}{4237h} & \frac{30720}{4237h} & & & \\
1 & 76 & 230 & 76 & 1 & & & & & & & & \\
 & & & & . & . & . & . & . & & & & \\
 & & & & & & & & 1 & 76 & 230 & 76 & 1 \\
 & & & \frac{-30720}{4237h} & \frac{-509952}{4237h} & \frac{3293184}{4237h} & \frac{-6782976}{4237h} & \frac{4558848}{4237h} & \frac{3256320}{4237h} & \frac{-76496896}{4237h} & \frac{-797568}{4237h} & \frac{810600}{4237h} & \frac{271000}{4237h} \\
 & & & & & & & & & 16 & 176 & 176 & 16
\end{bmatrix},$$

$$B_{24} = \frac{11}{23040h} \times$$
$$\begin{bmatrix}
16 & 176 & 176 & 16 & & & & & & & & & \\
\frac{-12665}{198h} & \frac{-3445}{18h} & \frac{18676}{99h} & \frac{768}{11h} & \frac{-265}{99h} & \frac{-371}{99h} & \frac{184}{33h} & \frac{-268}{99h} & \frac{83}{198h} & \frac{5}{198h} & & & \\
1 & 76 & 230 & 76 & 1 & & & & & & & & \\
 & . & . & . & . & . & & & & & & & \\
 & & 1 & 76 & 230 & 76 & 1 & & & & & & \\
 & & & \frac{-5}{198h} & \frac{-83}{198h} & \frac{268}{99h} & \frac{-184}{33h} & \frac{371}{99h} & \frac{265}{99h} & \frac{-768}{11h} & \frac{-18676}{99h} & \frac{3445}{18h} & \frac{12665}{198h} \\
 & & & & & & & & & 16 & 176 & 176 & 16
\end{bmatrix},$$

$$H = \begin{bmatrix}
16 & 176 & 176 & 16 & & & \\
\frac{-64}{h} & \frac{-192}{h} & \frac{192}{h} & \frac{64}{h} & & & \\
1 & 76 & 230 & 76 & 1 & & \\
 & . & . & . & . & . & \\
 & & 1 & 76 & 230 & 76 & 1 \\
 & & & \frac{-64}{h} & \frac{-192}{h} & \frac{192}{h} & \frac{64}{h} \\
 & & & 16 & 176 & 176 & 16
\end{bmatrix},$$

$$
\begin{aligned}
B_{25} &= \frac{-53}{2654208h}H, \quad B_{26} = \frac{-371}{13271040h}H,\\
B_{27} &= \frac{23}{552960}H, \quad B_{28} = \frac{-67}{3317760}H,\\
B_{29} &= \frac{83}{26542080}H, \quad B_{2,10} = \frac{1}{5308416}H,
\end{aligned}
$$

$$
B_{31} = \frac{1}{9216h^4}\times
$$

$$
\left[\begin{array}{cccccccccccc}
h^4 & 11h^4 & 11h^4 & h^4 & & & & & & & & \\
\frac{-11513h^3}{2880} & \frac{-34439h^3}{2880} & \frac{4237h^3}{360} & \frac{22h^3}{5} & \frac{-53h^3}{288} & \frac{-371h^3}{1440} & \frac{23h^3}{60} & \frac{-67h^3}{360} & \frac{83h^3}{2880} & \frac{h^3}{576} & & \\
\frac{39443}{120} & \frac{341533}{120} & \frac{7982}{3} & \frac{58577}{20} & \frac{835}{16} & \frac{32323}{60} & \frac{-9596}{15} & \frac{1406}{3} & \frac{-49829}{240} & \frac{6199}{120} & \frac{-143}{24} & \frac{1}{6} \\
\frac{-93}{40} & \frac{13579}{40} & \frac{169943}{60} & \frac{39989}{15} & \frac{28539}{10} & \frac{31367}{80} & \frac{-2036}{15} & \frac{1813}{15} & \frac{-6787}{120} & \frac{3319}{240} & \frac{-3}{2} & \frac{1}{24} \\
\frac{-269}{240} & \frac{47}{30} & \frac{27629}{80} & \frac{55617}{20} & \frac{33229}{12} & \frac{55617}{20} & \frac{27629}{80} & \frac{47}{30} & \frac{-269}{240} & & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
 & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & & & \frac{-269}{240} & \frac{47}{30} & \frac{27629}{80} & \frac{55617}{20} & \frac{33229}{12} & \frac{55617}{20} & \frac{27629}{80} & \frac{47}{30} & \frac{-269}{240} \\
\frac{-1}{24} & \frac{3}{2} & \frac{-3319}{240} & \frac{6787}{120} & \frac{-1813}{15} & \frac{2036}{15} & \frac{-31367}{80} & \frac{-28539}{10} & \frac{-39989}{15} & \frac{-169943}{60} & \frac{-13579}{40} & \frac{93}{40} \\
\frac{-1}{6} & \frac{143}{24} & \frac{-6199}{120} & \frac{49829}{240} & \frac{-1406}{3} & \frac{9596}{15} & \frac{-32323}{60} & \frac{-835}{16} & \frac{-58577}{20} & \frac{-7982}{3} & \frac{-341533}{120} & \frac{-39443}{120} \\
 & & \frac{-h^3}{576} & \frac{-83h^3}{2880} & \frac{67h^3}{360} & \frac{-23h^3}{60} & \frac{371h^3}{1440} & \frac{53h^3}{288} & \frac{-22h^3}{5} & \frac{-4237h^3}{360} & \frac{34439h^3}{2880} & \frac{11513h^3}{2880} \\
 & & & & & & & & h^4 & 11h^4 & 11h^4 & h^4
\end{array}\right],
$$

$$
B_{32} = \frac{19}{2304h^4} \times
$$

$$
\begin{bmatrix}
h^4 & 11h^4 & 11h^4 & h^4 & & & & & & & & \\
\frac{-11513h^3}{2880} & \frac{-34439h^3}{2880} & \frac{4237h^3}{360} & \frac{22h^3}{5} & \frac{-53h^3}{288} & \frac{-371h^3}{1440} & \frac{23h^3}{60} & \frac{-67h^3}{360} & \frac{83h^3}{2880} & \frac{h^3}{576} & & \\
\frac{341533}{9120} & \frac{-38493}{380} & \frac{-106939}{304} & \frac{97681}{456} & \frac{-5818057}{9120} & \frac{207197}{228} & \frac{-235622}{285} & \frac{28982}{57} & \frac{-933143}{4560} & \frac{22649}{456} & \frac{-143}{24} & \frac{1}{6} \\
\frac{-51}{152} & \frac{403589}{9120} & \frac{-23229}{152} & \frac{-636581}{4560} & \frac{-762049}{2280} & \frac{2870183}{9120} & \frac{-68168}{285} & \frac{40117}{285} & \frac{-124417}{2280} & \frac{11705}{912} & \frac{-3}{2} & \frac{1}{24} \\
\frac{-115}{912} & \frac{47}{30} & \frac{56983}{1824} & \frac{-230059}{2280} & \frac{-249157}{912} & \frac{-230059}{2280} & \frac{56983}{1824} & \frac{47}{30} & \frac{-115}{912} & & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
 & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & & & \frac{-115}{912} & \frac{47}{30} & \frac{56983}{1824} & \frac{-230059}{2280} & \frac{-249157}{912} & \frac{-230059}{2280} & \frac{56983}{1824} & \frac{47}{30} & \frac{-115}{912} \\
\frac{-1}{24} & \frac{3}{2} & \frac{-11705}{912} & \frac{124417}{2280} & \frac{-40117}{285} & \frac{68168}{285} & \frac{-2870183}{9120} & \frac{762049}{2280} & \frac{636581}{4560} & \frac{23229}{152} & \frac{-403589}{9120} & \frac{51}{152} \\
\frac{-1}{6} & \frac{143}{24} & \frac{-22649}{456} & \frac{933143}{4560} & \frac{-28982}{57} & \frac{235622}{285} & \frac{-207197}{228} & \frac{5818057}{9120} & \frac{-97681}{456} & \frac{106939}{304} & \frac{38493}{380} & \frac{-341533}{9120} \\
 & & \frac{-h^3}{576} & \frac{-83h^3}{2880} & \frac{67h^3}{360} & \frac{-23h^3}{60} & \frac{371h^3}{1440} & \frac{53h^3}{288} & \frac{-22h^3}{5} & \frac{-4237h^3}{360} & \frac{34439h^3}{2880} & \frac{11513h^3}{2880} \\
 & & & & & & & & h^4 & 11h^4 & 11h^4 & h^4
\end{bmatrix},
$$

$$
B_{33} = \frac{115}{4608h^4} \times
$$

$$
\begin{bmatrix}
h^4 & 11h^4 & 11h^4 & h^4 & & & & & & & & \\
\frac{-11513h^3}{2880} & \frac{-34439h^3}{2880} & \frac{4237h^3}{360} & \frac{22h^3}{5} & \frac{-53h^3}{288} & \frac{-371h^3}{1440} & \frac{23h^3}{60} & \frac{-67h^3}{360} & \frac{83h^3}{2880} & \frac{h^3}{576} & & \\
\frac{3991}{345} & \frac{-106939}{920} & \frac{18767}{92} & \frac{307583}{1380} & \frac{-777539}{1104} & \frac{1305893}{1380} & \frac{-291772}{345} & \frac{35362}{69} & \frac{-1127923}{5520} & \frac{136529}{2760} & \frac{-143}{24} & \frac{1}{6} \\
\frac{-123}{920} & \frac{24797}{1380} & \frac{-15787}{92} & \frac{603269}{1380} & \frac{-25778}{69} & \frac{74527}{240} & \frac{-17228}{69} & \frac{49259}{345} & \frac{-150053}{2760} & \frac{70289}{5520} & \frac{-3}{2} & \frac{1}{24} \\
\frac{-139}{5520} & \frac{47}{30} & \frac{3355}{1104} & \frac{-6551}{60} & \frac{19288}{69} & \frac{-6551}{60} & \frac{3355}{1104} & \frac{47}{30} & \frac{-139}{5520} & & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
 & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & & & \frac{-139}{5520} & \frac{47}{30} & \frac{3355}{1104} & \frac{-6551}{60} & \frac{19288}{69} & \frac{-6551}{60} & \frac{3355}{1104} & \frac{47}{30} & \frac{-139}{5520} \\
\frac{-1}{24} & \frac{3}{2} & \frac{-70289}{5520} & \frac{150053}{2760} & \frac{-49259}{345} & \frac{17228}{69} & \frac{-74527}{240} & \frac{25778}{69} & \frac{-603269}{1380} & \frac{15787}{92} & \frac{-24797}{1380} & \frac{123}{920} \\
\frac{-1}{6} & \frac{143}{24} & \frac{-136529}{2760} & \frac{1127923}{5520} & \frac{-35362}{69} & \frac{291772}{345} & \frac{-1305893}{1380} & \frac{777539}{1104} & \frac{-307583}{1380} & \frac{-18767}{92} & \frac{106939}{920} & \frac{-3991}{345} \\
 & & \frac{-h^3}{576} & \frac{-83h^3}{2880} & \frac{67h^3}{360} & \frac{-23h^3}{60} & \frac{371h^3}{1440} & \frac{53h^3}{288} & \frac{-22h^3}{5} & \frac{-4237h^3}{360} & \frac{34439h^3}{2880} & \frac{11513h^3}{2880} \\
 & & & & & & & & h^4 & 11h^4 & 11h^4 & h^4
\end{bmatrix},
$$

$$B_{34} = B_{32} + \frac{19}{2304h^4}\begin{bmatrix} 0 & 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & 0 & & & & & \\ \frac{9929}{9120} & \frac{719363}{2280} & \frac{935983}{912} & \frac{719363}{2280} & \frac{9929}{9120} & & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ & & \frac{9929}{9120} & \frac{719363}{2280} & \frac{935983}{912} & \frac{719363}{2280} & \frac{9929}{9120} \\ & & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_{35} = B_{31} + \frac{1}{9216h^4}\begin{bmatrix} 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & & & \\ \frac{-66361}{240} & \frac{-615959}{12} & \frac{-3951551}{24} & \frac{-615959}{12} & \frac{-66361}{240} & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & \frac{-66361}{240} & \frac{-615959}{12} & \frac{-3951551}{24} & \frac{-615959}{12} & \frac{-66361}{240} \\ & & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_{36} = \frac{1}{9216h^4}\begin{bmatrix} 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & & & \\ \frac{32323}{60} & \frac{207197}{3} & \frac{1305893}{6} & \frac{207197}{3} & \frac{32323}{60} & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & \frac{32323}{60} & \frac{207197}{3} & \frac{1305893}{6} & \frac{207197}{3} & \frac{32323}{60} \\ & & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_{37} = \frac{1}{2304h^4}\begin{bmatrix} 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & & & \\ \frac{-2399}{15} & \frac{-235622}{15} & \frac{-145886}{3} & \frac{-235622}{15} & \frac{-2399}{15} & & \\ & . & . & . & . & . & \\ & & \frac{-2399}{15} & \frac{-235622}{15} & \frac{-145886}{3} & \frac{-235622}{15} & \frac{-2399}{15} \\ & & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_{38} = \frac{1}{4608h^4}\begin{bmatrix} 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & & & \\ \frac{703}{3} & \frac{57964}{3} & \frac{176810}{3} & \frac{57964}{3} & \frac{703}{3} & & \\ & . & . & . & . & . & \\ & & \frac{703}{3} & \frac{57964}{3} & \frac{176810}{3} & \frac{57964}{3} & \frac{703}{3} \\ & & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_{39} = \frac{1}{9216h^4}\begin{bmatrix} 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & & & \\ \frac{-49829}{240} & \frac{-933143}{60} & \frac{-1127923}{24} & \frac{-933143}{60} & \frac{-49829}{240} & & \\ & . & . & . & . & . & \\ & & \frac{-49829}{240} & \frac{-933143}{60} & \frac{-1127923}{24} & \frac{-933143}{60} & \frac{-49829}{240} \\ & & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_{3,10} = \frac{1}{9216h^4}\begin{bmatrix} 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & & & \\ \frac{6199}{120} & \frac{22649}{6} & \frac{136529}{12} & \frac{22649}{6} & \frac{6199}{120} & & \\ & . & . & . & . & . & \\ & & \frac{6199}{120} & \frac{22649}{6} & \frac{136529}{12} & \frac{22649}{6} & \frac{6199}{120} \\ & & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_{3,11} = \frac{1}{9216h^4}\begin{bmatrix} 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & & & \\ \frac{-143}{24} & \frac{-2717}{6} & \frac{-16445}{12} & \frac{-2717}{6} & \frac{-143}{24} & & \\ & . & . & . & . & . & \\ & & \frac{-143}{24} & \frac{-2717}{6} & \frac{-16445}{12} & \frac{-2717}{6} & \frac{-143}{24} \\ & & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_{3,12} = \frac{1}{55296h^4}\begin{bmatrix} 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & & & \\ 1 & 76 & 230 & 76 & 1 & & \\ & . & . & . & . & . & \\ & & 1 & 76 & 230 & 76 & 1 \\ & & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$
\begin{aligned}
B_{41} &= \frac{1}{122880h^4}\begin{bmatrix}
0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & & & \\
-31 & -340 & -410 & -340 & -31 & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & & -31 & -340 & -410 & -340 & -31 \\
 & & & 0 & 0 & 0 & 0 \\
 & & & 0 & 0 & 0 & 0
\end{bmatrix}, \\
B_{42} &= B_{31},
\end{aligned}
$$

$$
B_{43} = B_{32} - \frac{1}{40960h^4}\begin{bmatrix}
0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & & & \\
61 & 17404 & 56590 & 17404 & 61 & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & & 61 & 17404 & 56590 & 17404 & 61 \\
 & & & 0 & 0 & 0 & 0 \\
 & & & 0 & 0 & 0 & 0
\end{bmatrix},
$$

$$
B_{44} = B_{33} + \frac{1}{138240h^4}\begin{bmatrix}
0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & & & \\
79 & 241876 & 804410 & 241876 & 79 & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & & 79 & 241876 & 804410 & 241876 & 79 \\
 & & & 0 & 0 & 0 & 0 \\
 & & & 0 & 0 & 0 & 0
\end{bmatrix},
$$

$$B_{45} \;=\; B_{34} - \frac{1}{184320h^4}\left[\begin{array}{ccccccccc} 0 & 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & 0 & & & & & \\ 1499 & 833636 & 2743810 & 833636 & 1499 & & & & \\ & . & . & . & . & . & & & \\ & & & 1499 & 833636 & 2743810 & 833636 & 1499 & \\ & & & & 0 & 0 & 0 & 0 & \\ & & & & 0 & 0 & 0 & 0 & \end{array}\right],$$

$$B_{46} \;=\; B_{35} + \frac{1}{30720h^4}\left[\begin{array}{ccccccccc} 0 & 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & 0 & & & & & \\ 1133 & 241340 & 778030 & 241340 & 1133 & & & & \\ & . & . & . & . & . & & & \\ & & & 1133 & 241340 & 778030 & 241340 & 1133 & \\ & & & & 0 & 0 & 0 & 0 & \\ & & & & 0 & 0 & 0 & 0 & \end{array}\right],$$

$$B_{47} \;=\; \frac{1}{138240h^4}\left[\begin{array}{ccccccccc} 0 & 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & 0 & & & & & \\ -2036 & -272672 & -861400 & -272672 & -2036 & & & & \\ & . & . & . & . & . & & & \\ & & & -2036 & -272672 & -861400 & -272672 & -2036 & \\ & & & & 0 & 0 & 0 & 0 & \\ & & & & 0 & 0 & 0 & 0 & \end{array}\right],$$

$$B_{48} = \frac{1}{138240h^4}\left[\begin{array}{ccccccc} 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & & & \\ 1813 & 160468 & 492590 & 160468 & 1813 & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & 1813 & 160468 & 492590 & 160468 & 1813 \\ & & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \end{array}\right],$$

$$B_{49} = \frac{1}{1105920h^4}\left[\begin{array}{ccccccc} 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & & & \\ -6787 & -497668 & -1500530 & -497668 & -6787 & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & -6787 & -497668 & -1500530 & -497668 & -6787 \\ & & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \end{array}\right],$$

$$B_{4,10} = \frac{1}{2211840h^4}\left[\begin{array}{ccccccc} 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & & & \\ 3319 & 234100 & 702890 & 234100 & 3319 & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & 3319 & 234100 & 702890 & 234100 & 3319 \\ & & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \end{array}\right],$$

$$B_{4,11} = \frac{-1}{6144h^4}\begin{bmatrix} 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ 1 & 76 & 230 & 76 & 1 & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & 1 & 76 & 230 & 76 & 1 \\ & & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_{4,12} = \frac{1}{209664h^4}\begin{bmatrix} 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ 1 & 76 & 230 & 76 & 1 & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & 1 & 76 & 230 & 76 & 1 \\ & & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{aligned} B_{51} &= B_{59} \\ &= \frac{-1}{2211840h^4}\begin{bmatrix} 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ 269 & 2300 & 1390 & 2300 & 269 & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & 269 & 2300 & 1390 & 2300 & 269 \\ & & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

$$
\begin{aligned}
B_{52} &= B_{58} \\
&= \frac{47}{276480h^4}\begin{bmatrix}
0 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & & & & \\
1 & 76 & 230 & 76 & 1 & & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
 & & 1 & 76 & 230 & 76 & 1 \\
 & & & 0 & 0 & 0 & 0 \\
 & & & 0 & 0 & 0 & 0
\end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
B_{53} &= B_{57} \\
&= B_{42} - \frac{1}{245760h^4}\begin{bmatrix}
0 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & & & & \\
-157 & 26372 & 91570 & 26372 & -157 & & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
 & & -157 & 26372 & 91570 & 26372 & -157 \\
 & & & 0 & 0 & 0 & 0 \\
 & & & 0 & 0 & 0 & 0
\end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
B_{54} &= B_{56} \\
&= B_{43} + \frac{1}{138240h^4}\begin{bmatrix}
0 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & & & & \\
-773 & 59188 & 215330 & 59188 & -773 & & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
 & & -773 & 59188 & 215330 & 59188 & -773 \\
 & & & 0 & 0 & 0 & 0 \\
 & & & 0 & 0 & 0 & 0
\end{bmatrix},
\end{aligned}
$$

$$
B_{55} = B_{44} - \frac{1}{184320h^4} \begin{bmatrix} 0 & 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & 0 & & & & & \\ -2063 & 203068 & 725030 & 203068 & -2063 & & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ & & -2063 & 203068 & 725030 & 203068 & -2063 & & \\ & & & 0 & 0 & 0 & 0 & & \\ & & & 0 & 0 & 0 & 0 & & \end{bmatrix},
$$

$$
\begin{aligned}
B_{i,i\pm2} &= B_{53}, \\
B_{i,i\pm1} &= B_{54}, \\
B_{i,i} &= B_{55},
\end{aligned}
$$

where $i = 5, \cdots, n$.

$$
\begin{aligned}
B_{n+1,n+5-i} &= B_{4,i}, \quad i = 1, 2, \cdots, 12. \\
B_{n+2,n+5-i} &= B_{3,i}, \quad i = 1, 2, \cdots, 12. \\
B_{n+3,n+5-i} &= -B_{2,i}, \quad i = 1, 2, \cdots, 10. \\
B_{n+4,n+5-i} &= B_{1,i}, \quad i = 1, 2, 3, 4.
\end{aligned}
$$

The coefficient matrix $\tilde{\tilde{A}}$ is a $(12(n+4), 12(n+4)$-banded sparse matrix. The Guassian-elimination method needs $O(n^4)$ flops to solve the linear system with $\tilde{\tilde{A}}$ as the coefficient matrix. In the numerical examples presented later in Chapter 8, we first re-scale the linear system so that each row vector of the coefficient matrix has unit infinity norm, which reduces the condition numbers of the coefficient matrix. Then the re-scaled linear system is solved by the "\" operator in MATLAB.

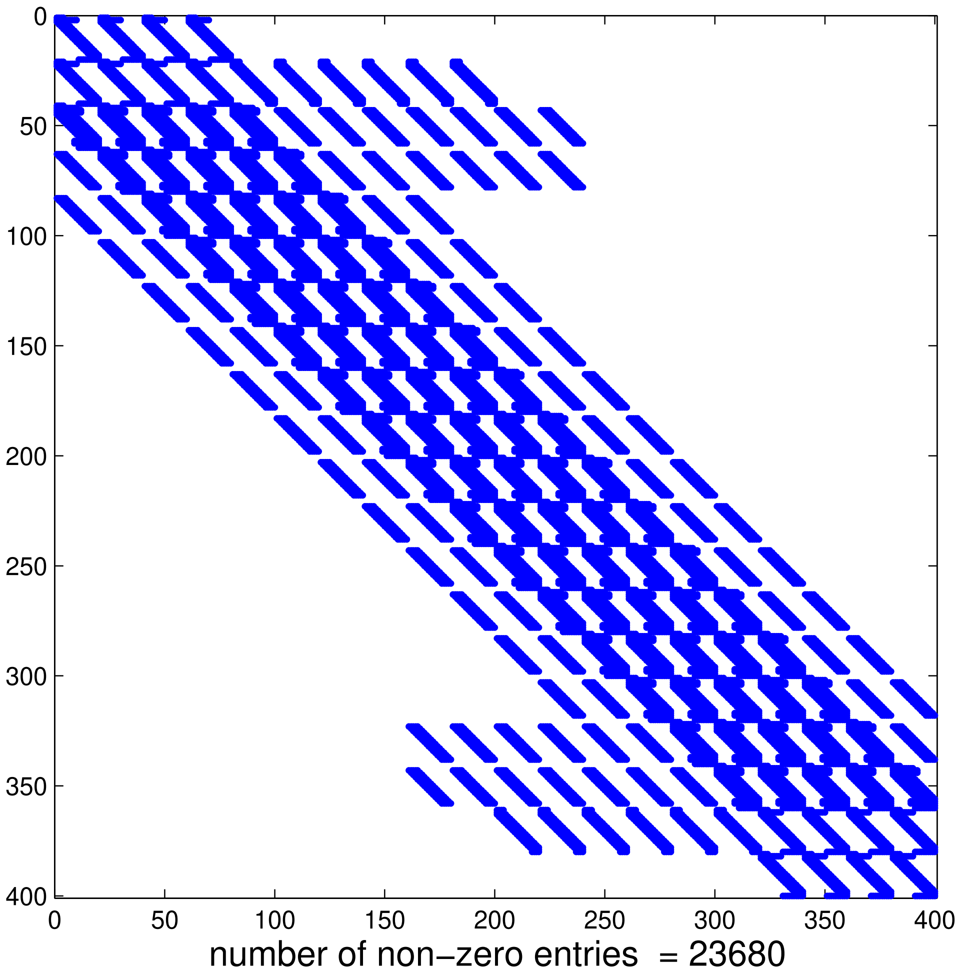


Figure 4.1: The sparsity pattern of the coefficient matrix for the one-step method when $n = 16$.

4.5 Biharmonic Problems with Homogeneous BCs

In this section, we apply the bi-quartic spline collocation method to two different biharmonic problems with homogeneous boundary conditions. The resulting linear systems play an essential role in our algorithm for solving the system (4.62).

In (4.4)-(4.8), when $g_1(x, y) = 0$ and $g_2(x, y) = 0$, we have the biharmonic Dirichlet problem with homogeneous boundary conditions,

$$
\begin{aligned}
Lu \equiv D_x^4 u + 2D_x^2 D_y^2 u + D_y^4 u &= f(x,y) \quad \text{in} \quad \Omega \equiv (0,1)\times(0,1), && (4.63)\\
B_1 u \equiv u &= 0, \quad \text{on} \quad \partial\Omega, && (4.64)\\
B_2 u \equiv \frac{\partial u}{\partial \mathrm{n}} &= 0 \quad \text{on} \quad \partial\Omega. && (4.65)
\end{aligned}
$$

The space of bi-quartic splines that satisfy the homogeneous boundary conditions has dimension n^2, since $8n+16$ degrees of freedom are absorbed by the boundary conditions. To form a set of quartic spline basis functions that satisfy the homogeneous boundary condition by construction, the standard quartic spline basis functions $\widehat{\phi}_i^x$ and $\widehat{\phi}_j^y$ are adjusted as follows:

$$
\begin{aligned}
\phi_1^x &= \widehat{\phi}_1^x + \frac{33}{4}\widehat{\phi}_{-1}^x - \frac{7}{4}\widehat{\phi}_0^x, & \phi_1^y &= \widehat{\phi}_1^y + \frac{33}{4}\widehat{\phi}_{-1}^y - \frac{7}{4}\widehat{\phi}_0^y,\\
\phi_2^x &= \widehat{\phi}_2^x + \frac{7}{4}\widehat{\phi}_{-1}^x - \frac{1}{4}\widehat{\phi}_0^x, & \phi_2^y &= \widehat{\phi}_2^y + \frac{7}{4}\widehat{\phi}_{-1}^y - \frac{1}{4}\widehat{\phi}_0^y,\\
\phi_i^x &= \widehat{\phi}_i^x, i = 3, \cdots, n, & \phi_j^y &= \widehat{\phi}_j^y, j = 3, \cdots, n, \qquad (4.66)\\
\phi_{n-1}^x &= \widehat{\phi}_{n-1}^x + \frac{7}{4}\widehat{\phi}_{n+2}^x - \frac{1}{4}\widehat{\phi}_{n+1}^x, & \phi_{n-1}^y &= \widehat{\phi}_{n-1}^y + \frac{7}{4}\widehat{\phi}_{n+2}^y - \frac{1}{4}\widehat{\phi}_{n+1}^y,\\
\phi_n^x &= \widehat{\phi}_n^x + \frac{33}{4}\widehat{\phi}_{n+2}^x - \frac{7}{4}\widehat{\phi}_{n+1}^x, & \phi_n^y &= \widehat{\phi}_n^y + \frac{33}{4}\widehat{\phi}_{n+2}^y - \frac{7}{4}\widehat{\phi}_{n+1}^y.
\end{aligned}
$$

We denote by S_{Δ_x} and $S_{\Delta_y}$the quartic spline spaces spanned by $\{\phi_i^x(x)\}_{i=1}^n$ and $\{\phi_j^y(y)\}_{i=1}^n$, respectively. The space $S_\Delta \equiv S_{\Delta_x} \otimes S_{\Delta_y}$ spanned by $\{\phi_i^x(x)\phi_j^y(y)\}_{i=1,j=1}^{n,n}$ satisfies the homogeneous boundary conditions automatically. To uniquely determine a bi-quartic spline $u_\Delta \in S_\Delta$, n^2 equations are needed.

When applying the three-step bi-quartic spline collocation method to the biharmonic Dirichlet problem with homogeneous boundary conditions, it is easy to see that the bi-quartic splines w_Δ in step 1 and v_Δ in step 2 satisfy homogeneous boundary conditions, therefore they can actually be found in the space S_Δ. The bi-quartic spline u_Δ, however, does not satisfy the homogeneous boundary conditions, which implies that it only belongs to the general space $\widehat{S}_\Delta$, not S_Δ. Hence, the three-step bi-quartic spline collocation method of Section 4.2 applied to (4.63)-(4.65) consists of the following steps:

Step 1: Determine a spline $w_\Delta \in S_\Delta$ by the equations

$$Lw_\Delta(\tau_i^x, \tau_j^y) = f(\tau_i^x, \tau_j^y), \quad i, j = 1, \cdots, n. \tag{4.67}$$

Step 2: Determine a spline $v_\Delta \in S_\Delta$ by the equations

$$Lv_\Delta(\tau_i^x, \tau_j^y) = \bar{f}(\tau_i^x, \tau_j^y), \quad i, j = 1, \cdots, n. \tag{4.68}$$

Step 3: Determine a spline $u_\Delta \in \widehat{S}_\Delta$ by the equations

$$\begin{aligned}
Lu_\Delta(\tau_i^x, \tau_j^y) &= \bar{\bar{f}}(\tau_i^x, \tau_j^y), \quad i, j = 1, \cdots, n, &(4.69)\\
u_\Delta(\tau_{\gamma_0}^x, \tau_j^y) &= g_1(\tau_{\gamma_0}^x, \tau_j^y), \quad j = 0, \cdots, n+1, &(4.70)\\
u_\Delta(\tau_i^x, \tau_{\gamma_0}^y) &= g_1(\tau_i^x, \tau_{\gamma_0}^y), \quad i = 1, \cdots, n &(4.71)\\
D_x u_\Delta(\tau_{\gamma_0}^x, \tau_j^y) &= \bar{\bar{g}}_2(\tau_{\gamma_0}^x, \tau_j^y), \quad j = 0, \cdots, n+1, &(4.72)\\
D_y u_\Delta(\tau_i^x, \tau_{\gamma_0}^y) &= \bar{\bar{g}}_2(\tau_i^x, \tau_{\gamma_0}^y), \quad i = 0, \cdots, n+1, &(4.73)\\
D_x D_y u_\Delta(\tau_{\gamma_0}^x, \tau_{\gamma_0}^y) &= \bar{\bar{g}}_c(\tau_{\gamma_0}^x, \tau_{\gamma_0}^y). &(4.74)
\end{aligned}$$

The linear systems arising from (4.67) and (4.68), respectively, share the same coefficient matrix which is an $n^2 \times n^2$ block-pentadiagonal matrix with pentadiagonal $n \times n$ blocks given by

$$A \equiv Q_4 \otimes Q_0 + 2Q_2 \otimes Q_2 + Q_0 \otimes Q_4, \tag{4.75}$$

where Q_i, $i = 0, 2, 4$, have the structures

$$Q_4 = \frac{1}{4h^4} \begin{bmatrix}
85 & -5 & 4 & & & & \\
-23 & 23 & -16 & 4 & & & \\
4 & -16 & 24 & -16 & 4 & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & & 4 & -16 & 24 & -16 & 4 \\
 & & & 4 & -16 & 23 & -23 \\
 & & & & 4 & -5 & 85
\end{bmatrix},$$

$$Q_2 = \frac{1}{32h^2}\begin{bmatrix} -35 & 19 & 4 & & & & \\ 9 & -41 & 16 & 4 & & & \\ 4 & 16 & -40 & 16 & 4 & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & 4 & 16 & -40 & 16 & 4 \\ & & & 4 & 16 & -41 & 9 \\ & & & & 4 & 19 & -35 \end{bmatrix},$$

$$Q_0 = \frac{1}{1536}\begin{bmatrix} 421 & 235 & 4 & & & & \\ 297 & 919 & 304 & 4 & & & \\ 4 & 304 & 920 & 304 & 4 & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & 4 & 304 & 920 & 304 & 4 \\ & & & 4 & 304 & 919 & 297 \\ & & & & 4 & 235 & 421 \end{bmatrix}.$$

The existence and uniqueness of w_Δ, v_Δ and u_Δ and the techniques for solving the linear systems with coefficient matrix A will be discussed later.

The second biharmonic problem with homogeneous boundary conditions is another type of biharmonic problem defined by

$$\begin{aligned} Lu \equiv D_x^4 u + 2D_x^2 D_y^2 u + D_y^4 u &= f(x,y) \quad \text{in} \quad \Omega \equiv (0,1)\times(0,1), && (4.76)\\ B_1 u \equiv u &= 0, \quad \text{on} \quad \partial\Omega, && (4.77)\\ B_2 u \equiv \Delta u &= 0 \quad \text{on} \quad \partial\Omega. && (4.78) \end{aligned}$$

Note that $\Delta u = 0$ on $\partial\Omega$ is equivalent to $\frac{\partial^2 u}{\partial n^2} = 0$, since $u = 0$ on $\partial\Omega$.

Suppose we want to find a bi-quartic spline that satisfies the equations (4.76)-(4.78) at the collocation points. Similar to the space S_Δ for the biharmonic Dirichlet problem with homogeneous boundary conditions (4.63)-(4.65), a space $\widetilde{S}_\Delta$ can be constructed, which satisfies the homogeneous boundary conditions (4.77)-(4.78) automatically. To construct

$\widetilde{S}_\Delta$, the standard quartic spline basis functions $\widehat{\phi}_i^x$ are adjusted as follows:

$$\begin{aligned}
\widetilde{\phi}_1^x &= \widehat{\phi}_1^x - \widehat{\phi}_0^x, & \widetilde{\phi}_1^y &= \widehat{\phi}_1^y - \widehat{\phi}_0^y, \\
\widetilde{\phi}_2^x &= \widehat{\phi}_2^x - \widehat{\phi}_{-1}^x, & \widetilde{\phi}_2^y &= \widehat{\phi}_2^y - \widehat{\phi}_{-1}^y, \\
\widetilde{\phi}_i^x &= \widehat{\phi}_i^x, i = 3, \cdots, n, & \widetilde{\phi}_j^y &= \widehat{\phi}_j^y, j = 3, \cdots, n, \\
\widetilde{\phi}_{n-1}^x &= \widehat{\phi}_{n-1}^x - \widehat{\phi}_{n+2}^x, & \widetilde{\phi}_{n-1}^y &= \widehat{\phi}_{n-1}^y - \widehat{\phi}_{n+2}^y, \\
\widetilde{\phi}_n^x &= \widehat{\phi}_n^x - \widehat{\phi}_{n+1}^x, & \widetilde{\phi}_n^y &= \widehat{\phi}_n^y - \widehat{\phi}_{n+1}^y.
\end{aligned} \tag{4.79}$$

The space $\widetilde{S}_\Delta$ spanned by $\{\widetilde{\phi}_i^x(x)\phi_j^y(y)\}_{i=1,j=1}^{n,n}$ satisfies the boundary conditions (4.77)-(4.78) by construction. Applied to the BVP (4.76)-(4.78), the standard Bi-Quartic SC method generates a linear system of the form

$$B\bar{\theta}' = \bar{g}', \tag{4.80}$$

where B is a block-pentadiagonal matrix with pentadiagonal $n \times n$ blocks, which takes the tensor product form

$$B \equiv \widetilde{Q}_4 \otimes Q_0 + 2\widetilde{Q}_2 \otimes Q_2 + \widetilde{Q}_0 \otimes Q_4, \tag{4.81}$$

where $\widetilde{Q}_i$, $i = 0, 2, 4$, have the structures

$$\widetilde{Q}_4 = \frac{1}{h^4} \begin{bmatrix}
10 & -5 & 1 & & & & \\
-5 & 6 & -4 & 1 & & & \\
1 & -4 & 6 & -4 & 1 & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & & 1 & -4 & 6 & -4 & 1 \\
 & & & 1 & -4 & 6 & -5 \\
 & & & & 1 & -5 & 10
\end{bmatrix},$$

$$\widetilde{Q}_2 = \frac{1}{8h^2}\begin{bmatrix} -14 & 3 & 1 & & & & & \\ 3 & -10 & 4 & 1 & & & & \\ 1 & 4 & -10 & 4 & 1 & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & 1 & 4 & -10 & 4 & 1 \\ & & & 1 & 4 & -10 & 3 \\ & & & & 1 & 3 & -14 \end{bmatrix},$$

$$\widetilde{Q}_0 = \frac{1}{384}\begin{bmatrix} 154 & 75 & 1 & & & & & \\ 75 & 230 & 76 & 1 & & & & \\ 1 & 76 & 230 & 76 & 1 & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & 1 & 76 & 230 & 76 & 1 \\ & & & 1 & 76 & 230 & 75 \\ & & & & 1 & 75 & 154 \end{bmatrix}.$$

To study the properties of the matrices $\widetilde{Q}_i$, $i = 0, 2, 4$, we recall the quadratic spline collocation matrices arising from second-order boundary value problems [9]. Consider the boundary value problem

$$\begin{aligned} u'' + u &= f \quad \text{in } (a, b), \\ u(a) &= 0, \\ u(b) &= 0, \end{aligned} \tag{4.82}$$

and its discretization by quadratic spline collocation, where the quadratic spline basis functions are adjusted so that the homogeneous Dirichlet boundary conditions are satisfied by construction. The matrix of the resulting linear system is $T_2 + T_0$, where the matrices T_i, $i = 0, 2$, arise from the terms u'' and u, respectively. Recall the forms of T_0

and T_2 as given in [9]:

$$T_2 = \frac{1}{h^2}\begin{bmatrix} -3 & 1 & & & \\ 1 & -2 & 1 & & \\ & \cdot & \cdot & \cdot & \\ & & 1 & -2 & 1 \\ & & & 1 & -3 \end{bmatrix}, \quad T_0 = \frac{1}{8}\begin{bmatrix} 5 & 1 & & & \\ 1 & 6 & 1 & & \\ & \cdot & \cdot & \cdot & \\ & & 1 & 6 & 1 \\ & & & 1 & 5 \end{bmatrix}.$$

It is easy to show [43] that

$$\begin{aligned} \widetilde{Q}_4 &= T_2^2, \\ \widetilde{Q}_2 &= T_2 \cdot T_0, \\ \widetilde{Q}_0 &= (T_0^2 + 8T_0 - 3I)/6. \end{aligned}$$

Since the matrices T_0 and T_2 are non-singular and their eigenvalues and eigenvectors are known [9], the eigenvalues and eigenvectors of the matrices $\widetilde{Q}_4$, $\widetilde{Q}_2$ and $\widetilde{Q}_0$ can be easily derived.

Chapter 5

Fast Solver for the Bi-Quartic SC Equations

The three-step Bi-Quartic SC method introduced in the preceding chapter for the biharmonic Dirichlet problem (4.1)-(4.3) involves solving three linear systems similar to (4.62). Thus solving these systems efficiently is critical for the application of the method. This chapter aims at developing an efficient method for the solution of the linear system (4.62).

5.1 Linear Transformation

The coefficient matrix $\widehat{A}$ of the linear system (4.62) is of size $(n+4)^2 \times (n+4)^2$. With appropriate row operations, however, we can reduce (4.62) to a linear system with the $n^2 \times n^2$ coefficient matrix A in (4.75) which is easier to solve. Upon solving the smaller linear system, the solution to the original linear system can be easily derived.

This reduction process consists of two stages. In the first stage, the degrees of freedom corresponding to the boundary conditions on $x = 0$ and $x = 1$ are eliminated and accordingly the blocks in the first two and last two rows and in the first two and last two columns of the coefficient matrix are transformed and integrated into the rest of the matrix. Thus a smaller linear system is generated with an $n(n+4) \times n(n+4)$ coefficient matrix. In addition, the coefficient matrix is a block pentadiagonal matrix with blocks of size $(n+4) \times (n+4)$. The second stage eliminates the degrees of freedom corresponding to the boundary conditions on $y = 0$ and $y = 1$. Accordingly the first two and last two rows, and the first two and last two columns in each block are eliminated. The coefficient matrix of the resulting linear system is exactly the matrix A in (4.75).

Recall in the linear system (4.62) we defined matrices W, and P_{ij}, $i = 1, \cdots, n$, $j = i-2, i-1, i, i+1, i+2$, which are of size $(n+4) \times (n+4)$. The non-singularity of matrix W can be shown similarly as the non-singularity of T in Lemma 2.2.2. Let K_1 and K_2 be defined as $K_1 = \text{blockdiag}(K_{1,1}, I, K_{1,3})$, and $K_2 = \text{blockdiag}(K_{2,1}, I, K_{2,3})$, where

$$K_{1,1} = \begin{bmatrix} I_{n+4} & & \\ I_{n+4} & \frac{h}{4} I_{n+4} & \\ -24P_{1,-1}W^{-1} & & I_{n+4} \end{bmatrix}, \quad K_{1,3} = \begin{bmatrix} I_{n+4} & -24P_{n,n+2}W^{-1} & \\ & I_{n+4} & \frac{-h}{4} I_{n+4} \\ & & I_{n+4} \end{bmatrix},$$

$$K_{2,1} = \begin{bmatrix} I_{n+4} & & & \\ & I_{n+4} & & \\ & -3(P_{1,0} - 11P_{1,-1})W^{-1} & I_{n+4} & \\ & -3P_{2,0}W^{-1} & & I_{n+4} \end{bmatrix},$$

and

$$K_{2,3} = \begin{bmatrix} I_{n+4} & & -3P_{n-1,n+1}W^{-1} & \\ & I_{n+4} & -3(P_{n,n+1} - 11P_{n,n+2})W^{-1} & \\ & & I_{n+4} & \\ & & & I_{n+4} \end{bmatrix}.$$

Using Lemma 2.2.1, it is trivial to show that K_1 and K_2 are nonsingular. Multiplying both sides of the system (4.62) by K_2K_1 generates

$$
\begin{vmatrix}
\frac{1}{24}W & \frac{11}{24}W & \frac{11}{24}W & \frac{1}{24}W & & & & & \\
 & \frac{1}{3}W & \frac{7}{12}W & \frac{1}{12}W & & & & & \\
 & & \tilde{P}_{1,1} & \tilde{P}_{1,2} & P_{1,3} & & & & \\
 & & \tilde{P}_{2,1} & \tilde{P}_{2,2} & P_{2,3} & P_{2,4} & & & \\
 & & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
 & & & P_{n-1,n-3} & P_{n-1,n-2} & \tilde{P}_{n-1,n-1} & \tilde{P}_{n-1,n} & & \\
 & & & & P_{n,n-2} & \tilde{P}_{n,n-1} & \tilde{P}_{n,n} & & \\
 & & & & & \frac{1}{12}W & \frac{7}{12}W & \frac{1}{3}W & \\
 & & & & & \frac{-1}{6h}W & \frac{-1}{2h}W & \frac{1}{2h}W & \frac{1}{6h}W
\end{vmatrix}
\begin{vmatrix}
\varsigma_{-1} \\ \varsigma_0 \\ \varsigma_1 \\ \varsigma_2 \\ \cdot \\ \varsigma_{n-1} \\ \varsigma_n \\ \varsigma_{n+1} \\ \varsigma_{n+2}
\end{vmatrix}
=
\begin{vmatrix}
\varepsilon_{-1} \\ \tilde{\varepsilon}_0 \\ \tilde{\varepsilon}_1 \\ \tilde{\varepsilon}_2 \\ \cdot \\ \tilde{\varepsilon}_{n-1} \\ \tilde{\varepsilon}_n \\ \tilde{\varepsilon}_{n+1} \\ \varepsilon_{n+2}
\end{vmatrix},
\tag{5.1}
$$

where

$$
\begin{aligned}
\tilde{P}_{1,1} = & \quad \tfrac{33}{4}P_{1,-1} - \tfrac{7}{4}P_{1,0} + P_{1,1}, & \tilde{P}_{1,2} = \frac{7}{4}P_{1,-1} - \frac{1}{4}P_{1,0} + P_{1,2}, \\
\tilde{P}_{n,n} = & \quad P_{n,n} - \tfrac{7}{4}P_{n,n+1} + \tfrac{33}{4}P_{n,n+2}, & \tilde{P}_{n,n-1} = P_{n,n-1} - \frac{1}{4}P_{n,n+1} + \frac{7}{4}P_{n,n+2}, \\
\tilde{P}_{2,1} = & \quad P_{2,1} - \tfrac{7}{4}P_{2,0}, & \tilde{P}_{2,2} = P_{2,2} - \frac{1}{4}P_{2,0}, \\
\tilde{P}_{n-1,n} = & \quad P_{n-1,n} - \tfrac{7}{4}P_{n-1,n+1}, & \tilde{P}_{n-1,n-1} = P_{n-1,n-1} - \frac{1}{4}P_{n-1,n+1},
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{\varepsilon}_0 &= \varepsilon_{-1} + \frac{h}{4}\varepsilon_0, \quad \tilde{\varepsilon}_2 = \varepsilon_2 - 3P_{2,0}W^{-1}\tilde{\varepsilon}_0, \\
\tilde{\varepsilon}_1 &= (\varepsilon_1 - 24P_{1,-1}W^{-1}\varepsilon_{-1}) - 3(P_{1,0} - 11P_{1,-1})W^{-1}\tilde{\varepsilon}_0, \\
\tilde{\varepsilon}_i &= \varepsilon_i, \quad i = 3, \cdots, n-2, \\
\tilde{\varepsilon}_{n+1} &= \varepsilon_{n+1} - \frac{h}{4}\varepsilon_{n+2}, \quad \tilde{\varepsilon}_{n-1} = \varepsilon_{n-1} - 3P_{n-1,n+1}W^{-1}\tilde{\varepsilon}_{n+1}, \\
\tilde{\varepsilon}_n &= (\varepsilon_n - 24P_{n,n+2}W^{-1}\varepsilon_{n+1}) - 3(P_{n,n+1} - 11P_{n,n+2})W^{-1}\tilde{\varepsilon}_{n+1}.
\end{aligned}
$$

Now consider the linear system

$$\begin{vmatrix} \tilde{P}_{1,1} & \tilde{P}_{1,2} & P_{1,3} & & & & \\ \tilde{P}_{2,1} & \tilde{P}_{2,2} & P_{2,3} & P_{2,4} & & & \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} & P_{3,5} & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & P_{n-2,n-4} & P_{n-2,n-3} & P_{n-2,n-2} & P_{n-2,n-1} & P_{n-2,n} \\ & & & P_{n-1,n-3} & P_{n-1,n-2} & \tilde{P}_{n-1,n-1} & \tilde{P}_{n-1,n} \\ & & & & P_{n,n-2} & \tilde{P}_{n,n-1} & \tilde{P}_{n,n} \end{vmatrix} \begin{vmatrix} \varsigma_1 \\ \varsigma_2 \\ \varsigma_3 \\ \cdot \\ \varsigma_{n-2} \\ \varsigma_{n-1} \\ \varsigma_n \end{vmatrix} = \begin{vmatrix} \tilde{\varepsilon}_1 \\ \tilde{\varepsilon}_2 \\ \tilde{\varepsilon}_3 \\ \cdot \\ \tilde{\varepsilon}_{n-2} \\ \tilde{\varepsilon}_{n-1} \\ \tilde{\varepsilon}_n \end{vmatrix}. \tag{5.2}$$

The linear system (5.2) determines the coefficients of the basis functions $\{\widehat{\phi}_i^x(x)\widehat{\phi}_j^y(y)\}_{i=1,j=-1}^{n,n+2}$. We remind readers that $\bar{c}_{i,j}$ is the coefficient of the basis function $\widehat{\phi}_i^x(x)\widehat{\phi}_j^y(y)$, and that $\varsigma_k = [\bar{c}_{k,-1}, \cdots, \bar{c}_{k,n+2}]^T$, $k = 1, \cdots, n$. Once the system (5.2) is solved, we can recover the remaining unknowns ς_{-1}, ς_0, ς_{n+1} and ς_{n+2} of (4.62) from the top two and bottom two block equations of (5.1). Figure 5.1 shows the sparsity pattern of the coefficient matrix of (5.2) when $n = 8$.

Notice that the coefficient matrix of (5.2) is of size $n(n+4) \times n(n+4)$. To further reduce (5.2) to a system with an $n^2 \times n^2$ coefficient matrix, the degrees of freedom corresponding to the boundary conditions on $y = 0$ and $y = 1$ are eliminated through the following process. First, we reorder the coefficients in (5.2) such that $\widehat{\varsigma}_k = [\bar{c}_{1,k}, \cdots, \bar{c}_{n,k}]^T$, $k = -1, \cdots, n+1$. Second, we change the ordering of the collocation points to "left-to-right" then "bottom-up", and present the collocation equations as described by the

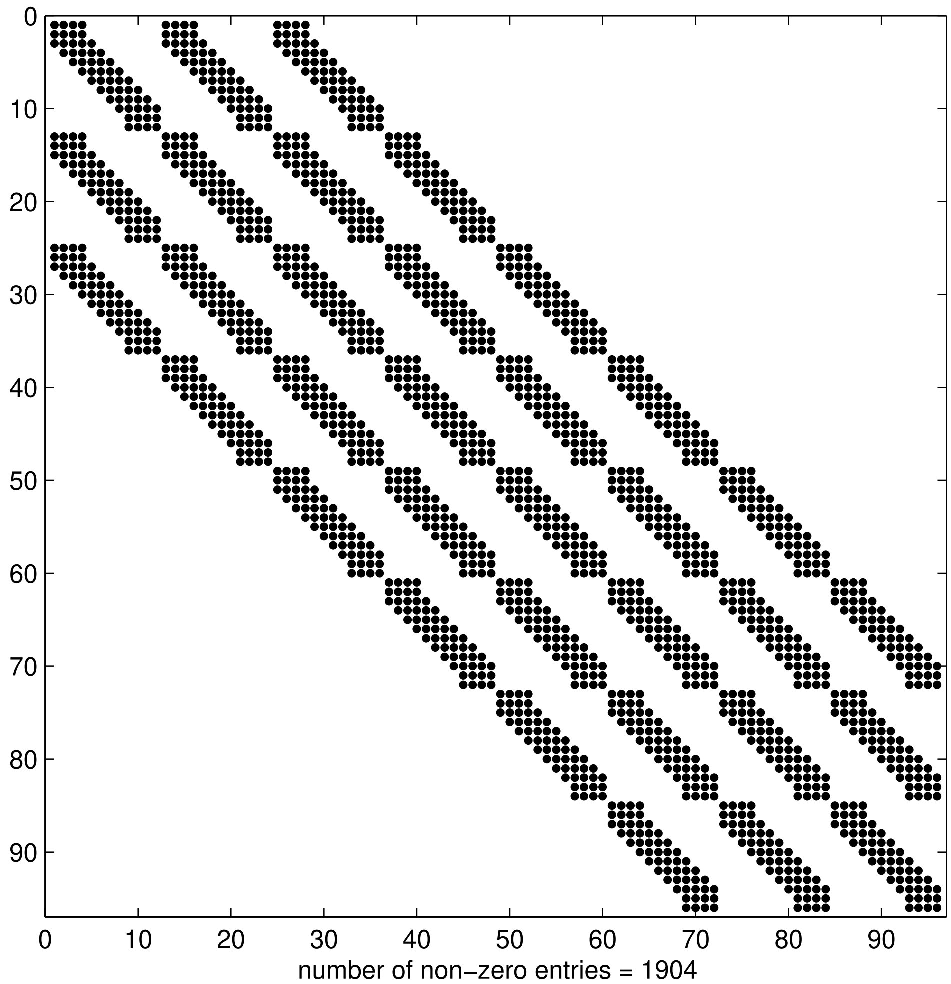


Figure 5.1: The sparsity pattern of the coefficient matrix in (5.2) when $n = 8$.

tableau

$$\left\{ \begin{array}{lllll} u_{1,0} & u_{2,0} & \cdots & u_{n-1,0} & u_{n,0} \\ D_y u_{1,0} & D_y u_{2,0} & \cdots & D_y u_{n-1,0} & D_y u_{n,0} \\ L u_{1,1} & L u_{2,1} & \cdots & L u_{n-1,1} & L u_{n,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ L u_{1,n} & L u_{2,n} & \cdots & L u_{n-1,n} & L u_{n,n} \\ D_y u_{1,n+1} & D_y u_{2,n+1} & \cdots & D_y u_{n-1,n+1} & D_y u_{n,n+1} \\ u_{1,n+1} & u_{2,n+1} & \cdots & u_{n-1,n+1} & u_{n,n+1} \end{array} \right\}.$$

The reordered matrix takes the following block form with $n + 4$ blocks of size $n \times n$ in each direction

$$
\left|\begin{array}{ccccccccc}
\frac{1}{24}\widehat{W} & \frac{11}{24}\widehat{W} & \frac{11}{24}\widehat{W} & \frac{1}{24}\widehat{W} & & & & & \\
\frac{-1}{6h}\widehat{W} & \frac{-1}{2h}\widehat{W} & \frac{1}{2h}\widehat{W} & \frac{1}{6h}\widehat{W} & & & & & \\
\widehat{P}_{1,-1} & \widehat{P}_{1,0} & \widehat{P}_{1,1} & \widehat{P}_{1,2} & \widehat{P}_{1,3} & & & & \\
 & \widehat{P}_{2,0} & \widehat{P}_{2,1} & \widehat{P}_{2,2} & \widehat{P}_{2,3} & \widehat{P}_{2,4} & & & \\
 & & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
 & & & \widehat{P}_{n-1,n-3} & \widehat{P}_{n-1,n-2} & \widehat{P}_{n-1,n-1} & \widehat{P}_{n-1,n} & \widehat{P}_{n-1,n+1} & \\
 & & & & \widehat{P}_{n,n-2} & \widehat{P}_{n,n-1} & \widehat{P}_{n,n} & \widehat{P}_{n,n+1} & \widehat{P}_{n,n+2} \\
 & & & & & \frac{-1}{6h}\widehat{W} & \frac{-1}{2h}\widehat{W} & \frac{1}{2h}\widehat{W} & \frac{1}{6h}\widehat{W} \\
 & & & & & \frac{1}{24}\widehat{W} & \frac{11}{24}\widehat{W} & \frac{11}{24}\widehat{W} & \frac{1}{24}\widehat{W}
\end{array}\right\|
\begin{array}{c}
\widehat{\varsigma}_{-1} \\ \widehat{\varsigma}_{0} \\ \widehat{\varsigma}_{1} \\ \widehat{\varsigma}_{2} \\ \cdot \\ \widehat{\varsigma}_{n-1} \\ \widehat{\varsigma}_{n} \\ \widehat{\varsigma}_{n+2} \\ \widehat{\varsigma}_{n+1}
\end{array}\right| = \left|\begin{array}{c}
\widehat{\varepsilon}_{-1} \\ \widehat{\varepsilon}_{0} \\ \widehat{\varepsilon}_{1} \\ \widehat{\varepsilon}_{2} \\ \cdot \\ \widehat{\varepsilon}_{n-1} \\ \widehat{\varepsilon}_{n} \\ \widehat{\varepsilon}_{n+2} \\ \widehat{\varepsilon}_{n+1}
\end{array}\right| \tag{5.3}
$$

where

$$
\widehat{W} = Q_0 = \frac{1}{1536}\begin{bmatrix}
421 & 235 & 4 & & & & \\
297 & 919 & 304 & 4 & & & \\
4 & 304 & 920 & 304 & 4 & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & & 4 & 304 & 920 & 304 & 4 \\
 & & & 4 & 304 & 919 & 297 \\
 & & & & 4 & 235 & 421
\end{bmatrix}.
$$

and $\widehat{P}_{ij}$, $i = 1, \cdots, n$, $j = i-2, i-1, i, i+1, i+2$, are defined by

$$
\widehat{P}_{i,i-2} = \widehat{P}_{i,i+2} = \frac{1}{768h^4}\begin{bmatrix}
43 & 229 & 28 & & & & \\
191 & 225 & 240 & 28 & & & \\
28 & 240 & 232 & 240 & 28 & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & & 28 & 240 & 232 & 240 & 28 \\
 & & & 28 & 240 & 225 & 191 \\
 & & & & 28 & 229 & 43
\end{bmatrix},
$$

$$\widehat{P}_{i,i-1} = \widehat{P}_{i,i+1} = \frac{1}{384h^4}\begin{bmatrix} 774 & -102 & 120 & & & & \\ -626 & -974 & -416 & 120 & & & \\ 120 & -416 & -944 & -416 & 120 & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & 120 & -416 & -944 & -416 & 120 \\ & & & 120 & -416 & -974 & -626 \\ & & & & 120 & -102 & 774 \end{bmatrix},$$

$$\widehat{P}_{i,i} = \frac{1}{384h^4}\begin{bmatrix} 6569 & -505 & 116 & & & & \\ -1147 & 3931 & -944 & 116 & & & \\ 116 & -944 & 3960 & -944 & 116 & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & 116 & -944 & 3960 & -944 & 116 \\ & & & 116 & -944 & 3931 & -1147 \\ & & & & 116 & -505 & 6569 \end{bmatrix}.$$

Noticing the similar pattern of (4.62) and (5.3), a similar block Gaussian elimination process can be applied to (5.3) as to (4.62). Since $\widehat{W}$ is strictly diagonally dominant matrix, $\widehat{W}$ is non-singular. Define $\widehat{K}_1$ and $\widehat{K}_2$ by $\widehat{K}_1 = \text{blockdiag}(\widehat{K}_{1,1}, I, \widehat{K}_{1,3})$ and $\widehat{K}_3 = \text{blockdiag}(\widehat{K}_{3,1}, I, \widehat{K}_{3,3})$, where

$$\widehat{K}_{1,1} = \begin{bmatrix} I_n & & \\ I_n & \frac{h}{4}I_n & \\ -24\widehat{P}_{1,-1}\widehat{W}^{-1} & & I_n \end{bmatrix}, \widehat{K}_{1,3} = \begin{bmatrix} I_n & & -24\widehat{P}_{n,n+2}\widehat{W}^{-1} \\ & \frac{-h}{4}I_n & I_n \\ & & I_n \end{bmatrix},$$

$$\widehat{K}_{2,1} = \begin{bmatrix} I_n & & & \\ & I_n & & \\ & -3(\widehat{P}_{1,0} - 11\widehat{P}_{1,-1})W^{-1} & I_n & \\ & -3\widehat{P}_{2,0}\widehat{W}^{-1} & & I_n \end{bmatrix}, \widehat{K}_{2,3} = \begin{bmatrix} I_n & & -3\widehat{P}_{n-1,n+1}\widehat{W}^{-1} & \\ & I_n & -3(\widehat{P}_{n,n+1} - 11\widehat{P}_{n,n+2})\widehat{W}^{-1} & \\ & & I_n & \\ & & & I_n \end{bmatrix}.$$

The non-singularity of $\widehat{K}_1$ and $\widehat{K}_2$ can be shown by Lemma 2.2.1. Multiplying both sides of the system (5.3) by $\widehat{K}_2\widehat{K}_1$ generates

$$\left|\begin{array}{ccccccccc}
\frac{1}{24}\widehat{W} & \frac{11}{24}\widehat{W} & \frac{11}{24}\widehat{W} & \frac{1}{24}\widehat{W} & & & & & \\
 & \frac{1}{3}\widehat{W} & \frac{7}{12}\widehat{W} & \frac{1}{12}\widehat{W} & & & & & \\
 & & \tilde{\widehat{P}}_{1,1} & \tilde{\widehat{P}}_{1,2} & \widehat{P}_{1,3} & & & & \\
 & & \tilde{\widehat{P}}_{2,1} & \tilde{\widehat{P}}_{2,2} & \widehat{P}_{2,3} & \widehat{P}_{2,4} & & & \\
 & & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
 & & & \widehat{P}_{n-1,n-3} & \widehat{P}_{n-1,n-2} & \tilde{\widehat{P}}_{n-1,n-1} & \tilde{\widehat{P}}_{n-1,n} & & \\
 & & & & \widehat{P}_{n,n-2} & \tilde{\widehat{P}}_{n,n-1} & \tilde{\widehat{P}}_{n,n} & & \\
 & & & & & \frac{1}{12}\widehat{W} & \frac{7}{12}\widehat{W} & \frac{1}{3}\widehat{W} & \\
 & & & & & \frac{1}{24}\widehat{W} & \frac{11}{24}\widehat{W} & \frac{11}{24}\widehat{W} & \frac{1}{24}\widehat{W}
\end{array}\right\|\begin{array}{c}
\widehat{\varsigma}_{-1} \\ \widehat{\varsigma}_0 \\ \widehat{\varsigma}_1 \\ \widehat{\varsigma}_2 \\ \cdot \\ \widehat{\varsigma}_{n-1} \\ \widehat{\varsigma}_n \\ \widehat{\varsigma}_{n+1} \\ \widehat{\varsigma}_{n+2}
\end{array}\right| = \left|\begin{array}{c}
\widehat{\varepsilon}_{-1} \\ \tilde{\widehat{\varepsilon}}_0 \\ \tilde{\widehat{\varepsilon}}_1 \\ \tilde{\widehat{\varepsilon}}_2 \\ \cdot \\ \tilde{\widehat{\varepsilon}}_{n-1} \\ \tilde{\widehat{\varepsilon}}_n \\ \tilde{\widehat{\varepsilon}}_{n+1} \\ \widehat{\varepsilon}_{n+2}
\end{array}\right|, \qquad (5.4)$$

where

$$\begin{aligned}
\tilde{\widehat{P}}_{1,1} = &\ \tfrac{33}{4}\widehat{P}_{1,-1} - \tfrac{7}{4}\widehat{P}_{1,0} + \widehat{P}_{1,1}, \quad \tilde{\widehat{P}}_{1,2} = \frac{7}{4}\widehat{P}_{1,-1} - \frac{1}{4}\widehat{P}_{1,0} + \widehat{P}_{1,2}, \\
\tilde{\widehat{P}}_{n,n} = &\ \widehat{P}_{n,n} - \tfrac{7}{4}\widehat{P}_{n,n+1} + \tfrac{33}{4}\widehat{P}_{n,n+2}, \quad \tilde{\widehat{P}}_{n,n-1} = \widehat{P}_{n,n-1} - \frac{1}{4}\widehat{P}_{n,n+1} + \frac{7}{4}\widehat{P}_{n,n+2},
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\widehat{P}}_{2,1} = &\ \widehat{P}_{2,1} - \tfrac{7}{4}\widehat{P}_{2,0}, \quad \tilde{\widehat{P}}_{2,2} = \widehat{P}_{2,2} - \frac{1}{4}\widehat{P}_{2,0}, \\
\tilde{\widehat{P}}_{n-1,n} = &\ \widehat{P}_{n-1,n} - \tfrac{7}{4}\widehat{P}_{n-1,n+1}, \quad \tilde{\widehat{P}}_{n-1,n-1} = \widehat{P}_{n-1,n-1} - \frac{1}{4}\widehat{P}_{n-1,n+1},
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\widehat{\varepsilon}}_0 &= \widehat{\varepsilon}_{-1} + \frac{h}{4}\widehat{\varepsilon}_0, \quad \tilde{\widehat{\varepsilon}}_2 = \widehat{\varepsilon}_2 - 3\widehat{P}_{2,0}\widehat{W}^{-1}\tilde{\widehat{\varepsilon}}_0, \\
\tilde{\widehat{\varepsilon}}_1 &= (\widehat{\varepsilon}_1 - 24\widehat{P}_{1,-1}\widehat{W}^{-1}\widehat{\varepsilon}_{-1}) - 3(\widehat{P}_{1,0} - 11\widehat{P}_{1,-1})\widehat{W}^{-1}\tilde{\widehat{\varepsilon}}_0, \\
\tilde{\widehat{\varepsilon}}_i &= \widehat{\varepsilon}_i, \quad i = 3, \cdots, n-2, \\
\tilde{\widehat{\varepsilon}}_{n+1} &= \widehat{\varepsilon}_{n+1} - \frac{h}{4}\widehat{\varepsilon}_{n+2}, \quad \tilde{\widehat{\varepsilon}}_{n-1} = \widehat{\varepsilon}_{n-1} - 3\widehat{P}_{n-1,n+1}\widehat{W}^{-1}\tilde{\widehat{\varepsilon}}_{n+1}, \\
\tilde{\widehat{\varepsilon}}_n &= (\widehat{\varepsilon}_n - 24\widehat{P}_{n,n+2}\widehat{W}^{-1}\widehat{\varepsilon}_{n+1}) - 3(\widehat{P}_{n,n+1} - 11\widehat{P}_{n,n+2})\widehat{W}^{-1}\tilde{\widehat{\varepsilon}}_{n+1}.
\end{aligned}$$

Now consider the linear system

$$\begin{vmatrix} \widehat{\widetilde{P}}_{1,1} & \widehat{\widetilde{P}}_{1,2} & \widehat{P}_{1,3} & & & & \\ \widehat{\widetilde{P}}_{2,1} & \widehat{\widetilde{P}}_{2,2} & \widehat{P}_{2,3} & \widehat{P}_{2,4} & & & \\ \widehat{P}_{3,1} & \widehat{P}_{3,2} & \widehat{P}_{3,3} & \widehat{P}_{3,4} & \widehat{P}_{3,5} & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & \widehat{P}_{n-2,n-4} & \widehat{P}_{n-2,n-3} & \widehat{P}_{n-2,n-2} & \widehat{P}_{n-2,n-1} & \widehat{P}_{n-2,n} \\ & & & \widehat{P}_{n-1,n-3} & \widehat{P}_{n-1,n-2} & \widehat{\widetilde{P}}_{n-1,n-1} & \widehat{\widetilde{P}}_{n-1,n} \\ & & & & \widehat{P}_{n,n-2} & \widehat{\widetilde{P}}_{n,n-1} & \widehat{\widetilde{P}}_{n,n} \end{vmatrix} \begin{vmatrix} \widehat{\varsigma}_1 \\ \widehat{\varsigma}_2 \\ \widehat{\varsigma}_3 \\ \cdot \\ \widehat{\varsigma}_{n-2} \\ \widehat{\varsigma}_{n-1} \\ \widehat{\varsigma}_n \end{vmatrix} = \begin{vmatrix} \widehat{\widetilde{\varepsilon}}_1 \\ \widehat{\widetilde{\varepsilon}}_2 \\ \widehat{\widetilde{\varepsilon}}_3 \\ \cdot \\ \widehat{\widetilde{\varepsilon}}_{n-2} \\ \widehat{\widetilde{\varepsilon}}_{n-1} \\ \widehat{\widetilde{\varepsilon}}_n \end{vmatrix}. \tag{5.5}$$

Clearly, once we solve (5.5), we can recover the remaining unknowns $\widehat{\varsigma}_{-1}$, $\widehat{\varsigma}_0$, $\widehat{\varsigma}_{n+1}$ and $\widehat{\varsigma}_{n+2}$ of (5.3) from the top two and bottom two block equations of (5.4). Simple calculations show that the coefficient matrix in (5.5) is exactly the matrix A in (4.75). Therefore we have successfully reduced the linear system (4.62) to another smaller linear system. We will analyze its solvability and present a fast solver for it in the following sections.

5.2 Fast Solver for the Second Biharmonic Problem

In this section, we present a fast solver for the linear system (4.81). It is known [9, 43] that $\widetilde{Q}_i$, $i = 0, 2, 4$ have common orthonormal eigenvectors δ_i, $i = 1, \cdots, n$, given by

$$\delta_{ij} = \kappa_i \sin\frac{(2j-1)i\pi}{2n}, j = 1, \cdots, n, \tag{5.6}$$

where

$$\kappa_i = \sqrt{\frac{2}{n}},\ i = 1, \cdots, n-1, \text{and}\ \kappa_n = \sqrt{\frac{1}{n}}. \tag{5.7}$$

We define the matrix $Q = (q_{ij})_{n\times n} \equiv (\delta_{ji})_{n\times n}$, and notice that Q is the inverse of the Discrete Sine Transform II (DST-II) [33] matrix of order n. We have $\widetilde{Q}_0 = Q\Lambda^{(0)}Q^T$, $\widetilde{Q}_2 = Q\Lambda^{(2)}Q^T$, and $\widetilde{Q}_4 = Q\Lambda^{(4)}Q^T$, where $\Lambda^{(i)}$, $i = 0, 2, 4$, are diagonal matrices, whose diagonal entries are the eigenvalues of $\widetilde{Q}_i$, $i = 0, 2, 4$, respectively. In particular,

$$\begin{aligned} \Lambda_j^{(4)} &= \frac{16}{h^4}\sin^4\frac{j\pi}{2n}, \quad \Lambda_j^{(2)} = \frac{1}{h^2}(-2 + 2\cos^4\frac{j\pi}{2n}), \\ \Lambda_j^{(0)} &= \frac{5}{24} + \frac{3}{4}\cos^2\frac{j\pi}{2n} + \frac{1}{24}\cos^4\frac{j\pi}{2n}, \end{aligned} \tag{5.8}$$

where $j = 1, \cdots, n$, and where, for brevity, we used Λ_j instead of $\Lambda_{j,j}$. Let I_n denote the identity matrix of order n, and

$$W \equiv \Lambda^{(4)} \otimes Q_0 + 2\Lambda^{(2)} \otimes Q_2 + \Lambda^{(0)} \otimes Q_4. \tag{5.9}$$

Obviously, W is a block-diagonal matrix, consisting of n pentadiagonal blocks w_i, $i = 1, \cdots, n$, each of size $n \times n$. By properties of tensor products, we have the block-diagonalization of B by $Q \otimes I_n$, that is,

$$B = (Q \otimes I_n) W (Q^T \otimes I_n).$$

Assuming W is non-singular, we also have

$$B^{-1} = (Q \otimes I_n) W^{-1} (Q^T \otimes I_n).$$

In Section 5.4, we show that B (and W) are non-singular.

In [10], an algorithm based on the Fast Fourier Transform (FFT) is presented for computing the solution to the Bi-Quadratic SC equations. The algorithm applies Fast Sine Transforms II (FST-II) and the corresponding inverse (iFST-II) in one-dimension to block-diagonalize the matrix, and solves the block-diagonal matrix W by Gauss elimination. A similar algorithm can be applied to solve the equations (4.80). Before describing the algorithm, we adopt a convenient notation from [10]. For any $mn \times 1$ vector g, let $g_{n\times m}$ denote an $n \times m$ matrix with entries the components of g laid out in n rows and m columns, column by column.

Algorithm FFTSC(n, $\bar{g}'$)

Step 1: Perform **FST-II** of size n to each of the n columns of $(\bar{g}'_{n\times n})^T$ to obtain $g^{(1)}_{n\times n} = Q^T (g'_{n\times n})^T$, or equivalently, $g^{(1)} = (Q^T \otimes I_n) g'$

Step 2: Solve the block-diagonal system $W g^{(2)} = g^{(1)}$, where W is given in (5.9).

Step 3: Perform **iFST-II** of size n to each of the n columns of $(g^{(2)}_{n\times n})^T$ to obtain $\bar{\theta}'_{n\times n} = Q (g^{(2)}_{n\times n})^T$, or equivalently, $\bar{\theta}' = (Q \otimes I_n) g^{(2)} = (Q \otimes I_n) W^{-1} (Q^T \otimes I_n) \bar{g}' = B^{-1} \bar{g}'$.

Due to the form of W, Step 2 involves solving n pentadiagonal systems of size $n \times n$. Thus it requires approximately $20n^2$ real single flops, of which $10n^2$ are attributed to LU factorization and $10n^2$ to back-and-forward substitutions. Each of Steps 1 and 3 requires $2.5n^2 \log(n)$ real single flops. Hence, the FFTSC algorithm requires approximately $5n^2 \log_2(n) + 20n^2$ real single flops.

5.3 Fast Solver for the Biharmonic Dirichlet Problem

We now derive a fast algorithm for the linear system (5.5). For later convenience, we define some matrices as follows:

$$U \equiv \begin{bmatrix} 3 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 3 \end{bmatrix} \in R^{2\times n}, \quad U_4 \equiv \frac{1}{h^4} \begin{bmatrix} 15/4 & 0 \\ -1/4 & 0 \\ 0 & \vdots \\ \vdots & 0 \\ 0 & -1/4 \\ 0 & 15/4 \end{bmatrix} \in R^{n\times 2},$$

$$U_2 \equiv \frac{1}{h^2} \begin{bmatrix} 7/32 & 0 \\ -1/32 & 0 \\ 0 & \vdots \\ \vdots & 0 \\ 0 & -1/32 \\ 0 & 7/32 \end{bmatrix} \in R^{n\times 2}, \quad U_0 \equiv \begin{bmatrix} -65/1536 & 0 \\ -1/1536 & 0 \\ 0 & \vdots \\ \vdots & 0 \\ 0 & -1/1536 \\ 0 & -65/1536 \end{bmatrix} \in R^{n\times 2}.$$

It is easy to see that $Q_4 - \widetilde{Q}_4 = U_4 U$, $Q_2 - \widetilde{Q}_2 = U_2 U$ and $Q_0 - \widetilde{Q}_0 = U_0 U$. Then we have

$$A - B = (U_4 \otimes Q_0 + 2U_2 \otimes Q_2 + U_0 \otimes Q_4)(U \otimes I_n). \tag{5.10}$$

Let $S \equiv U_4 \otimes Q_0 + 2U_2 \otimes Q_2 + U_0 \otimes Q_4$, and $T \equiv U \otimes I_n$. Then, $A = B + ST$. From the *Sherman-Morrison-Woodbury formula*, assuming B and $I_{2n} + TB^{-1}S$ are non-singular, we get

$$A^{-1} = B^{-1} - B^{-1}S(I_{2n} + TB^{-1}S)^{-1}TB^{-1}. \tag{5.11}$$

Let $D \equiv I_{2n} + TB^{-1}S$ and notice that D is $2n \times 2n$. To develop a fast solver for system (5.5), we study fast solution techniques for the system with coefficient matrix D.

Let $\Gamma_1 \equiv \frac{15}{4h^4}Q_0 + \frac{7}{16h^2}Q_2 - \frac{65}{1536}Q_4$ and $\Gamma_2 \equiv \frac{1}{4h^4}Q_0 + \frac{1}{16h^2}Q_2 + \frac{1}{1536}Q_4$. Then, recalling the definition of the matrix $Q = (q_{ij})_{n\times n} \equiv (\delta_{ji})_{n\times n}$ where $\delta_{i,j}$ are defined in (5.6), we have

$$\begin{aligned} D &= I_{2n} + TB^{-1}S \\ &= I_{2n} + (UQ \otimes I_n) \begin{bmatrix} w_1^{-1} & & \\ & \ddots & \\ & & w_n^{-1} \end{bmatrix} (Q^T U_4 \otimes Q_0 + 2Q^T U_2 \otimes Q_2 + Q^T U_0 \otimes Q_4) \\ &= I_{2n} + (UQ \otimes I_n) \begin{bmatrix} w_1^{-1} & & \\ & \ddots & \\ & & w_n^{-1} \end{bmatrix} \begin{bmatrix} q_{11}\Gamma_1 - q_{21}\Gamma_2, & q_{n,1}\Gamma_1 - q_{n-1,1}\Gamma_2 \\ \vdots & \vdots \\ q_{1,i}\Gamma_1 - q_{2,i}\Gamma_2, & q_{n,i}\Gamma_1 - q_{n-1,i}\Gamma_2 \\ \vdots & \vdots \\ q_{1,n}\Gamma_1 - q_{2,n}\Gamma_2, & q_{n,n}\Gamma_1 - q_{n-1,n}\Gamma_2 \end{bmatrix} \\ &= I_{2n} + \\ & \begin{bmatrix} \sum_{j=1}^{n}(3q_{1,j} + q_{2,j})w_j^{-1}(q_{1,j}\Gamma_1 - q_{2,j}\Gamma_2) & \sum_{j=1}^{n}(3q_{1,j} + q_{2,j})w_j^{-1}(q_{n,j}\Gamma_1 - q_{n-1,j}\Gamma_2) \\ \sum_{j=1}^{n}(3q_{n,j} + q_{n-1,j})w_j^{-1}(q_{1,j}\Gamma_1 - q_{2,j}\Gamma_2) & \sum_{j=1}^{n}(3q_{n,j} + q_{n-1,j})w_j^{-1}(q_{n,j}\Gamma_1 - q_{n-1,j}\Gamma_2) \end{bmatrix}. \end{aligned}$$

Notice that $q_{i,j} = (-1)^{j+1}q_{n-i+1,j}$. Letting

$$V_1 = \sum_{j=1,3}^{n-1}(3q_{1,j} + q_{2,j})w_j^{-1}(q_{1,j}\Gamma_1 - q_{2,j}\Gamma_2), \quad V_2 = \sum_{j=2,4}^{n}(3q_{1,j} + q_{2,j})w_j^{-1}(q_{1,j}\Gamma_1 - q_{2,j}\Gamma_2),$$

we have

$$D = I_{2n} + \begin{bmatrix} V_1 + V_2 & V_1 - V_2 \\ V_1 - V_2 & V_1 + V_2 \end{bmatrix}.$$

A linear system of the form $D\gamma = \beta$, with D as above, can be decomposed into two subsystems

$$\begin{cases} T_1(\gamma_1+\gamma_2) &= \beta_1+\beta_2, \\ T_2(\gamma_1-\gamma_2) &= \beta_1-\beta_2, \end{cases}$$

where $\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$, $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$, $T_1 = I_n + 2V_1$, and $T_2 = I_n + 2V_2$. Also, D is invertible, iff T_1 and T_2 are invertible.

The two subsystems are of size $n \times n$, and can be solved by the preconditioned GMRES method. To achieve good convergence performance, finding appropriate preconditioners is essential. Now we introduce two matrices that can play the role of preconditioners.

Let $C \equiv \widetilde{Q}_4 \otimes \widetilde{Q}_0 + 2\widetilde{Q}_2 \otimes \widetilde{Q}_2 + \widetilde{Q}_0 \otimes \widetilde{Q}_4$, and $\widetilde{W} \equiv \Lambda^{(4)} \otimes \widetilde{Q}_0 + 2\Lambda^{(2)} \otimes \widetilde{Q}_2 + \Lambda^{(0)} \otimes \widetilde{Q}_4$. Obviously, $\widetilde{W}$ is a block-diagonal matrix, consisting of n pentadiagonal blocks $\widetilde{w}_i$, $i = 1, \cdots, n$, each of size $n \times n$. Assuming $\widetilde{W}$ is non-singular, we have $C^{-1} = (Q \otimes I_n)\widetilde{W}^{-1}(Q^T \otimes I_n)$. We now define a few more matrices: $\widetilde{S} \equiv U_4 \otimes \widetilde{Q}_0 + 2U_2 \otimes \widetilde{Q}_2 + U_0 \otimes \widetilde{Q}_4$, $\widetilde{\Gamma}_1 \equiv \frac{15}{4h^4}\widetilde{Q}_0 + \frac{7}{16h^2}\widetilde{Q}_2 - \frac{65}{1536}\widetilde{Q}_4$ and $\widetilde{\Gamma}_2 \equiv \frac{1}{4h^4}\widetilde{Q}_0 + \frac{1}{16h^2}\widetilde{Q}_2 + \frac{1}{1536}\widetilde{Q}_4$. Let also $\widetilde{D} \equiv I_{2n} + TC^{-1}\widetilde{S}$. Then we have

$$\begin{aligned}
\widetilde{D} &= I_{2n} + TC^{-1}\widetilde{S} \\
&= I_{2n} + (UQ \otimes I_n) \begin{bmatrix} \widetilde{w}_1^{-1} & & \\ & \ddots & \\ & & \widetilde{w}_n^{-1} \end{bmatrix} (Q^T U_4 \otimes \widetilde{Q}_0 + 2Q^T U_2 \otimes \widetilde{Q}_2 + Q^T U_0 \otimes \widetilde{Q}_4) \\
&= I_{2n} + \\
&\quad \begin{bmatrix} \sum_{j=1}^{n}(3q_{1,j}+q_{2,j})\widetilde{w}_j^{-1}(q_{1,j}\widetilde{\Gamma}_1 - q_{2,j}\widetilde{\Gamma}_2), & \sum_{j=1}^{n}(3q_{1,j}+q_{2,j})\widetilde{w}_j^{-1}(q_{n,j}\widetilde{\Gamma}_1 - q_{n-1,j}\widetilde{\Gamma}_2) \\ \sum_{j=1}^{n}(3q_{n,j}+q_{n-1,j})w_j^{-1}(q_{1,j}\widetilde{\Gamma}_1 - q_{2,j}\widetilde{\Gamma}_2), & \sum_{j=1}^{n}(3q_{n,j}+q_{n-1,j})\widetilde{w}_j^{-1}(q_{n,j}\widetilde{\Gamma}_1 - q_{n-1,j}\widetilde{\Gamma}_2) \end{bmatrix} \\
&= I_{2n} + \begin{bmatrix} \widetilde{V}_1 + \widetilde{V}_2 & \widetilde{V}_1 - \widetilde{V}_2 \\ \widetilde{V}_1 - \widetilde{V}_2 & \widetilde{V}_1 + \widetilde{V}_2 \end{bmatrix},
\end{aligned}$$

where

$$\widetilde{V}_1 = \sum_{j=1,3}^{n-1}(3q_{1,j}+q_{2,j})\widetilde{w}_j^{-1}(q_{1,j}\widetilde{\Gamma}_1 - q_{2,j}\widetilde{\Gamma}_2), \quad \widetilde{V}_2 = \sum_{j=2,4}^{n}(3q_{1,j}+q_{2,j})\widetilde{w}_j^{-1}(q_{1,j}\widetilde{\Gamma}_1 - q_{2,j}\widetilde{\Gamma}_2).$$

By using a similar idea as the one applied to D, we introduce two matrices $\widetilde{T}_1 = I_n + 2\widetilde{V}_1$, and $\widetilde{T}_2 = I_n + 2\widetilde{V}_2$. Then we have

$$\begin{aligned} Q^T\widetilde{T}_1Q = I_n + 2\sum_{j=1,3}^{n-1}&\Big\{(3q_{1,j}+q_{2,j})(\Lambda_j^{(4)}\Lambda^{(0)} + 2\Lambda_j^{(2)}\Lambda^{(2)} + \Lambda_j^{(0)}\Lambda^{(4)})^{-1} \\ \times&\Big[q_{1,j}(\frac{15}{4h^4}\Lambda^{(0)} + \frac{7}{16h^2}\Lambda^{(2)} - \frac{65}{1536}\Lambda^{(4)}) - q_{2,j}(\frac{1}{4h^4}\Lambda^{(0)} + \frac{1}{16h^2}\Lambda^{(2)} + \frac{1}{1536}\Lambda^{(4)})\Big]\Big\}, \end{aligned} \tag{5.12}$$

and

$$\begin{aligned} Q^T\widetilde{T}_2Q = I_n + 2\sum_{j=2,4}^{n}&\Big\{(3q_{1,j}+q_{2,j})(\Lambda_j^{(4)}\Lambda^{(0)} + 2\Lambda_j^{(2)}\Lambda^{(2)} + \Lambda_j^{(0)}\Lambda^{(4)})^{-1} \\ \times&\Big[q_{1,j}(\frac{15}{4h^4}\Lambda^{(0)} + \frac{7}{16h^2}\Lambda^{(2)} - \frac{65}{1536}\Lambda^{(4)}) - q_{2,j}(\frac{1}{4h^4}\Lambda^{(0)} + \frac{1}{16h^2}\Lambda^{(2)} + \frac{1}{1536}\Lambda^{(4)})\Big]\Big\}. \end{aligned} \tag{5.13}$$

Obviously, both $Q^T\widetilde{T}_1Q$ and $Q^T\widetilde{T}_2Q$ are diagonal. Therefore, a system with matrix $\widetilde{T}_r$, $r = 1, 2$, can be solved by FFT. This fact and the noticeable similarities between $\widetilde{T}_r$ and T_r, suggest that $\widetilde{T}_r$ maybe a good preconditioner of T_r. In Section 6, we study the properties of the preconditioned matrices $\widetilde{T}_r^{-1}T_r$, $r = 1, 2$. We now give the algorithm for computing $\bar{\theta} = A^{-1}\bar{g}$.

Algorithm BiHarmComp(n, $\bar{g}$)

Step 1: Solve $B\alpha = \bar{g}$ by the algorithm **FFTSC** to get $\alpha = B^{-1}\bar{g}$.

Step 2: Compute $\beta = T\alpha$. Let $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$.

Step 3: Solve $T_1(\gamma_1+\gamma_2) = \beta_1+\beta_2$ and $T_2(\gamma_1-\gamma_2) = \beta_1-\beta_2$, both by PGMRES, with $\widetilde{T}_1$ and $\widetilde{T}_2$ as preconditioners, respectively. Set $\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$.

Step 4: Compute $\zeta = S\gamma$.

Step 5: Solve $B\eta = \zeta$ by algorithm **FFTSC** to get $\eta = B^{-1}\zeta$.

Step 6: Compute $\bar{\theta} = \alpha + \eta$.

As we mentioned before, each of Steps 1 and 5 takes $5n^2\log_2(n) + 20n^2$ flops. The special structures of matrices T and S make each of Steps 2 and 4 require $O(n)$ flops. The complexity of Step 3 is $15mn^2$, where m is the number of iterations. Therefore, this algorithm requires $10n^2\log_2(n) + (40 + 15m)n^2$ real single flops. Clearly, m plays an important role in determining the overall complexity of **BiHarmComp**.

5.4 Existence and Uniqueness of Solution

It is easy to see that (5.5) is uniquely solvable, as long as B and D are invertible. Recall also that D is invertible as long as T_r, $r = 1, 2$, are invertible.

We first show that B is invertible. Let $\widetilde{B} = Q_0 \otimes \widetilde{Q}_4 + 2Q_2 \otimes \widetilde{Q}_2 + Q_4 \otimes \widetilde{Q}_0$. It is easy to see that $B = P\widetilde{B}P^T$, for some appropriate permutation matrix P. It is also easy to see that

$$\widetilde{B} - C = (U_4 \otimes \widetilde{Q}_0 + 2U_2 \otimes \widetilde{Q}_2 + U_0 \otimes \widetilde{Q}_4)(U \otimes I_n) = \widetilde{S}T, \tag{5.14}$$

from which we conclude that $\widetilde{B}$ is invertible, as long as C and $\widetilde{D}$ $(= I_{2n} + TC^{-1}\widetilde{S})$ are invertible. Similarly, as above, $\widetilde{D}$ is invertible, when $\widetilde{T}_r$, $r = 1, 2$, are invertible. It is easy to see that the eigenvalues of C are $\Lambda_j^{(4)}\Lambda_i^{(0)} + 2\Lambda_j^{(2)}\Lambda_i^{(2)} + \Lambda_j^{(0)}\Lambda_i^{(4)}$, and they are all positive, from which the invertibility of C follows. The eigenvalues of $\widetilde{T}_r$, $r = 1, 2$, can be deduced from the diagonalizations (5.12)-(5.13), and can be shown to be positive (also see equations (6.2) and (6.27)), thus showing the invertibility of $\widetilde{T}_r$, $r = 1, 2$.

The invertibility of T_r, $r = 1, 2$, can be deduced from the convergence analysis of GMRES on $\widetilde{T}_r^{-1}T_r$, $r = 1, 2$, which appears in Section 6. More specifically, in Section 6, it is shown that the eigenvalues of $\widetilde{T}_r^{-1}T_r$, $r = 1, 2$, are non-zero.

5.5 Remarks

It can be easily shown that the vector γ computed in Step 3 of **BiHarmComp** satisfies $\gamma = T\bar{\theta}$, that is, $(\gamma_1)_i = 3\bar{\theta}_i + \bar{\theta}_{i+n}$, and $(\gamma_2)_i = 3\bar{\theta}_{n^2-n+i} + \bar{\theta}_{n^2-2n+i}$, for $i = 1, \cdots, n$. Thus solving $D\gamma = \beta$ computes quantities related to the coefficients of the spline approximation corresponding to collocation points near the vertical boundary lines. Among the fast solvers for the biharmonic Dirichlet problem found in the literature, the one that is closest to **BiHarmComp** is the method in [7]. However, in [7], the discretization is obtained by standard second order finite differences, the arising matrices are symmetric and the PCG method is used for the solution of the problem related to the two opposite boundaries. It is worth noting that the linear systems $A\bar{\theta} = \bar{g}$ or, equivalently, $(B + ST)\bar{\theta} = \bar{g}$, and $D\gamma = \beta$ can be seen as Schur complement problems arising from the same augmented problem

$$\begin{bmatrix} I_{2n} & -T \\ S & B \end{bmatrix} \begin{bmatrix} \gamma \\ \bar{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{g} \end{bmatrix}.$$

In that sense, the solver proposed here resembles the methods in [30, 6] which also solve a Schur complement problem related to the two opposite boundaries. However, the methods in [30, 6] reduce the biharmonic to a coupled system of second order PDEs, and apply fourth order discretizations to it, while we apply a sixth order discretization directly to the fourth order derivatives.

Chapter 6

Complexity Analysis of the Fast Solver

In this chapter, we present a theoretical analysis of the performance of **BiHarmComp**. As we mentioned before, the number of iterations of the PGMRES method, m, is critical. Therefore we focus on analyzing the properties of matrices $\widetilde{T}_r^{-1}T_r$, $r = 1, 2$. With the analysis, it will be seen that m is independent of the gridsize n, which makes the complexity of **BiHarmComp** be $O(n^2 \log(n))$.

6.1 Introduction

First, we remind readers of a theorem which guides the analysis. The notation $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the minimum and maximum eigenvalues, respectively, of the indicated matrix, and $\lambda(\cdot)$ denotes any eigenvalue of the indicated matrix.

Theorem 6.1.1 ([39], Page 134 **and Page** 193**).** *Let $\mathcal{A}$ be a real positive definite (not necessarily symmetric) matrix. Then, the residual norm achieved at the i-th step of*

GMRES satisfies the inequality

$$\|r_i\|_2 \leq \left[1 - \frac{\lambda_{\min}^2\left(\frac{\mathcal{A}^T+\mathcal{A}}{2}\right)}{\lambda_{\max}(\mathcal{A}^T\mathcal{A})}\right]^{i/2} \|r_0\|_2 \,. \tag{6.1}$$

In **BiHarmComp**, the coefficient matrices to which GMRES is applied are $\widetilde{T}_r^{-1}T_r$, $r = 1, 2$. Recall that Q is an orthogonal matrix, therefore, it is equivalent to consider the matrices $(Q^T\widetilde{T}_r^{-1}Q)Q^TT_rQ$ instead. Let $\mathcal{A}^{(r)} = (Q^T\widetilde{T}_r^{-1}Q)Q^TT_rQ$, $r = 1, 2$. The analysis of the convergence of GMRES is on estimating bounds for the eigenvalues of $(\mathcal{A}^{(r)})^T\mathcal{A}^{(r)}$ and $\frac{(\mathcal{A}^{(r)})^T+\mathcal{A}^{(r)}}{2}$. The general idea in our analysis is as follows. We first derive expressions for all entries of the matrices under consideration, i.e., $\mathcal{A}^{(r)}$. With these expressions, we manage to find approximate formulae for the entries of $\mathcal{A}^{(r)}$. With the help of these approximations, we elicit the bounds for the eigenvalues of the needed matrices.

6.2 Matrix Representation

In this section, we focus on deriving explicit expressions for the entries in matrices $\mathcal{A}^{(r)}$. For convenience, we assume that the gridsize n is even. Please note, however, that in practice n does not have to be even. The assumption that n is even is only for the analysis purpose. From (5.12) and (5.13), it can be seen that the diagonal matrix $Q^T\widetilde{T}_rQ$ can be written as

$$Q^T\widetilde{T}_rQ \;=\; \begin{bmatrix} 1+t_1^{(r)} & & \\ & \ddots & \\ & & 1+t_n^{(r)} \end{bmatrix}, \tag{6.2}$$

where

$$t_i^{(r)} = \sum_{j=r,r+2}^{n-1} \frac{2(3q_{1,j}+q_{2,j})p_{j,i}}{\lambda_{i,j}}, \quad i = 1, 2, \cdots, n, \quad r = 1, 2, \tag{6.3}$$

$$\lambda_{i,j} \equiv \Lambda_i^{(4)}\Lambda_j^{(0)} + 2\Lambda_i^{(2)}\Lambda_j^{(2)} + \Lambda_i^{(0)}\Lambda_j^{(4)}, \quad i, j = 1, \cdots, n, \tag{6.4}$$

and

$$p_{i,j} \equiv q_{1,i}(\frac{15}{4h^4}\Lambda_j^{(0)}+\frac{7}{16h^2}\Lambda_j^{(2)}-\frac{65}{1536}\Lambda_j^{(4)})-q_{2,i}(\frac{1}{4h^4}\Lambda_j^{(0)}+\frac{1}{16h^2}\Lambda_j^{(2)}+\frac{1}{1536}\Lambda_j^{(4)}), \quad i,j=1,\cdots,n. \tag{6.5}$$

Let $P_n \in R^{n\times n}$ be the permutation matrix such that if $\mathcal{D} \in R^{(n/2)\times(n/2)}$, $\mathcal{E} \in R^{2\times 2}$ then

$$P_n(\mathcal{D}\otimes\mathcal{E})P_n^T = \mathcal{E}\otimes\mathcal{D}.$$

Theorem 6.2.1. *The matrix $\mathcal{A}^{(r)}$ is block-diagonalized as*

$$P_n\mathcal{A}^{(r)}P_n^T = I_n - \Psi^{(r)} \equiv I_n - \begin{bmatrix} \Psi^{(r,1)} & 0 \\ 0 & \Psi^{(r,2)} \end{bmatrix}, \tag{6.6}$$

for $r=1,2$. Moreover, if $j_r = 2(j-1)+r$, $k_r = 2(k-1)+r$, and $l_r = 2(l-1)+r$, then $\Psi^{(r,s)}$ is an $n/2\times n/2$ matrix with elements

$$\begin{aligned}
\psi_{k,l}^{(r,s)} \quad = \quad & \frac{2(3q_{1,l_s}+q_{2,l_s})}{t_{k_s}^{(r)}+1} \\
& \times\frac{q_{1,k_s}}{1536}\sum_{j=1}^{n/2}\frac{2(3q_{1,j_r}+q_{2,j_r})}{\lambda_{j_r,k_s}}\left[(349+17\cos\frac{k_s\pi}{n})q_{1,j_r}+(5-7\cos\frac{k_s\pi}{n})q_{2,j_r}\right] \\
& +\frac{2(3q_{1,l_s}+q_{2,l_s})}{t_{k_s}^{(r)}+1}\times\sum_{j=1}^{n/2}\frac{2(3q_{1,j_r}+q_{2,j_r})}{t_{j_r}^{(s)}+1}\times\frac{p_{k_s,j_r}}{\lambda_{j_r,k_s}}\times\frac{p_{j_r,l_s}}{\lambda_{j_r,l_s}} \\
& -\frac{2(3q_{1,l_s}+q_{2,l_s})}{t_{k_s}^{(r)}+1}\sum_{j=1}^{n/2}\frac{2(3q_{1,j_r}+q_{2,j_r})}{t_{j_r}^{(s)}+1}\times\frac{p_{k_s,j_r}}{\lambda_{j_r,k_s}} \\
& \times\left\{\frac{q_{1,j_r}}{1536}\left[(349+17\cos\frac{j_r\pi}{n})\xi_{j_r}^{(s)}+(5-7\cos\frac{j_r\pi}{n})\tau_{j_r}^{(s)}\right]\right\},
\end{aligned} \tag{6.7}$$

where

$$\xi_j^{(s)} = \sum_{i=s,s+2}^{n}\frac{2(3q_{1,i}+q_{2,i})q_{1,i}}{\lambda_{i,j}} = \sum_{i=s,s+2}^{n}\frac{2\kappa_i^2(1-\cos\frac{i\pi}{n})(2+\cos\frac{i\pi}{n})}{\lambda_{i,j}}, \tag{6.8}$$

$$\tau_j^{(s)} = \sum_{i=s,s+2}^{n}\frac{2(3q_{1,i}+q_{2,i})q_{2,i}}{\lambda_{i,j}} = \sum_{i=s,s+2}^{n}\frac{2\kappa_i^2(1-\cos\frac{i\pi}{n})(2+\cos\frac{i\pi}{n})(1+2\cos\frac{i\pi}{n})}{\lambda_{i,j}}. \tag{6.9}$$

Proof. We consider the definitions of T_r and $\widetilde{T}_r$, and apply the *Sherman-Morrison-Woodbury formula* to them. The calculations are straightforward but lengthy, so we omit them. $\square$

For later convenience, we let

$$
\begin{aligned}
\rho_1^{(r,s)}(j,k,l,n) &= \frac{2(3q_{1,l_s}+q_{2,l_s})}{t_{k_s}^{(r)}+1} \times \frac{q_{1,k_s}}{1536} \times \frac{2(3q_{1,j_r}+q_{2,j_r})}{\lambda_{j_r,k_s}} \\
&\quad \times \left[(349+17\cos\frac{k_s\pi}{n})q_{1,j_r} + (5-7\cos\frac{k_s\pi}{n})q_{2,j_r}\right], \qquad (6.10) \\
\rho_2^{(r,s)}(j,k,l,n) &= \frac{2(3q_{1,l_s}+q_{2,l_s})}{t_{k_s}^{(r)}+1} \times \frac{2(3q_{1,j_r}+q_{2,j_r})}{t_{j_r}^{(s)}+1} \times \frac{p_{k_s,j_r}}{\lambda_{j_r,k_s}} \times \frac{p_{j_r,l_s}}{\lambda_{j_r,l_s}}, \qquad (6.11) \\
\rho_3^{(r,s)}(j,k,l,n) &= -\frac{2(3q_{1,l_s}+q_{2,l_s})}{t_{k_s}^{(r)}+1}\frac{2(3q_{1,j_r}+q_{2,j_r})}{t_{j_r}^{(s)}+1} \times \frac{p_{k_s,j_r}}{\lambda_{j_r,k_s}} \\
&\quad \times \left\{\frac{q_{1,j_r}}{1536}\left[(349+17\cos\frac{j_r\pi}{n})\xi_{j_r}^{(s)} + (5-7\cos\frac{j_r\pi}{n})\tau_{j_r}^{(s)}\right]\right\}. \qquad (6.12)
\end{aligned}
$$

From (6.7)-(6.12), we have

$$
\psi_{k,l}^{(r,s)} = \sum_{j=1}^{n/2}[\rho_1^{(r,s)}(j,k,l,n) + \rho_2^{(r,s)}(j,k,l,n) + \rho_3^{(r,s)}(j,k,l,n)]. \qquad (6.13)
$$

6.3 Matrix Approximation

In this section, we find convenient matrices to approximate $\Psi^{(1)}$ and $\Psi^{(2)}$. By "convenient" we mean that the eigenvalue information of the matrices is easy to obtain. The approximate matrices help analyze the properties of the original matrices.

We start our journey with finding approximations to $t_j^{(1)}$ and $t_j^{(2)}$. To do that, we have to derive the closed forms of $\xi_j^{(s)}$ and $\tau_j^{(s)}$ first. Recalling (6.4), we have

$$
\begin{aligned}
\lambda_{i,j} &= \frac{7}{12}\left(\left(\cos\frac{j\pi}{n}+\frac{15+2\sqrt{37}}{7}\right)\cos\frac{i\pi}{n}+\frac{15+2\sqrt{37}}{77}(11\cos\frac{j\pi}{n}-37+2\sqrt{37})\right) \\
&\quad \left(\left(\cos\frac{j\pi}{n}+\frac{15-2\sqrt{37}}{7}\right)\cos\frac{i\pi}{n}+\frac{15-2\sqrt{37}}{77}(11\cos\frac{j\pi}{n}-37-2\sqrt{37})\right).
\end{aligned}
\qquad (6.14)
$$

Using (6.14), it is easy to see that, when $\cos\frac{j\pi}{n} \neq \frac{2\sqrt{37}-15}{7}$, we have

$$\begin{aligned}\frac{(1-\cos\frac{i\pi}{n})(2+\cos\frac{i\pi}{n})}{\lambda_{i,j}} &= -\frac{12}{7\cos^2\frac{j\pi}{n}+30\cos\frac{j\pi}{n}+11} \\ &\quad + \frac{\mu_j}{(7\cos\frac{j\pi}{n}+15+2\sqrt{37})\cos\frac{i\pi}{n}+(15+2\sqrt{37})(\cos\frac{j\pi}{n}-\frac{37-2\sqrt{37}}{11})} \\ &\quad + \frac{\nu_j}{(7\cos\frac{j\pi}{n}+15-2\sqrt{37})\cos\frac{i\pi}{n}+(15-2\sqrt{37})(\cos\frac{j\pi}{n}-\frac{37+2\sqrt{37}}{11})},\end{aligned} \tag{6.15}$$

where

$$\begin{aligned}\mu_j &= \frac{6\sqrt{37}(85\cos\frac{j\pi}{n}+23\sqrt{37}\cos\frac{j\pi}{n}-1033-155\sqrt{37})}{37(\cos\frac{j\pi}{n}-1)(7\cos\frac{j\pi}{n}+15+2\sqrt{37})}, \\ \nu_j &= \frac{6\sqrt{37}(-85\cos\frac{j\pi}{n}+23\sqrt{37}\cos\frac{j\pi}{n}+1033-155\sqrt{37})}{37(\cos\frac{j\pi}{n}-1)(7\cos\frac{j\pi}{n}+15-2\sqrt{37})}.\end{aligned}$$

Let

$$\begin{aligned}a_j &= 7\cos\frac{j\pi}{n}+15+2\sqrt{37}, \quad b_j = (15+2\sqrt{37})(\cos\frac{j\pi}{n}-\frac{37-2\sqrt{37}}{11}) \\ c_j &= 7\cos\frac{j\pi}{n}+15-2\sqrt{37}, \quad d_j = (15-2\sqrt{37})(\cos\frac{j\pi}{n}-\frac{37+2\sqrt{37}}{11}).\end{aligned}$$

It is trivial to see that

$$\begin{cases} -1 < \frac{a_j}{b_j} < 0 & \text{when} \quad i \in [1,\ldots,n-1], \text{ i.e. } \cos\frac{j\pi}{n} \in (-1,1), \\ -1 < \frac{c_j}{d_j} < 0 & \text{when} \quad \cos\frac{j\pi}{n} \in (\frac{2\sqrt{37}-15}{7}, 1), \\ c_j = 0 & \text{when} \quad \cos\frac{j\pi}{n} = \frac{2\sqrt{37}-15}{7}, \\ 0 < \frac{c_j}{d_j} < 1 & \text{when} \quad \cos\frac{j\pi}{n} \in (-1, \frac{2\sqrt{37}-15}{7}). \end{cases}$$

Therefore, when $\cos\frac{j\pi}{n} \neq 1$ and $\cos\frac{j\pi}{n} \neq \frac{2\sqrt{37}-15}{7}$, substituting (6.15) into (6.8), we have

$$\xi_j^{(s)} = 2\mu_j \sum_{i=s,s+2}^{n} \frac{\kappa_i^2}{a_j\cos\frac{i\pi}{n}+b_j} + 2\nu_j \sum_{i=s,s+2}^{n} \frac{\kappa_i^2}{c_j\cos\frac{i\pi}{n}+d_j} - \frac{24}{11+30\cos\frac{j\pi}{n}+7\cos^2\frac{j\pi}{n}} \sum_{i=s,s+2}^{n} \kappa_i^2. \tag{6.16}$$

Notice that $\xi_j^{(s)}$ has summation terms involving $\cos\frac{i\pi}{n}$. To derive the closed form of $\xi_j^{(s)}$, we need some formulae to calculate those summation terms. The formulae we need can be derived by the Poisson Summation Formula, which, in turn, can be obtained from Fourier's theorem [5].

Lemma 6.3.1 (Fourier's theorem). *Let $f(x)$ be a function which is piecewise continuous on $[-L, L]$. Its Fourier series is given by*

$$\frac{f(x^-)+f(x^+)}{2} = \frac{1}{2L}\int_{-L}^{L} f(t)dt + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(t)\cos\frac{n\pi(t-x)}{L}dt$$

at each point $-L < x < L$ where the one sided derivatives $f'_R(x)$ and $f'_L(x)$ both exist.

Suppose $f(x)$ also has definitions on the domains $[L, 3L], \ldots, [(2m-1)L, (2m+1)L]$, $m \in Z^+$, then for $k = 1, \cdots, m$, we have

$$\begin{aligned}
&\frac{f(2kL+x^-)+f(2kL+x^+)}{2} \\
&= \frac{1}{2L}\int_{-L}^{L} f(2kL+t)dt + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(2kL+t)\cos\frac{n\pi(t-x)}{L}dt \\
&= \frac{1}{2L}\int_{(2k-1)L}^{(2k+1)L} f(t)dt + \frac{1}{L}\sum_{n=1}^{\infty}\int_{(2k-1)L}^{(2k+1)L} f(t)\cos\frac{n\pi(t-2kL-x)}{L}dt \\
&= \frac{1}{2L}\int_{(2k-1)L}^{(2k+1)L} f(t)dt + \frac{1}{L}\sum_{n=1}^{\infty}\int_{(2k-1)L}^{(2k+1)L} f(t)\cos\frac{n\pi(t-x)}{L}dt.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\sum_{k=0}^{m}\frac{f(2kL+x^-)+f(2kL+x^+)}{2} \\
&= \frac{1}{2L}\int_{-L}^{(2m+1)L} f(t)dt + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{(2m+1)L} f(t)\cos\frac{n\pi(t-x)}{L}dt.
\end{aligned} \tag{6.17}$$

Next we give the Poisson Summation Formula which is slightly different from what can be found in a basic analysis textbook [23]. We tailor it to our problem's need.

Theorem 6.3.2 (Poisson Summation Formula). *Given a continuous function $f(x)$ defined on domain $[0, a]$, we have*

$$\frac{1}{2}f(0)+\sum_{k=1}^{m} f(\frac{ka}{m+1})+\frac{1}{2}f(a) = \frac{m+1}{a}\left[\int_0^a f(t)dt + 2\sum_{n=1}^{\infty}\int_0^a f(t)\cos\frac{2n\pi(m+1)t}{a}dt\right]. \tag{6.18}$$

Proof. We construct a function $p(x)$ defined by

$$p(x) = \begin{cases} f(x+a), & x < 0 \\ f(x), & x \geq 0 \end{cases}.$$

For $x = 0$, we have

$$\sum_{k=0}^{m} \frac{p(\frac{ka}{m+1} + 0^-) + p(\frac{ka}{m+1} + 0^+)}{2} = \frac{1}{2}f(0) + \frac{1}{2}f(a) + \sum_{k=1}^{m} f(\frac{ka}{m+1}), \tag{6.19}$$

and for $L = \frac{a}{2(m+1)}$, we have

$$\int_{-L}^{a-L} p(t)dt = \int_0^a p(t)dt = \int_0^a f(t)dt, \tag{6.20}$$

$$\begin{aligned} \int_{-L}^{a-L} p(t)\cos\frac{2n\pi(m+1)(t-x)}{a}dt &= \int_0^a p(t)\cos\frac{2n\pi(m+1)(t-x)}{a}dt \\ &= \int_0^a f(t)\cos\frac{2n\pi(m+1)(t-x)}{a}dt. \end{aligned} \tag{6.21}$$

Relations (6.19), (6.17), (6.20) and (6.21), lead to (6.18). □

Using Theorem 6.3.2, we obtain the following theorem and corollaries.

Theorem 6.3.3. *For* $0 < a^2 < 1$,

$$\frac{1}{2(a+1)} + \sum_{i=1}^{n-1} \frac{1}{a\cos\frac{i\pi}{n} + 1} + \frac{1}{2(1-a)} = \frac{n}{\sqrt{1-a^2}} \frac{a^{2n} + (1-\sqrt{1-a^2})^{2n}}{a^{2n} - (1-\sqrt{1-a^2})^{2n}}.$$

Proof. We define a function $f(x) = \dfrac{1}{a\cos x + 1}$. Theorem 6.3.2 gives the relation

$$\frac{1}{2}f(0) + \sum_{i=1}^{n-1} f(\frac{i\pi}{n}) + \frac{1}{2}f(\pi) = \frac{n}{\pi}\left(\int_0^\pi f(t)dt + 2\sum_{k=1}^{\infty}\int_0^\pi f(t)\cos(2knt)dt\right).$$

In addition, the following integrals can be found in [22] (Page 409):

$$\begin{aligned} \int_0^\pi f(x)dx &= \int_0^\pi \frac{dx}{a\cos x + 1} = \frac{\pi}{\sqrt{1-a^2}}, \\ \int_0^\pi f(x)\cos(2knx)dx &= \int_0^\pi \frac{\cos(2knx)dx}{a\cos x + 1} = \frac{\pi}{\sqrt{1-a^2}}\left(\frac{1-\sqrt{1-a^2}}{a}\right)^{2kn}. \end{aligned}$$

The derivation of the desired result is now trivial. □

The following corollary is a straightforward consequence of Theorem 6.3.3.

Corollary 6.3.4. *For* $0 < |a| < |b|$,

$$\frac{1}{2(a+b)} + \sum_{i=1}^{n-1} \frac{1}{a\cos\frac{i\pi}{n} + b} + \frac{1}{2(b-a)} = \frac{|b|}{b}\frac{n}{\sqrt{b^2-a^2}} \frac{a^{2n} + (|b|-\sqrt{b^2-a^2})^{2n}}{a^{2n} - (|b|-\sqrt{b^2-a^2})^{2n}}.$$

We next give two corollaries that are derived from Corollary 6.3.4.

Corollary 6.3.5. *For* $0 < |a| < |b|$,

$$\frac{1}{2(a+b)} + \sum_{i=2,4}^{n-1} \frac{1}{a\cos\frac{i\pi}{n} + b} + \frac{1}{2(b-a)} = \frac{|b|}{b}\frac{n}{2\sqrt{b^2-a^2}}\frac{|a|^n + (|b| - \sqrt{b^2-a^2})^n}{|a|^n - (|b| - \sqrt{b^2-a^2})^n}.$$

Corollary 6.3.6. *For* $0 < |a| < |b|$,

$$\sum_{i=1,3}^{n-1} \frac{1}{a\cos\frac{i\pi}{n} + b} = \frac{|b|}{b}\frac{n}{2\sqrt{b^2-a^2}}\frac{|a|^n - (|b| - \sqrt{b^2-a^2})^n}{|a|^n + (|b| - \sqrt{b^2-a^2})^n}.$$

Applying Corollaries 6.3.5 and 6.3.6 in (6.16), we derive the following closed form expression for $\xi_j^{(s)}$:

$$\begin{aligned}\xi_j^{(s)} &= -\frac{24}{11 + 30\cos\frac{j\pi}{n} + 7\cos^2\frac{j\pi}{n}} \\ &\quad -\frac{2\mu_j}{\sqrt{b_j^2 - a_j^2}} \times \frac{a_j^n + (-1)^s(-b_j - \sqrt{b_j^2 - a_j^2})^n}{a_j^n - (-1)^s(-b_j - \sqrt{b_j^2 - a_j^2})^n} \\ &\quad -\frac{2\nu_j}{\sqrt{d_j^2 - c_j^2}} \times \frac{|c_j^n| + (-1)^s(-d_j - \sqrt{d_j^2 - c_j^2})^n}{|c_j^n| - (-1)^s(-d_j - \sqrt{d_j^2 - c_j^2})^n}.\end{aligned} \tag{6.22}$$

Similarly, we have the following closed form expression for $\tau_j^{(s)}$:

$$\begin{aligned}\tau_j^{(s)} &= \frac{24(39\cos^2\frac{j\pi}{n} - 194\cos\frac{j\pi}{n} - 181)}{(7\cos^2\frac{j\pi}{n} + 30\cos\frac{j\pi}{n} + 11)^2} \\ &\quad -\left(1 - \frac{2b_j}{a_j}\right)\frac{2\mu_j}{\sqrt{b_j^2 - a_j^2}} \times \frac{a_j^n + (-1)^s(-b_j - \sqrt{b_j^2 - a_j^2})^n}{a_j^n - (-1)^s(-b_j - \sqrt{b_j^2 - a_j^2})^n} \\ &\quad -\left(1 - \frac{2d_j}{c_j}\right)\frac{2\nu_j}{\sqrt{d_j^2 - c_j^2}} \times \frac{|c_j^n| + (-1)^s(-d_j - \sqrt{d_j^2 - c_j^2})^n}{|c_j^n| - (-1)^s(-d_j - \sqrt{d_j^2 - c_j^2})^n}.\end{aligned} \tag{6.23}$$

In the above derivation procedure, we have assumed that $\cos\frac{j\pi}{n} \neq \frac{2\sqrt{37}-15}{7}$. However, it is trivial to verify that the formulae (6.22) and (6.23) hold even when $\cos\frac{j\pi}{n} = \frac{2\sqrt{37}-15}{7}$. The next step is to derive Taylor's expansions for $\xi_j^{(s)}$ and $\tau_j^{(s)}$. Because of the extraordinary complexity of the closed form expressions of $\xi_j^{(s)}$ and $\tau_j^{(s)}$, it is not practicable to elicit the Taylor's expansions directly. Therefore, we use a "divide and conquer" strategy: we calculate the expansion for every term in $\xi_j^{(s)}$ and $\tau_j^{(s)}$, then combine these expansions together and give the Taylor's expansions of $\xi_j^{(s)}$ and $\tau_j^{(s)}$. Maple is employed in this process, the details of which, although straightforward, are tedious, therefore omitted. Here we include only some of the results. The Taylor's expansions for $\xi_j^{(s)}$ and $\tau_j^{(s)}$ are given by

$$\begin{aligned}
\xi_j^{(s)} &= \frac{3(1+(-1)^s e^{-j\pi})}{4(1-(-1)^s e^{-j\pi})}\left(\frac{n}{j\pi}\right) - (-1)^s \frac{3ne^{-j\pi}}{2(1+e^{-j\pi})^2} - \frac{1}{2} \\
&\quad + \frac{6j^2\pi^2 e^{-j\pi}(e^{-j\pi}+(-1)^s) + 78j\pi e^{-j\pi}(e^{-j\pi}-(-1)^s)}{128(1-(-1)^s e^{-j\pi})^3}\left(\frac{j\pi}{n}\right) \qquad (6.24) \\
&\quad + \frac{49(1-(-1)^s e^{-j\pi})(1-e^{-2j\pi})}{128(1-(-1)^s e^{-j\pi})^3}\left(\frac{j\pi}{n}\right) + O\left(\frac{1}{n^2}\right). \\
\tau_j^{(s)} &= \frac{9(1+(-1)^s e^{-j\pi})}{4(1-(-1)^s e^{-j\pi})}\left(\frac{n}{j\pi}\right) - (-1)^s \frac{9ne^{-j\pi}}{2(1+e^{-j\pi})^2} - \frac{7}{2} \\
&\quad + (-1)^s \frac{3[6j^2\pi^2 e^{-j\pi}(1+(-1)^s e^{-j\pi}) - 142j\pi e^{-j\pi}(1-(-1)^s e^{-j\pi})]}{128(1-(-1)^s e^{-j\pi})^3}\left(\frac{j\pi}{n}\right) \\
&\quad + \frac{435(1-(-1)^s e^{-j\pi})(1-e^{-2j\pi})}{128(1-(-1)^s e^{-j\pi})^3}\left(\frac{j\pi}{n}\right) + O\left(\frac{1}{n^2}\right). \qquad (6.25)
\end{aligned}$$

From (6.3), using (6.5), (6.8), (6.9), (5.8), and Taylor's expansion, we have

$$\begin{aligned}
t_j^{(s)} &= \left(\frac{15}{4h^4}\Lambda_j^{(0)} + \frac{7}{16h^2}\Lambda_j^{(2)} - \frac{65}{1536}\Lambda_j^{(4)}\right)\xi_j^{(s)} - \left(\frac{1}{4h^4}\Lambda_j^{(0)} + \frac{1}{16h^2}\Lambda_j^{(2)} + \frac{1}{1536}\Lambda_j^{(4)}\right)\tau_j^{(s)} \\
&= \frac{269+434\cos\frac{j\pi}{n}+17\cos^2\frac{j\pi}{n}}{192}\xi_j^{(s)} - \frac{11+30\cos\frac{j\pi}{n}+7\cos^2\frac{j\pi}{n}}{192}\tau_j^{(s)} \\
&= \left(\frac{15}{4} - \frac{39j^2\pi^2}{32n^2} + O\left(\frac{1}{n^4}\right)\right)\xi_j^{(s)} - \left(\frac{1}{4} - \frac{11j^2\pi^2}{96n^2} + O\left(\frac{1}{n^4}\right)\right)\tau_j^{(s)}. \qquad (6.26)
\end{aligned}$$

Substituting (6.24) and (6.25) into (6.26), we get

$$\begin{aligned}
t_j^{(s)} &= \frac{9(1+(-1)^s e^{-j\pi})}{4(1-(-1)^s e^{-j\pi})}\left(\frac{n}{j\pi}\right) - (-1)^s \frac{9ne^{-j\pi}}{2(1-(-1)^s e^{-j\pi})^2} - 1 \\
&\quad + \frac{9(-1)^s[2j^2\pi^2 e^{-j\pi}(1+(-1)^s e^{-j\pi})]}{128[1-(-1)^s e^{-j\pi}]^3} \times \left(\frac{j\pi}{n}\right) \qquad (6.27) \\
&\quad - \frac{9[1+(-1)^s 2j\pi e^{-j\pi} - e^{-2j\pi}][1-(-1)^s e^{-j\pi}]}{128[1-(-1)^s e^{-j\pi}]^3} \times \left(\frac{j\pi}{n}\right) + O\left(\frac{1}{n^2}\right).
\end{aligned}$$

Using (6.27), we can show that

$$\begin{aligned}
\frac{1}{t_j^{(s)}+1} &= \frac{4(1-(-1)^s e^{-j\pi})^2}{9(1-(-1)^s 2j\pi e^{-j\pi}-e^{-2j\pi})}\frac{j\pi}{n}+\frac{1-(-1)^s e^{-j\pi}}{72(1-(-1)^s 2j\pi e^{-j\pi}-e^{-2j\pi})^2}\frac{j^3\pi^3}{n^3} \\
&\times[(1+(-1)^s 2j\pi e^{-j\pi}-e^{-2j\pi})(1-(-1)^s e^{-j\pi}) \qquad (6.28)\\
&-(-1)^s 2j^2\pi^2 e^{-j\pi}(1+(-1)^s e^{-j\pi})]+O(\frac{1}{n^4}).
\end{aligned}$$

We now consider $\psi_{k,l}^{(r,s)}$ as given by (6.13). Using (6.28), (6.5), (6.14), and Maple, we get

$$\rho_1^{(r,s)}(j,k,l,n) = O(\frac{1}{n^3}), \quad \rho_3^{(r,s)}(j,k,l,n) = O(\frac{1}{n^3}), \tag{6.29}$$

$$\rho_2^{(r,s)}(j,k,l,n) = f^{(r,s)}(j_r,k_s,l_s)+\hat{f}^{(r,s)}(j_r,k_s,l_s,n)+O(\frac{1}{n^3}), \tag{6.30}$$

where $j_r = 2(j-1)+r$, $k_r = 2(k-1)+r$, $l_r = 2(l-1)+r$, and the functions $f^{(r,s)}(j,k,l)$ and $\hat{f}^{(r,s)}(j,k,l,n)$ are defined by

$$f^{(r,s)}(j,k,l) = \frac{16\kappa_j^2\kappa_k\kappa_l n^2}{\pi^2}\times\frac{j^3k^2l}{(k^2+j^2)^2(l^2+j^2)^2}\times\frac{(a_j^{(s)}a_k^{(r)})^2}{b_j^{(s)}b_k^{(r)}}, \tag{6.31}$$

$$\begin{aligned}
\hat{f}^{(r,s)}(j,k,l,n) &= -4\kappa_j^2\kappa_k\kappa_l n^2\times\frac{j^3k^2l}{(k^2+j^2)^2(l^2+j^2)^2}\times\frac{(a_j^{(s)}a_k^{(r)})^2}{b_j^{(s)}b_k^{(r)}} \\
&\times\left[\frac{2l^2}{n^2}+\frac{11j^2}{6n^2}-\frac{k^2}{6n^2}-\frac{4j^4+11j^2k^2+4k^4}{6(k^2+j^2)n^2}-\frac{4j^4+11j^2l^2+4l^4}{6(j^2+l^2)n^2}\right. \\
&\left.-\frac{k^2}{8n^2}\times\frac{c_k^{(r)}}{a_k^{(r)}b_k^{(r)}}-\frac{j^2}{8n^2}\times\frac{c_j^{(s)}}{a_j^{(s)}b_j^{(s)}}\right], \qquad (6.32)
\end{aligned}$$

and $a_k^{(s)}$, $b_k^{(s)}$, and $c_k^{(s)}$ are given by

$$\begin{aligned}
a_k^{(s)} &= 1+(-1)^{s-1}e^{-k\pi}, \quad b_k^{(s)} = 1+2(-1)^{s-1}k\pi e^{-k\pi}-e^{-2k\pi}, \qquad (6.33)\\
c_k^{(s)} &= 2(-1)^{s-1}k^2\pi^2e^{-k\pi}[1+(-1)^s e^{-k\pi}]+[1+2(-1)^s k\pi e^{-k\pi}-e^{-2k\pi}][1-(-1)^s e^{-k\pi}].
\end{aligned}$$

We emphasize that, taking into account (5.7), the term $\kappa_j^2\kappa_k\kappa_l n^2$, which appears in relations (6.31)-(6.32), does not depend on n. It is worth noting that $a_k^{(s)}$, $b_k^{(s)}$, and $c_k^{(s)}$ are all exponentially close to one in k, and $f^{(r,s)}(j,k,l)$ are positive. Taking into account relations (6.29)-(6.30), relation (6.13) results in the following theorem.

Theorem 6.3.7. *For $k, l \in Z^+$, and for $r = 1, 2,\ s = 1, 2,$*

$$\psi_{k,l}^{(r,s)} = \sum_{j=1}^{n/2} \left[f^{(r,s)}(j_r, k_s, l_s) + \hat{f}^{(r,s)}(j_r, k_s, l_s, n) + O(\frac{1}{n^3}) \right]. \tag{6.34}$$

This theorem indicates that, as the order n becomes large, the entries of the matrix $\Psi^{(r,s)}$ converge to the respective ones of a limit matrix $\Phi_\infty^{(r,s)}$ defined by

$$\varphi_{k,l}^{(r,s)} = \sum_{j=1}^{\infty} \left[f^{(r,s)}(j_r, k_s, l_s) + \hat{f}^{(r,s)}(j_r, k_s, l_s, \infty) \right]. \tag{6.35}$$

To illustrate this theorem, we compare the 3×3 leading principal submatrix of the limit matrix $\Phi_\infty^{(r,s)}$ (here $N = 2048$ is used to approximate infinity) with the corresponding submatrix of $\Psi_{64}^{(r,s)}$ for $n = 64$ in Table 6.1. It can be seen that the approximation is already quite good for $n = 64$.

	0.3147	0.0530	0.0182		0.3147	0.0534	0.0186
$\Psi_{64}^{(1,1)}$	0.1872	0.0849	0.0466	$\Phi_\infty^{(1,1)}$	0.1865	0.0852	0.0474
	0.1096	0.0795	0.0531		0.1083	0.0791	0.0536
	0.1630	0.0571	0.0279		0.1632	0.0578	0.0287
$\Psi_{64}^{(1,2)}$	0.1184	0.0678	0.0415	$\Phi_\infty^{(1,2)}$	0.1178	0.0682	0.0424
	0.0890	0.0639	0.0446		0.0876	0.0636	0.0452
	0.1100	0.0566	0.0230		0.1101	0.0570	0.0235
$\Psi_{64}^{(2,1)}$	0.1365	0.0914	0.0483	$\Phi_\infty^{(2,1)}$	0.1360	0.0917	0.0492
	0.0944	0.0822	0.0536		0.0933	0.0818	0.0541
	0.1330	0.0582	0.0288		0.1332	0.0588	0.0296
$\Psi_{64}^{(2,2)}$	0.1159	0.0683	0.0416	$\Phi_\infty^{(2,2)}$	0.1154	0.0687	0.0425
	0.0885	0.0641	0.0446		0.0871	0.0638	0.0452

Table 6.1: Leading principal submatrix of $\Psi_{64}^{(r,s)}$ and $\Phi_\infty^{(r,s)}$

6.4 Estimation of the Bounds for Eigenvalues

In the last section, we derived approximate expressions for the entries of $\Psi^{(r,s)}$, and therefore for those of $\Psi^{(r)}$. In this section, we discuss some properties of the functions $f^{(r,s)}(j,k,l)$ and $\hat{f}^{(r,s)}(j,k,l,n)$, which can be used to obtain bounds on the column and row norms of $\Psi^{(r,s)}$, and consequently of $\Psi^{(r)}$. From these, we elicit an upper bound for the eigenvalues of $(\mathcal{A}^{(r)})^T\mathcal{A}^{(r)}$ and a lower bound for the eigenvalues of $\frac{(\mathcal{A}^{(r)})^T+\mathcal{A}^{(r)}}{2}$.

To obtain bounds on the column and row norms of $\Psi^{(r,s)}$, we study the quantities $\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} f^{(r,s)}(j_r,k_s,l_s)$, $\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \hat{f}^{(r,s)}(j_r,k_s,l_s,\infty)$, $\sum_{l=1}^{\infty}\sum_{j=1}^{\infty} f^{(r,s)}(j_r,k_s,l_s)$, and $\sum_{l=1}^{\infty}\sum_{j=1}^{\infty} \hat{f}^{(r,s)}(j_r,k_s,l_s,\infty)$, where $f^{(r,s)}(j_r,k_s,l_s)$ and $\hat{f}^{(r,s)}(j_r,k_s,l_s,n)$ are given by (6.31)-(6.32). Due to the complicated form of the above sums, we first consider the quantities

$$F_1^{(r,s)}(l) = \sum_{k=s,s+2}^{\infty}\sum_{j=r,r+2}^{\infty} \frac{j^3k^2l}{(k^2+j^2)^2(l^2+j^2)^2}, \qquad (6.36)$$

$$F_2^{(r,s)}(l) = \sum_{k=s,s+2}^{\infty}\sum_{j=r,r+2}^{\infty} \frac{j^3l^2k}{(k^2+j^2)^2(l^2+j^2)^2}, \qquad (6.37)$$

where $r=1,2$, $s=1,2$, and l is a positive integer, and notice that a quantity similar to $F_1^{(r,s)}(l)$ appears in $\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} f^{(r,s)}(j_r,k_s,l_s)$, while a quantity similar to $F_2^{(r,s)}(l)$ appears in $\sum_{l=1}^{\infty}\sum_{j=1}^{\infty} f^{(r,s)}(j_r,k_s,l_s)$.

Lemma 6.4.1. *For $l\in\mathbb{Z}^+$, and for $r=1,2$, $s=1,2$,*

$$\begin{aligned}
\frac{\pi^2}{64}-\frac{1}{l}\frac{\pi}{32}-\frac{1}{l}\frac{1}{16}-\frac{1}{l^2}\frac{3\sqrt{3}}{64} &\le F_1^{(r,s)}(l) \le \frac{\pi^2}{64}+\frac{1}{l}\frac{\pi}{32}+\frac{1}{l}\frac{1}{16}+\frac{1}{l^2}\frac{3\sqrt{3}}{64},\\
\frac{1}{16}-\frac{1}{l}\frac{3\sqrt{3}}{64}-\frac{1}{l}\frac{3\sqrt{3}\pi}{128}-\frac{1}{l^2}\frac{3\sqrt{3}}{16} &\le F_2^{(r,s)}(l) \le \frac{1}{16}+\frac{1}{l}\frac{3\sqrt{3}}{64}+\frac{1}{l}\frac{3\sqrt{3}\pi}{128}+\frac{1}{l^2}\frac{3\sqrt{3}}{16}.
\end{aligned}$$

Proof. Consider $F_1^{(1,1)}(l)$. We have

$$F_1^{(1,1)}(l) = \sum_{k=1,3}^{\infty}\sum_{j=1,3}^{\infty}\frac{j^3k^2l}{(k^2+j^2)^2(l^2+j^2)^2} = \frac{1}{l}\sum_{j=1,3}^{\infty}\frac{1}{l}\sum_{k=1,3}^{\infty}\frac{(j/l)^3(k/l)^2}{((k/l)^2+(j/l)^2)^2(1+(j/l)^2)^2}.$$

Let

$$f(x,y) = \frac{x^3y^2}{(x^2+y^2)^2(1+x^2)^2}.$$

Clearly,

$$\lim_{l\to\infty}\frac{1}{l}\sum_{k=1,3}^{\infty}\frac{(j/l)^3(k/l)^2}{((k/l)^2+(j/l)^2)^2(1+(j/l)^2)^2}=\frac{1}{2}\int_0^{\infty}f(x,y)dy=\frac{\pi x^2}{8(1+x^2)^2},$$

where $x=\frac{j}{l}$. Given $x=\frac{j}{l}$, $f(x,y)$ gets its maximum $f_{\max}(x,y)=\frac{x}{4(1+x^2)^2}$ when $y=x$, and $f(x,y)$ is monotone on each side of its maximizer. By considering the discrete sum for finite l and the above facts, it follows that

$$\frac{\pi x^2}{8(1+x^2)^2}-\frac{1}{l}\frac{x}{4(1+x^2)^2}\le\frac{1}{l}\sum_{k=1,3}^{\infty}\frac{(j/l)^3(k/l)^2}{((k/l)^2+(j/l)^2)^2(1+(j/l)^2)^2}\le\frac{\pi x^2}{8(1+x^2)^2}+\frac{1}{l}\frac{x}{4(1+x^2)^2}.$$

Therefore,

$$\frac{1}{l}\sum_{j=1,3}^{\infty}\frac{\pi x^2}{8(1+x^2)^2}-\frac{1}{l^2}\sum_{j=1,3}^{\infty}\frac{x}{4(1+x^2)^2}\le F_1^{(1,1)}(l)\le\frac{1}{l}\sum_{j=1,3}^{\infty}\frac{\pi x^2}{8(1+x^2)^2}+\frac{1}{l^2}\sum_{j=1,3}^{\infty}\frac{x}{4(1+x^2)^2}.$$

Noticing that

$$\begin{aligned}\max_x\left\{\frac{\pi x^2}{8(1+x^2)^2}\right\}&=\frac{\pi}{32},\quad\text{when}\quad x=1,\\ \max_x\left\{\frac{x}{4(1+x^2)^2}\right\}&=\frac{3\sqrt{3}}{64},\quad\text{when}\quad x=\frac{\sqrt{3}}{3},\end{aligned}$$

we have

$$\frac{1}{2}\int_0^{\infty}\frac{\pi x^2}{8(1+x^2)^2}dx-\frac{1}{l}\frac{\pi}{32}\le\frac{1}{l}\sum_{j=1,3}^{\infty}\frac{\pi x^2}{8(1+x^2)^2}\le\frac{1}{2}\int_0^{\infty}\frac{\pi x^2}{8(1+x^2)^2}dx+\frac{1}{l}\frac{\pi}{32},$$

$$\frac{1}{2}\int_0^{\infty}\frac{x}{4(1+x^2)^2}dx-\frac{1}{l}\frac{3\sqrt{3}}{64}\le\frac{1}{l}\sum_{j=1,3}^{\infty}\frac{x}{4(1+x^2)^2}\le\frac{1}{2}\int_0^{\infty}\frac{x}{4(1+x^2)^2}dx+\frac{1}{l}\frac{3\sqrt{3}}{64},$$

that is,

$$\begin{aligned}\frac{\pi^2}{64}-\frac{1}{l}\frac{\pi}{32}&\le\frac{1}{l}\sum_{j=1,3}^{\infty}\frac{\pi x^2}{8(1+x^2)^2}\le\frac{\pi^2}{64}+\frac{1}{l}\frac{\pi}{32},\\ \frac{1}{16}-\frac{1}{l}\frac{3\sqrt{3}}{64}&\le\frac{1}{l}\sum_{j=1,3}^{\infty}\frac{x}{4(1+x^2)^2}\le\frac{1}{16}+\frac{1}{l}\frac{3\sqrt{3}}{64}.\end{aligned}$$

Hence,

$$\frac{\pi^2}{64}-\frac{1}{l}\frac{\pi}{32}-\frac{1}{l}\frac{1}{16}-\frac{1}{l^2}\frac{3\sqrt{3}}{64}\le F_1^{(1,1)}(l)\le\frac{\pi^2}{64}+\frac{1}{l}\frac{\pi}{32}+\frac{1}{l}\frac{1}{16}+\frac{1}{l^2}\frac{3\sqrt{3}}{64}.$$

The other inequalities in the lemma can be shown by a similar technique. $\square$

Lemma 6.4.1 gives upper and lower bounds for $F_1^{(r,s)}(l)$ and $F_2^{(r,s)}(l)$. However, these bounds are not tight enough when $l = 1, 2, 3$, for $F_1^{(r,\ s)}(l)$, or when $k = 1, 2, \cdots, 100$, for $F_2^{(r,\ s)}(k)$. In the following, we give special treatment to these cases.

Lemma 6.4.2. *For $r = 1, 2,\ s = 1, 2,$*

$$\begin{aligned} 0 < F_1^{(r,s)}(l) &< 0.1964, && \textit{when } l = 1, 2, 3, 4, \\ 0 < F_2^{(r,s)}(l) &< 0.085, && \textit{when } l \le 100. \end{aligned}$$

Proof. Consider $F_1^{(1,1)}(l)$, when $l = 1, 2, 3, 4$. We have

$$\begin{aligned} F_1^{(1,1)}(l) &= l \sum_{j=1,3}^{\infty} \sum_{k=1,3}^{\infty} \frac{j^3k^2}{(k^2+j^2)^2(l^2+j^2)^2} \\ &= l \sum_{j=1,3}^{1023} \sum_{k=1,3}^{1023} \frac{j^3k^2}{(k^2+j^2)^2(l^2+j^2)^2} + l \sum_{j=1,3}^{1023} \sum_{k=1025,1027}^{\infty} \frac{j^3k^2}{(k^2+j^2)^2(l^2+j^2)^2} \\ &+ l \sum_{j=1025,1027}^{\infty} \sum_{k=1,3}^{\infty} \frac{j^3k^2}{(k^2+j^2)^2(l^2+j^2)^2}. \end{aligned} \tag{6.38}$$

It is easy to calculate the value of the first double sum in the right-hand side of (6.38). We have

$$l \sum_{j=1,3}^{1023} \sum_{k=1,3}^{1023} \frac{j^3k^2}{(k^2+j^2)^2(l^2+j^2)^2} = \begin{cases} 0.1946 & \text{when } l = 1, \\ 0.0797 & \text{when } l = 2, \\ 0.0507 & \text{when } l = 3, \\ 0.0374 & \text{when } l = 4. \end{cases}$$

Next, we focus on estimating the values of the other two double sums. By techniques similar to those used in the proof of Lemma 6.4.1, it follows that

$$
\begin{aligned}
& l \sum_{j=1,3}^{1023} \sum_{k=1025,1027}^{\infty} \frac{j^3k^2}{(k^2+j^2)^2(l^2+j^2)^2} \\
\leq\ & l \sum_{j=1,3}^{1023} \left(\frac{1}{2} \int_{k=1025}^{\infty} \frac{j^3k^2}{(k^2+j^2)^2(l^2+j^2)^2} dk + \frac{j^3 \times 1025^2}{(l^2+j^2)^2(j^2+1025^2)^2} \right) \\
=\ & l \sum_{j=1,3}^{1023} \frac{j^2(-2j^2 \arctan \frac{1025}{j} + \pi j^2 + 2 \times 1025 j + 1025^2\pi - 2 \times 1025^2 \arctan \frac{1025}{j})}{8(l^2+j^2)^2(j^2+1025^2)} \\
& + l \sum_{j=1,3}^{1023} \frac{j^3 \times 1025^2}{(l^2+j^2)^2(j^2+1025^2)^2} \\
=\ & \begin{cases} 0.0015 & \text{when } l = 1, \\ 0.0027 & \text{when } l = 2, \\ 0.0037 & \text{when } l = 3, \\ 0.0047 & \text{when } l = 4. \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
& l \sum_{j=1025,1027}^{\infty} \sum_{k=1,3}^{\infty} \frac{j^3k^2}{(k^2+j^2)^2(l^2+j^2)^2} \leq l \sum_{j=1025,1027}^{\infty} \left(\frac{1}{2} \int_{k=0}^{\infty} \frac{j^3k^2}{(k^2+j^2)^2(l^2+j^2)^2} dk + \frac{j}{4(l^2+j^2)^2} \right) \\
=\ & l \sum_{j=1025,1027}^{\infty} \frac{\pi j^2}{8(l^2+j^2)^2} + l \sum_{j=1025,1027}^{\infty} \frac{j}{4(l^2+j^2)^2} \\
\leq\ & \frac{l}{2} \int_{1025}^{\infty} \frac{\pi j^2}{8(l^2+j^2)^2} dj + \frac{\pi \times 1025^2 l}{8(l^2+1025^2)^2} + \frac{l}{2} \int_{1025}^{\infty} \frac{j}{4(l^2+j^2)^2} dj + \frac{1025 l}{4(l^2+1025^2)^2} \\
=\ & \frac{\pi(2 \times 1025 l - 2 \arctan(1025/l) l^2 - 2 \arctan(1025/l) \times 1025^2 + \pi l^2 + 1025^2\pi)}{64(l^2+1025^2) l} \\
& + \frac{\pi \times 1025^2 l}{8(l^2+1025^2)^2} + \frac{l}{16(l^2+1025^2)} + \frac{1025 l}{4(l^2+1025^2)^2} \\
=\ & \begin{cases} 1.9199 \times 10^{-4} & \text{when } l = 1, \\ 1.9243 \times 10^{-4} & \text{when } l = 2, \\ 1.9286 \times 10^{-4} & \text{when } l = 3, \\ 1.9329 \times 10^{-4} & \text{when } l = 4. \end{cases}
\end{aligned}
$$

By combining the above results and the positivity of $F_1^{(1,1)}(l)$, we have

$$0 < F_1^{(1,1)}(l) < 0.1964, \quad \text{when } l = 1, 2, 3, 4.$$

The rest of the bounds in Lemma 6.4.2 can be derived similarly. $\square$

The combination of Lemmas 6.4.1 and 6.4.2 gives the following bounds for $F_1^{(r,s)}$ and $F_2^{(r,s)}$.

Theorem 6.4.3. *For $l \in \mathbb{Z}^+$, and for $r = 1, 2,\ s = 1, 2,$*

$$\begin{aligned} 0 &< F_1^{(r,s)}(l) &< 0.1964, \\ 0 &< F_2^{(r,s)}(l) &< 0.085. \end{aligned}$$

We now consider the quantities

$$G_1^{(r,s)}(l) \;=\; \sum_{k=s,s+2}^{\infty} \sum_{j=r,r+2}^{\infty} \frac{j^3 k^2 l}{(k^2+j^2)^2(l^2+j^2)^2} \times \frac{(a_j^{(s)} a_k^{(r)})^2}{b_j^{(s)} b_k^{(r)}}, \tag{6.39}$$

$$G_2^{(r,s)}(l) \;=\; \sum_{k=s,s+2}^{\infty} \sum_{j=r,r+2}^{\infty} \frac{j^3 l^2 k}{(k^2+j^2)^2(l^2+j^2)^2} \times \frac{(a_j^{(s)} a_l^{(r)})^2}{b_j^{(s)} b_l^{(r)}}, \tag{6.40}$$

where $r = 1, 2$, $s = 1, 2$, and l is a positive integer, and notice that $G_1^{(r,s)}(l)$ and $G_2^{(r,s)}(k)$ are essentially $\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} f^{(r,s)}(j_r, k_s, l_s)$ and $\sum_{l=1}^{\infty}\sum_{j=1}^{\infty} f^{(r,s)}(j_r, k_s, l_s)$, respectively, if we ignore the (independent of n) factor $\frac{16\kappa_j^2\kappa_k\kappa_l n^2}{\pi^2}$ that accompanies each term of the sums.

Theorem 6.4.4. *For $l \in \mathbb{Z}^+$, and for $r = 1, 2,\ s = 1, 2,$*

$$0 < G_1^{(r,s)}(l) < 0.1964, \tag{6.41}$$

$$0 < G_2^{(r,s)}(l) < 0.085. \tag{6.42}$$

Proof. It is worth noticing that $G_t^{(r,s)}(l)$ only slightly differ from $F_t^{(r,s)}(l)$, where $t = 1, 2$, in that $G_t^{(r,s)}(l)$ involve the quantities $a_l^{(r,s)}$ and $b_l^{(r,s)}$, which were given in (6.33). Numerical calculations show that, for small l, say $l < 50$, the inequalities (6.41) and (6.42) hold. Since the factors $a_l^{(r,s)}$ and $b_l^{(r,s)}$ are exponentially close to one on l, for larger l, the bounds cannot be exceeded. $\square$

Up to now, we have concentrated on the function $f^{(r,s)}(j, k, l)$. Next, we investigate some properties of the function $\hat{f}^{(r,s)}(j,k,l,n)$. Let

$$\begin{aligned} z^{(r,s)}(j,k,l,n) &= \frac{j^3k^2l}{(k^2+j^2)^2(l^2+j^2)^2} \times \frac{(a_j^{(s)}a_k^{(r)})^2}{b_j^{(s)}b_k^{(r)}} \\ &\quad \times \left[\frac{2l^2}{n^2} + \frac{11j^2}{6n^2} - \frac{k^2}{6n^2} - \frac{4j^4+11j^2k^2+4k^4}{6(k^2+j^2)n^2} - \frac{4j^4+11j^2l^2+4l^4}{6(j^2+l^2)n^2} \right. \\ &\quad \left. -\frac{k^2}{8n^2} \times \frac{c_k^{(r)}}{a_k^{(r)}b_k^{(r)}} - \frac{j^2}{8n^2} \times \frac{c_j^{(s)}}{a_j^{(s)}b_j^{(s)}}\right], \end{aligned} \tag{6.43}$$

and notice that $\hat{f}^{(r,s)}(j,k,l,n) = -4\kappa_j^2\kappa_k\kappa_l n^2 z^{(r,s)}(j,k,l,n)$.

Lemma 6.4.5. *For $l \in \mathbb{Z}^+$, and for $r = 1,2$, $s = 1,2$,*

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} z^{(r,s)}(j,k,l,\infty) = 0, \text{ and } \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} z^{(r,s)}(j,l,k,\infty) = 0.$$

Proof. Consider $z^{(r,s)}(j,k,l,n)$ in (6.43). Since $a_j^{(r)}$, $b_j^{(r)}$ and $c_j^{(r)}$ are all exponentially close to one on j, and also bounded, we drop these factors, and consider the following function instead,

$$\begin{aligned} \hat{z}^{(r,s)}(j,k,l,n) &= \frac{j^3k^2l}{(k^2+j^2)^2(l^2+j^2)^2} \\ &\quad \times \left[\frac{2l^2}{n^2} + \frac{11j^2}{6n^2} - \frac{k^2}{6n^2} - \frac{4j^4+11j^2k^2+4k^4}{6(k^2+j^2)n^2} - \frac{4j^4+11j^2l^2+4l^4}{6(j^2+l^2)n^2} \right. \\ &\quad \left. -\frac{k^2}{8n^2} - \frac{j^2}{8n^2}\right] \\ &= \frac{j^3k^2l}{(k^2+j^2)^2(l^2+j^2)^2}\left[\frac{4l^2}{3n^2} + \frac{3j^2}{8n^2} - \frac{23k^2}{24n^2} - \frac{j^2k^2}{2(k^2+j^2)n^2} - \frac{j^2l^2}{2(j^2+l^2)n^2}\right]. \end{aligned}$$

Consider

$$\begin{aligned} &\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{j^3k^2l}{(k^2+j^2)^2(l^2+j^2)^2} \times \frac{l^2}{n^2} \\ &= \frac{l^3}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{j^3k^2}{(k^2+j^2)^2(l^2+j^2)^2} = \frac{l^3}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{j^3k^2}{(k^2+j^2)^{5/3}(k^2+j^2)^{1/3}(l^2+j^2)^2} \\ &< \frac{l^3}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{j^3k^2}{k^{2\times5/3}j^{2/3}j^4} = \frac{l^3}{n^2}\sum_{j=1}^{n}\frac{1}{j^{5/3}}\sum_{k=1}^{n}\frac{1}{k^{4/3}}. \end{aligned}$$

Clearly, both $\sum_{j=1}^{\infty}\frac{1}{j^{5/3}}$ and $\sum_{k=1}^{\infty}\frac{1}{k^{4/3}}$ converge. Therefore,

$$\lim_{n\to\infty}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{j^3k^2l}{(k^2+j^2)^2(l^2+j^2)^2} \times \frac{l^2}{n^2} = 0.$$

Similarly, it be shown that

$$\begin{aligned}
\lim_{n\to\infty}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{j^3k^2l}{(k^2+j^2)^2(l^2+j^2)^2}\times\frac{j^2}{n^2} &= 0,\\
\lim_{n\to\infty}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{j^3k^2l}{(k^2+j^2)^2(l^2+j^2)^2}\times\frac{k^2}{n^2} &= 0,\\
\lim_{n\to\infty}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{j^3k^2l}{(k^2+j^2)^2(l^2+j^2)^2}\times\frac{j^2k^2}{(j^2+k^2)n^2} &= 0,\\
\lim_{n\to\infty}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{j^3k^2l}{(k^2+j^2)^2(l^2+j^2)^2}\times\frac{j^2l^2}{(j^2+l^2)n^2} &= 0.
\end{aligned}$$

Hence,

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\hat{z}^{(r,s)}(j,k,l,\infty)=0,\ \text{ and }\ \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\hat{z}^{(r,s)}(j,l,k,\infty)=0,$$

from which the result of the lemma follows. □

Theorem 6.4.6. *For $l\in\mathbb{Z}^+$, and for $r=1,2,\ s=1,2,$*

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\hat{f}^{(r,s)}(j_r,k_s,l_s,\infty)=0,\ \textit{ and }\ \sum_{l=1}^{\infty}\sum_{j=1}^{\infty}\hat{f}^{(r,s)}(j_r,k_s,l_s,\infty)=0.$$

Proof. The proof of this theorem is trivial, after considering Lemma 6.4.5, and the fact that $|-4\kappa_j^2\kappa_k\kappa_l n^2|\le 16$. □

We now derive the bounds on the column and row norms of $\Psi^{(r)}$.

Theorem 6.4.7. *The column and row norms of $\Psi^{(r)}$ are bounded as follows:*

$$\begin{aligned}
0<\left\|\Psi^{(r)}\right\|_1&\le 1.2736,\\
0<\left\|\Psi^{(r)}\right\|_\infty&\le 0.5512.
\end{aligned}$$

Proof. By combining Theorems 6.4.4 and 6.4.6, and taking into account that $\frac{16\kappa_j^2\kappa_k\kappa_l n^2}{\pi^2}\le\frac{64}{\pi^2}$, we have

$$\begin{aligned}
0<\left\|\Psi^{(r,s)}\right\|_1\le\sum_{k=1}^{\infty}\psi_{k,l}^{(r,s)}\le\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}f^{(r,s)}(j_r,k_s,l_s)<\frac{64}{\pi^2}\times 0.1964=1.2736,\\
0<\left\|\Psi^{(r,s)}\right\|_\infty\le\sum_{l=1}^{\infty}\psi_{k,l}^{(r,s)}\le\sum_{l=1}^{\infty}\sum_{j=1}^{\infty}f^{(r,s)}(j_r,k_s,l_s)<\frac{64}{\pi^2}\times 0.085=0.5512.
\end{aligned}$$

Recalling that $\Psi^{(r)} = \begin{bmatrix} \Psi^{(r,1)} & 0 \\ 0 & \Psi^{(r,2)} \end{bmatrix}$, the bounds on $\left\|\Psi^{(r)}\right\|_1$ and $\left\|\Psi^{(r)}\right\|_\infty$ are trivially derived. □

6.5 Convergence of PGMRES

In this section, we present the main result on the convergence of PGMRES applied to $T_r x = y$ with preconditioner $\widetilde{T}_r$. We begin with a lemma.

Lemma 6.5.1. *Let $\mathcal{A}$ be a real $n \times n$ matrix, not necessarily symmetric, which satisfies $||I_n - \mathcal{A}||_1 \le \hat{\kappa}_1$ and $||I_n - \mathcal{A}||_\infty \le \hat{\kappa}_\infty$, with $\hat{\kappa}_1 \hat{\kappa}_\infty < 1$. Then,*

$$1 - \sqrt{\hat{\kappa}_1 \hat{\kappa}_\infty} \le \lambda \left(\frac{\mathcal{A}^T + \mathcal{A}}{2} \right) \tag{6.44}$$

and

$$(1 - \sqrt{\hat{\kappa}_1 \hat{\kappa}_\infty})^2 \le \lambda(\mathcal{A}^T \mathcal{A}) \le (1 + \sqrt{\hat{\kappa}_1 \hat{\kappa}_\infty})^2. \tag{6.45}$$

Moreover, $\mathcal{A}$ is non-singular and positive definite.

Proof. For convenience, let $\mathcal{B} = I_n - \mathcal{A}$. Since $\mathcal{A} = I_n - \mathcal{B}$, we have

$$\lambda \left(\frac{\mathcal{A}^T + \mathcal{A}}{2} \right) = 1 - \frac{1}{2}\lambda(\mathcal{B}^T + \mathcal{B}) \ge 1 - \frac{1}{2}||\mathcal{B} + \mathcal{B}^T||_2 \ge 1 - \frac{1}{2}(||\mathcal{B}||_2 + ||\mathcal{B}^T||_2) = 1 - ||\mathcal{B}||_2 \ge$$
$$1 - \sqrt{||\mathcal{B}||_1 ||\mathcal{B}||_\infty} \ge 1 - \sqrt{\hat{\kappa}_1 \hat{\kappa}_\infty} > 0,$$

which proves (6.44), and the non-singularity and positive definiteness of $\mathcal{A}$. Using the well-known result on the sensitivity of the singular values ([21], Page 449), we have

$$-\sigma_1(\mathcal{B}) \le \sigma(I_n - \mathcal{B}) - 1 \le \sigma_1(\mathcal{B}),$$

that is,

$$1 - \sigma_1(\mathcal{B}) \le \sigma(I_n - \mathcal{B}) \le 1 + \sigma_1(\mathcal{B}).$$

Since

$$\sigma_1(\mathcal{B}) \le \sqrt{\|\mathcal{B}\|_1 \|\mathcal{B}\|_\infty} \le \sqrt{\hat{\kappa}_1 \hat{\kappa}_\infty},$$

and $(\sigma(I_n - \mathcal{B}))^2 = (\sigma(\mathcal{A}))^2 = \lambda(\mathcal{A}^T \mathcal{A})$, relation (6.45) is proved. □

At this point, we can derive the required bounds for the eigenvalues of $\frac{(\mathcal{A}^{(r)})^T+\mathcal{A}^{(r)}}{2}$ and $(\mathcal{A}^{(r)})^T\mathcal{A}^{(r)}$, and describe the convergence of PGMRES applied to $T_r x = y$ with preconditioner $\widetilde{T}_r$.

Theorem 6.5.2. *The matrices T_r, $r = 1, 2$, are non-singular. The preconditioned matrices $\widetilde{T}_r^{-1}T_r$, $r = 1, 2$, satisfy*

$$0.1621 \leq \lambda\left(\frac{(\widetilde{T}_r^{-1}T_r)^T + \widetilde{T}_r^{-1}T_r}{2}\right) \quad \textit{and} \quad 0.0263 \leq \lambda\left((\widetilde{T}_r^{-1}T_r)^T\widetilde{T}_r^{-1}T_r\right) \leq 3.3777. \tag{6.46}$$

Moreover, if the GMRES method is applied to the linear system $T_r x = y$ with $\widetilde{T}_r$ as the preconditioner, then the initial error is reduced by a factor ϵ after at most

$$m = \frac{2\ln\epsilon}{\ln\left(1 - \frac{0.1621^2}{3.3777}\right)} \approx 256\ln\left(\frac{1}{\epsilon}\right) \tag{6.47}$$

iterations.

Proof. Applying Lemma 6.5.1 to $\mathcal{A}^{(r)} = I_n - P_n^T\Psi^{(r)}P_n$ and taking into account Theorem 6.4.7, it follows that $\mathcal{A}^{(r)}$ is non-singular and

$$0.1621 = 1 - \sqrt{1.2736 \times 0.5512} \leq \lambda\left(\frac{(\mathcal{A}^{(r)})^T + \mathcal{A}^{(r)}}{2}\right) \tag{6.48}$$

and

$$0.0263 = \left(1 - \sqrt{1.2736 \times 0.5512}\right)^2 \leq \lambda\left((\mathcal{A}^{(r)})^T\mathcal{A}^{(r)}\right) \leq \left(1 + \sqrt{1.2736 \times 0.5512}\right)^2 = 3.3777. \tag{6.49}$$

Taking into account that $\mathcal{A}^{(r)} = (Q^T\widetilde{T}_r^{-1}Q)Q^T T_r Q$, the orthogonality of Q, and the non-singularity of $\widetilde{T}_r$, we have the non-singularity of T_r, and relations (6.46). Then, from Theorem 6.1.1, we have relation (6.47). □

This theorem ascertains convergence to any specified accuracy in a number of iterations independent of the gridsize n.

In practice, the numerical experiments show that, for $\epsilon = 10^{-12}$, ten iterations are usually enough. This means that the result of Theorem 6.5.2 is pessimistic. The reason that leads to the pessimism is twofold. First, the calculated bounds for the eigenvalues are not tight enough. For example, when $n = 256$, numerical experiments show that

$$\begin{aligned} \lambda\left(\frac{(\mathcal{A}^{(r)})^T + \mathcal{A}^{(r)}}{2}\right) &\geq 0.52, \\ \lambda((\mathcal{A}^{(r)})^T \mathcal{A}^{(r)}) &\leq 1.02. \end{aligned}$$

Clearly, the mathematically derived bounds are far from the bounds obtained numerically. Second, the result of Theorem 6.1.1 does not tightly describe the error convergence. This can been seen when we substitute the real bounds into Theorem 6.1.1, in which case the expected number of iterations is still much larger than the actual number. We emphasize, however, that the main focus of the analysis is to derive the independence of the number of iterations from the grid size and not to derive the precise number of iterations required.

Chapter 7

Discretization Error Analysis

In the previous chapters, we have seen the applications of the three-step and the one-step methods to certain biharmonic problems. In this chapter we analyze the discretization errors of these two methods when applied to the biharmonic Dirichlet problem.

7.1 Error Analysis of One-Step Method

The error analysis is based on the properties of the coefficient matrix $\tilde{\tilde{A}}$. Through numerical experiments we arrive at the following conjecture.

Conjecture 7.1.1. *The inverse of matrix $\tilde{\tilde{A}}$ exists, and its infinity norm, $\left\|\tilde{\tilde{A}}^{-1}\right\|_\infty$, is bounded from above by a positive constant independent of the gridsize n.*

Table 7.1: Numerically calculated values of $\left\|\tilde{\tilde{A}}^{-1}\right\|_\infty$

n	16	24	32	48	64
Order of $\tilde{\tilde{A}}$	$20^2 \times 20^2$	$28^2 \times 28^2$	$36^2 \times 36^2$	$52^2 \times 52^2$	$68^2 \times 68^2$
$\left\|\tilde{\tilde{A}}^{-1}\right\|_\infty$	4919.5	4879.1	4859	4838.8	4828.8

In Table 7.1, we present numerical evidence that the conjecture is true. More specifically, we show the values of $\left\|\widetilde{\overline{A}}^{-1}\right\|_{\infty}$ calculated numerically for various gridsizes n. Notice that $\left\|\widetilde{\overline{A}}^{-1}\right\|_{\infty}$ slightly decreases as n increases.

Based on Conjecture 7.1.1, we can establish the convergence analysis of the one-step bi-quartic spline collocation method for the biharmonic Dirichlet problem using techniques similar to those in [27] and [9].

Theorem 7.1.2. *If $u \in C^{10}(\Omega)$, and Conjecture 7.1.1 is true, then the collocation approximation u_Δ satisfies the following a priori bounds:*

(a) global errors:

$$\begin{aligned}
\left\|D^k_{\{x,\,y\}}(u-u_\Delta)\right\|_\infty &= O(h^{5-k}), \quad k=0,\cdots,4, && (7.1)\\
\left\|D^k_x D^l_y(u-u_\Delta)\right\|_\infty &= O(h^{6-(k+l)}), \quad 2\le k+l\le 4,\ k\ne 0,\ l\ne 0; && (7.2)
\end{aligned}$$

(b) discrete errors at mid points: for $i,j=1,\cdots,n$,

$$\left\|D^k_{\{x,\,y\}}(u-u_\Delta)(\tau^x_i,\tau^y_j)\right\|_\infty = O(h^{6-k}), \quad k=0,2,4, \qquad (7.3)$$

(c) discrete errors at grid points: for $i,j=0,\cdots,n$,

$$\left\|D^k_{\{x,\,y\}}(u-u_\Delta)(x_i,y_j)\right\|_\infty = O(h^{6-k}), \quad k=0,2. \qquad (7.4)$$

$$(7.5)$$

Proof. We prove (7.2). The rest of the error bounds can be shown similarly. Let S and u_Δ be written as

$$\begin{aligned}
S(x,y) &= \sum_{i=-1}^{n+2}\sum_{j=-1}^{n+2} c^S_{i,j}\widetilde{\phi}^x_i(x)\widetilde{\phi}^y_j(y),\\
u_\Delta(x,y) &= \sum_{i=-1}^{n+2}\sum_{j=-1}^{n+2} c^u_{i,j}\widetilde{\phi}^x_i(x)\widetilde{\phi}^y_j(y).
\end{aligned}$$

From (3.23)-(3.24), and (4.9)-(4.12), we have

$$\begin{aligned}(L + P_L)(S - u_\Delta)(\tau_i^x, \tau_j^y) &= O(h^6), \quad (\tau_i^x, \tau_j^y) \in \Pi - \Pi_\partial,\\ (B_k + P_{B_k})(S - u_\Delta)(\tau_i^x, \tau_j^y) &= O(h^6), \quad (\tau_i^x, \tau_j^y) \in \Pi_\partial, k = 1, 2,\\ (D_x D_y + P_{B_c})(S - u_\Delta)(\tau_i^x, \tau_j^y) &= O(h^6), \quad i, j = 0, n + 1.\end{aligned}$$

which is equivalent to

$$\widetilde{\overline{A}}(\bar{c}^S - \bar{c}^u) = O(h^6).$$

This means that

$$\left\|\bar{c}^S - \bar{c}^u\right\|_\infty \le \left\|\widetilde{\overline{A}}^{-1}\right\|_\infty \cdot O(h^6) \Rightarrow \left\|\bar{c}^S - \bar{c}^w\right\|_\infty = O(h^6).$$

This relation and the boundedness of the basis functions $\widetilde{\phi}_i^x(x)\widetilde{\phi}_j^x(y)$ imply that

$$\left\|D_x^k D_y^l (S - u_\Delta)\right\|_\infty = O(h^{6-(k+l)}), \quad 0 \le k + l \le 4.$$

Using Theorem 2.2.9 and the triangle inequality, we obtain

$$\left\|D_x^k D_y^l (u - u_\Delta)\right\|_\infty = O(h^{\min\{6-(k+l),\, 5-\max\{k,\, l\}\}}),$$

which leads to (7.2). □

7.2 Error Analysis of Three-Step Method

For the three-step method, we have shown that $\widehat{A}$ is non-singular and have developed a fast solver for the linear system (4.62). Therefore the collocation approximations w_Δ, v_Δ, and u_Δ in $\widehat{S}_\Delta$ exist and are uniquely defined.

The error analysis of the three-step method follows a similar technique as the analysis for the one-step method. Through numerical experiments we arrive at the following conjecture.

Conjecture 7.2.1. *The infinity norm of $\widehat{A}^{-1}$, $\left\|\widehat{A}^{-1}\right\|_\infty$, is bounded from above by a positive constant which is independent of the gridsize n.*

Table 7.2: Numerically calculated values of $\left\|\widehat{A}^{-1}\right\|_\infty$

n	8	16	24	32	48	64
Order of $\widehat{A}$	$12^2 \times 12^2$	$20^2 \times 20^2$	$28^2 \times 28^2$	$36^2 \times 36^2$	$52^2 \times 52^2$	$68^2 \times 68^2$
$\left\|\widehat{A}^{-1}\right\|_\infty$	2271.2	2164.1	2129.1	2111.7	2094.3	2085.7

In Table 7.2 we present numerical evidence that the conjecture is true. We show the values of $\left\|\widehat{A}^{-1}\right\|_\infty$ calculated numerically for various gridsizes n. Notice that $\left\|\widehat{A}^{-1}\right\|_\infty$ slightly decreases as n increases. Another interesting observation is that $\left\|\widetilde{\widetilde{A}}^{-1}\right\|_\infty \approx 2.3\left\|\widehat{A}^{-1}\right\|_\infty$

By Conjecture 7.2.1, we can establish the convergence analysis of the three-step biquartic spline collocation method for the biharmonic Dirichlet problem.

Theorem 7.2.2. *If $u \in C^{10}(\Omega)$, and Conjecture 7.2.1 is true, then the collocation approximations w_Δ, v_Δ, and u_Δ satisfy the following a priori bounds:*

$$\begin{aligned}
\left\|D^k_{\{x,\,y\}}(u - w_\Delta)\right\|_\infty &= O(h^{2-k}), \quad k = 0, 1, && (7.6)\\
\left\|D^k_x D^l_y(u - v_\Delta)\right\|_\infty &= O(h^{4-(k+l)}), \quad 0 \le k+l \le 3, && (7.7)\\
\left\|D^k_{\{x,\,y\}}(u - u_\Delta)\right\|_\infty &= O(h^{5-k}), \quad k = 0, \cdots, 4, && (7.8)\\
\left\|D^k_x D^l_y(u - u_\Delta)\right\|_\infty &= O(h^{6-(k+l)}), \quad 2 \le k+l \le 4,\ k \ne 0,\ l \ne 0, && (7.9)\\
\left\|D^k_{\{x,\,y\}}(u - u_\Delta)(\tau^x_i, \tau^y_j)\right\|_\infty &= O(h^{6-k}), \quad k = 0, 2, 4, && (7.10)\\
\left\|D^k_{\{x,\,y\}}(u - u_\Delta)(x_i, y_j)\right\|_\infty &= O(h^{6-k}), \quad k = 0, 2. && (7.11)
\end{aligned}$$

Proof. Let S and w_Δ be written as

$$\begin{aligned}
S(x,y) &= \sum_{i=-1}^{n+2}\sum_{j=-1}^{n+2} c^S_{i,j}\widetilde{\phi}^x_i(x)\widetilde{\phi}^y_j(y),\\
w_\Delta(x,y) &= \sum_{i=-1}^{n+2}\sum_{j=-1}^{n+2} c^w_{i,j}\widetilde{\phi}^x_i(x)\widetilde{\phi}^y_j(y).
\end{aligned}$$

From (3.13)-(3.14), (4.4)-(4.8), and (4.9)-(4.12), we have

$$\begin{aligned} L(S-w_\Delta)(\tau_i^x,\tau_j^y) &= O(h^2), \quad (\tau_i^x,\tau_j^y)\in\Pi-\Pi_\partial, \\ B_k(S-w_\Delta)(\tau_i^x,\tau_j^y) &= O(h^2), \quad (\tau_i^x,\tau_j^y)\in\Pi_\partial, k=1,2, \\ D_xD_y(S-w_\Delta)(\tau_{\gamma_0}^x,\tau_{\gamma_0}^y) &= O(h^4), \end{aligned}$$

which are equivalent to

$$\widehat{A}(\bar{c}^S-\bar{c}^w)=O(h^2).$$

This means that

$$\left\|\bar{c}^S-\bar{c}^w\right\|_\infty \le \left\|\widehat{A}^{-1}\right\|_\infty \cdot O(h^2) \Rightarrow \left\|\bar{c}^S-\bar{c}^w\right\|_\infty = O(h^2).$$

This relation and the boundedness of the basis functions $\widetilde{\phi}_i^x(x)\widetilde{\phi}_j^x(y)$ imply that

$$\left\|D_{\{x,\,y\}}^k(S-w_\Delta)\right\|_\infty = O(h^{2-k}), \quad k=0,1.$$

Using Theorem 2.2.9 and the triangle inequality, we obtain (7.6).

Similarly, let

$$v_\Delta(x,y)=\sum_{i=-1}^{n+2}\sum_{j=-1}^{n+2} c_{i,j}^v\widetilde{\phi}_i^x(x)\widetilde{\phi}_j^y(y)$$

be the representations of v_Δ with respect to the basis functions. Combining equations (4.17), (4.23)-(4.31), we obtain

$$\begin{aligned} L(S-v_\Delta)(\tau_i^x,\tau_j^y) &= P_L(S-w_\Delta)(\tau_i^x,\tau_j^y)+O(h^4), \quad (\tau_i^x,\tau_j^y)\in\Pi-\Pi_\partial, \\ B_k(S-v_\Delta)(\tau_i^x,\tau_j^y) &= P_{B_k}(S-w_\Delta)(\tau_i^x,\tau_j^y)+O(h^4), \quad (\tau_i^x,\tau_j^y)\in\Pi_\partial, k=1,2, \\ D_xD_y(S-v_\Delta)(\tau_{\gamma_0}^x,\tau_{\gamma_0}^y) &= P_{B_c}(S-w_\Delta)(\tau_{\gamma_0}^x,\tau_{\gamma_0}^y)+O(h^4). \end{aligned}$$

By assuming the approximations of the relevant derivatives in the terms P_L, P_{B_1}, P_{B_2}, and P_{B_c} are $O(h^2)$ (see Remark 1), we get

$$\begin{aligned} L(S-v_\Delta)(\tau_i^x,\tau_j^y) &= O(h^4), \quad (\tau_i^x,\tau_j^y)\in\Pi-\Pi_\partial, \\ B_k(S-v_\Delta)(\tau_i^x,\tau_j^y) &= O(h^4), \quad (\tau_i^x,\tau_j^y)\in\Pi_\partial, k=1,2, \\ D_xD_y(S-v_\Delta)(\tau_{\gamma_0}^x,\tau_{\gamma_0}^y) &= O(h^4), \end{aligned}$$

which can be equivalently written in matrix form

$$\widehat{A}(\bar{c}^S - \bar{c}^v) = O(h^4),$$

resulting in

$$\left\|\bar{c}^S - \bar{c}^v\right\|_\infty = O(h^4).$$

This result and the properties of the basis functions prove that

$$\left\|D_x^k D_y^l (S - v_\Delta)\right\|_\infty = O(h^{4-(k+l)}), \quad 0 \le k + l \le 3.$$

Using Theorem 2.2.9 and the triangle inequality, we obtain

$$\left\|D_x^k D_y^l (u - v_\Delta)\right\|_\infty = O(h^{\min\{4-(k+l),\, 5-\max\{k,\, l\}\}}),$$

which leads to (7.7). The rest of equations can be shown in the same manner. □

7.3 Remarks

In this section, we mention several points in respect to the error analysis in the last two sections.

Remark 1: From equations (4.44)-(4.52), we know that $P_L w_\Delta(\tau_i^x, \tau_j^y)$ consists of two terms which approximate $D_x^6 u_{i,j}$ and $D_y^6 u_{i,j}$, respectively. These two approximations are expected to be of second order. Although at the moment this assumption cannot be theoretically proved, it is justified by numerical experiments. In Chapter 8, we apply the three-step method to Problem 8.1.2 and obtain w_Δ in the first step. Table (7.3) shows the order of convergence of $D_x^6 w_\Delta$ and $D_y^6 w_\Delta$. From $n = 8$ to $n = 16$, we see fifth order of convergence, but from $n = 16$ to $n = 32$, the order of convergence reduces significantly, a fact we believe is caused by rounding errors.

Table 7.3: Errors and orders of convergence for the approximations of $D_x^6 w_\Delta$ and $D_y^6 w_\Delta$ for Problem 8.1.2

$u(x,y) = [1-\cos(2\pi x)][1-\cos(2\pi y)]$, $\Omega = (0,1)\times(0,1)$				
	$D_x^6 u_{i,j}$		$D_y^6 u_{i,j}$	
n	error	order	error	order
8	1.5919e+05		1.5919e+05	
16	5.1769e+03	4.9425	5.1779e+03	4.9425
32	4.5707e+03	0.1797	4.5707e+03	0.1797

Remark 2: The error bounds of the cross-derivative $D_x^2 D_y^2 u_\Delta$ in the one-step method and three-step method shown in Theorem 7.1.2 and Theorem 7.2.2 are pessimistic. In our numerical experiments, we have seen a higher order of convergence,

$$\left\| D_x^2 D_y^2 (u_\Delta - u)(\tau_i^x, \tau_j^y) \right\|_\infty = O(h^3), \tag{7.12}$$

In the next chapter, we will present the related numerical results.

Remark 3: In Section 7.1, we provide an almost complete proof of the convergence of the one-step method for the biharmonic Dirichlet problem. By "almost", we mean that, while the proof of the boundedness of the inverse collocation matrix is given numerically, the rest of the proof follows clear mathematical arguments. The main steps of the analysis are:

(a) define an appropriate interpolant S; let c^S be the vector of degrees of freedom of S;

(b) prove that, at the collocation points, $S - u = O(h^r)$, and $(L + P_L)S - Lu = O(h^r)$ (interior points), $(B_k + P_{B_k})S - B_k u = O(h^r)$ (boundary points), where r is the maximum order of convergence targeted in the proof; that is, the error and the perturbed residual of S at the collocation points are $O(h^r)$;

(c) define the spline u_Δ by the collocation method: $(L + P_L)u_\Delta = Lu$ (interior points); $(B_k + P_{B_k})u_\Delta = B_k u$ (boundary points); thus the difference between the collocation and the interpolant (perturbed) residual equations at the collocation points is $O(h^r)$; let c^u be the vector of degrees of freedom of u_Δ; then $||A(c^S - c^u)||_\infty = O(h^r)$;

(d) show that the inverse A^{-1} of the collocation matrix A is bounded independently of the stepsize h (this is shown numerically), therefore, $||c^S - c^u||_\infty = O(h^r)$;

(e) with the boundedness of the basis functions, obtain $\|S - u_\Delta\|_\infty = O(h^r)$. By the triangle inequality, get $u - u_\Delta = O(h^r)$ at the collocation points;

Generally speaking, this is the approach used in [27]. However, in that paper, the definition of the interpolant is not appropriate as far as the boundary conditions are concerned, and the collocation equations on the boundary do not include the appropriate perturbations either. See also [1]. The choice of interpolant in [27] is the standard bicubic interpolant, for which the necessary error expansions at the collocation points have not been shown. The correct interpolant arises from the work in [17], also from [12] where the formulae are given for non-uniform grid partitions. In our derivation of the one-step method, both steps (a) and (b) were done correctly; we also included the appropriate perturbations in the boundary collocation equations.

Chapter 8

Numerical Results

In this chapter, we present some numerical results to demonstrate the performance of the Bi-Quartic SC methods. For most of the test problems considered, the exact solution function is known, thus, once the collocation approximation u_Δ is obtained, the errors $||u - u_\Delta||$ can be calculated. In the tables below, the errors at "grid points" and at "midpoints" correspond to the maximum errors in absolute value of the Quartic SC approximations at the discretization grid and at the center points of each sub-rectangle, respectively, and are both expected to be of sixth order, while the "sampled global" error corresponds to the error on an uniform grid of 251×251 points, which do not fall on special locations relative to the mesh points, and is expected to be of fifth order. The "sampled global" error is taken as an approximation to the maximum error over the whole domain.

8.1 Three-Step Method

This section shows several numerical experiments that demonstrate the discretization accuracy of the Three-Step Bi-Quartic SC method and the efficiency of the algorithm **BiHarmComp** for the biharmonic Dirichlet problems. As mentioned earlier, the efficiency of **BiHarmComp** is mainly determined by the number of iterations of PGMRES.

Therefore we present the numbers of iterations of PGMRES in the tables as an indication of the efficiency of **BiharmComp**.

The stopping criterion of PGMRES is the relative Euclidean norm of the residual and the tolerance is set to 10^{-12}. The initial guess for Step 1 is the zero vector, and Steps 2 and 3 use the solution from the preceding stage as the initial guess. All experiments are implemented with MATLAB in double precision.

Problem 8.1.1: Consider Problem (3.1)-(3.2) with

$$\begin{aligned}
Lu &\equiv \Big[\left(1+e^{-(x+y)}\right) D_x^4 + \frac{x+y}{10} D_x^3 D_y + \left(3 + \frac{1}{1+x+y}\right) D_x^2 D_y^2 \\
&\quad + \frac{1}{5+xy} D_x D_y^3 + \left(1+\frac{xy}{4}\right) D_y^4 - (x+y) D_x^3 + (1+xy) D_y^3 \\
&\quad + e^{x+y} D_x D_y - x D_y + (x+y)\Big] u, \text{ on } \Omega \equiv (0,2)\times(0,2), \\
B_1 u &\equiv u, \text{ on } \partial\Omega, \\
B_2 u &\equiv \frac{\partial u}{\partial \mathrm{n}}, \text{ on } \partial\Omega,
\end{aligned}$$

and f, g_1, and g_2 chosen so that the exact solution $u(x,y) = e^{x+y}$. This problem has non-constant coefficients and also involves more derivatives than the biharmonic problems.

Table 8.1: Errors and orders of convergence at various discrete points from the three-step method for Problem 8.1.1

	grid points		midpoints		sampled global	
n	error	order	error	order	error	order
8	4.18e-05		1.48e-05		5.28e-05	
16	6.37e-07	6.04	2.29e-07	6.02	1.11e-06	5.58
24	4.98e-08	6.28	1.56e-08	6.63	1.21e-07	5.46
32	8.38e-09	6.20	2.27e-09	6.70	2.64e-08	5.28
48	7.37e-10	6.00	2.21e-10	5.74	3.27e-09	5.16

We apply the three-step method to this problem and use the "\" operator in Matlab to solve the resulting linear systems. From Table 8.1, we can see that the three-step method achieves sixth order of convergence at the grid points and the midpoints, and fifth order at the sampled global points.

Besides the errors and orders of convergence at the grid points, midpoints, and sampled points, we also test the errors and order of convergence at the Gaussian points. For a sub-rectangular domain defined by four corners, (x_i, y_j), (x_{i+1}, y_{j+1}), (x_i, y_{j+1}), and (x_{i+1}, y_{j+1}), we consider the following four Gaussian points, $(x_i + \frac{3-\sqrt{3}}{6}h, y_j + \frac{3-\sqrt{3}}{6}h)$, $(x_i + \frac{3-\sqrt{3}}{6}h, y_{j+1} - \frac{3-\sqrt{3}}{6}h)$, $(x_{i+1} - \frac{3-\sqrt{3}}{6}h, y_j + \frac{3-\sqrt{3}}{6}h)$, and $(x_{i+1} - \frac{3-\sqrt{3}}{6}h, y_{j+1} - \frac{3-\sqrt{3}}{6}h)$.

Tables 8.2 and 8.3 show the errors and orders of convergence of the derivatives of the approximation from the three-step method at the sampled points and the Gaussian points. Due to the space limitation, in Tables 8.2 and 8.3 we omit the results for $n = 12$. Notice that the **boldface** entries in Table 8.2 have higher orders of convergence at the sampled points than the theoretical analysis has predicted in the last chapter. Another interesting point is that $D^3_{\{x,y\}}u_\Delta$ exhibits superconvergence at the Gaussian points.

We also use Problem 8.1.1 to verify our error analysis of the bi-quartic spline interpolant in Chapter 3, as well as the relations between the interpolant and the collocation approximations in Chapter 7. Due to the space limitation, we present only the orders of convergence from $n = 8$ to $n = 16$, and from $n = 16$ to $n = 32$ in Tables 8.4-8.6. These results are consistent with our previous theoretical analysis.

Now we consider several biharmonic problems. In the tables below, the errors shown are from Step 3 of the three-step method.

Problem 8.1.2: Consider Problem (4.1)-(4.3) with f, g_1 and g_2 chosen so that the exact solution is $u(x, y) = [1 - \cos(2\pi x)][1 - \cos(2\pi y)]$ on $\Omega = (0, 1) \times (0, 1)$.

Table 8.7 shows results from the application of **BiHarmComp** to the bi-Quartic SC system arising from the discretization of Problem 2. The error and convergence results are presented to verify the performance of the sixth order Bi-Quartic SC method. The number of PGMRES iterations for solving $T_1(\gamma_1 + \gamma_2) = \beta_1 + \beta_2$ is presented as a triple, each component corresponding to a step of the Bi-Quartic SC method. For this problem,

Table 8.2: Errors and orders of convergence of u_Δ (three-step method) and derivatives at sampled points for Problem 8.1.1

	errors					convergence			
n	8	16	24	32	48	16	24	32	48
$\|u-u_\Delta\|_\infty$	5.28e-05	1.11e-06	1.21e-07	2.64e-08	3.27e-09	5.58	5.46	5.28	5.16
$\|D_x(u-u_\Delta)\|_\infty$	7.42e-04	2.63e-05	4.34e-06	1.27e-06	2.35e-07	4.82	4.44	4.28	4.15
$\|D_y(u-u_\Delta)\|_\infty$	5.91e-04	2.38e-05	3.96e-06	1.16e-06	2.29e-07	4.64	4.42	4.28	3.99
$\left\|D_x^2(u-u_\Delta)\right\|_\infty$	2.79e-02	1.18e-03	2.86e-04	1.07e-04	3.25e-05	4.56	3.49	3.41	2.95
$\|D_xD_y(u-u_\Delta)\|_\infty$	1.02e-02	4.31e-04	6.21e-05	1.52e-05	2.00e-06	**4.57**	**4.78**	**4.89**	**5.01**
$\left\|D_y^2(u-u_\Delta)\right\|_\infty$	1.05e-02	9.37e-04	2.61e-04	1.07e-04	3.17e-05	3.49	3.15	3.10	3.00
$\left\|D_x^3(u-u_\Delta)\right\|_\infty$	5.94e-01	9.68e-02	3.70e-02	1.95e-02	8.22e-03	2.62	2.37	2.23	2.13
$\left\|D_x^2D_y(u-u_\Delta)\right\|_\infty$	3.05e-01	2.71e-02	6.03e-03	2.05e-03	4.52e-04	**3.49**	**3.70**	**3.75**	**3.73**
$\left\|D_xD_y^2(u-u_\Delta)\right\|_\infty$	2.52e-01	2.15e-02	4.67e-03	1.54e-03	3.06e-04	**3.55**	**3.76**	**3.87**	**3.98**
$\left\|D_y^3(u-u_\Delta)\right\|_\infty$	3.43e-01	7.45e-02	3.19e-02	1.78e-02	7.81e-03	2.20	2.09	2.04	2.03
$\left\|D_x^4(u-u_\Delta)\right\|_\infty$	8.04e-00	3.59e-00	2.32e-00	1.72e-00	1.14e-00	1.16	1.07	1.04	1.02
$\left\|D_x^3D_y(u-u_\Delta)\right\|_\infty$	3.83e-00	7.10e-01	2.47e-01	1.16e-01	4.10e-02	**2.43**	**2.61**	**2.62**	**2.56**
$\left\|D_x^2D_y^2(u-u_\Delta)\right\|_\infty$	7.46e-00	1.33e-00	4.44e-01	2.00e-01	6.55e-02	**2.49**	**2.71**	**2.77**	**2.76**
$\left\|D_xD_y^3(u-u_\Delta)\right\|_\infty$	3.42e-00	5.60e-01	1.74e-01	7.43e-02	2.81e-02	**2.61**	**2.88**	**2.96**	**2.40**
$\left\|D_y^4(u-u_\Delta)\right\|_\infty$	6.68e-00	3.34e-00	2.24e-00	1.68e-00	1.13e-00	1.00	0.99	0.99	0.99

because u is separable and symmetric in the x and y directions, the vectors β_1 and β_2 of **BiHarmComp** are equal, so PGMRES applied to $T_2(\gamma_1 - \gamma_2) = \beta_1 - \beta_2$ produces the exact (zero) solution in no iterations. This problem has also been used in [6], which has some recently published research results from a Hermite cubic piecewise polynomial collocation method to the biharmonic Dirichlet problem on a rectangular domain. In Table 8.8, we present the results from [6] for comparison. We include the number of flops (in thousands) that arises from the respective flops counts formulae. The results show that the sixth-order algorithm achieves higher accuracy with a smaller gridsize and fewer real single flops than the fourth-order method in [6]. Notice that, for a fair comparison of the number of flops, we present the flops counts for our algorithm assuming the T_2

Table 8.3: Errors and orders of convergence of u_Δ (three-step method) and derivatives at Gaussian points for Problem 8.1.1

	errors					convergence			
n	8	16	24	32	48	16	24	32	48
$\|u-u_\Delta\|_\infty$	5.13e-05	1.11e-06	1.22e-07	2.64e-08	3.25e-09	5.52	5.47	5.30	5.17
$\|D_x(u-u_\Delta)\|_\infty$	2.13e-04	8.06e-06	1.20e-06	3.25e-07	5.56e-08	4.72	4.70	4.53	4.36
$\|D_y(u-u_\Delta)\|_\infty$	2.93e-04	8.92e-06	1.31e-06	3.54e-07	6.00e-08	5.04	4.73	4.55	4.37
$\|D_x^2(u-u_\Delta)\|_\infty$	1.19e-02	1.07e-03	2.82e-04	1.13e-04	3.25e-05	3.48	3.29	3.16	3.08
$\|D_xD_y(u-u_\Delta)\|_\infty$	2.19e-03	1.06e-04	1.63e-05	4.34e-06	7.76e-07	4.37	4.61	4.60	4.24
$\|D_y^2(u-u_\Delta)\|_\infty$	8.84e-03	9.34e-04	2.61e-04	1.08e-04	3.19e-05	3.24	3.14	3.06	3.02
$\|D_x^3(u-u_\Delta)\|_\infty$	1.71e-01	1.42e-02	3.06e-03	1.05e-03	2.69e-04	**3.59**	**3.79**	**3.70**	**3.36**
$\|D_x^2D_y(u-u_\Delta)\|_\infty$	1.14e-01	9.77e-03	2.11e-03	6.91e-04	1.42e-04	3.54	3.79	3.87	3.90
$\|D_xD_y^2(u-u_\Delta)\|_\infty$	3.15e-02	3.42e-03	8.64e-04	3.28e-04	8.81e-05	3.20	3.39	3.36	3.24
$\|D_y^3(u-u_\Delta)\|_\infty$	9.57e-02	9.93e-03	2.39e-03	8.93e-04	2.49e-04	**3.27**	**3.51**	**3.42**	**3.15**
$\|D_x^4(u-u_\Delta)\|_\infty$	4.57e-00	2.03e-00	1.32e-00	9.79e-01	6.50e-01	1.17	1.07	1.03	1.01
$\|D_x^3D_y(u-u_\Delta)\|_\infty$	2.31e-00	4.17e-01	1.41e-01	6.47e-02	2.21e-02	2.47	2.68	2.71	2.64
$\|D_x^2D_y^2(u-u_\Delta)\|_\infty$	1.37e-00	2.42e-01	8.09e-02	3.65e-02	1.20e-02	2.50	2.71	2.76	2.75
$\|D_xD_y^3(u-u_\Delta)\|_\infty$	7.00e-01	1.39e-01	4.88e-02	2.33e-02	8.64e-03	2.33	2.58	2.58	2.44
$\|D_y^4(u-u_\Delta)\|_\infty$	3.90e-00	1.91e-00	1.28e-00	9.62e-01	6.44e-01	1.03	1.00	0.99	0.99

solution takes the same number of flops as the T_1 one. We also omitted the $O(n^2)$ terms, since they are not given explicitly in [6].

Problem 8.1.3: Consider Problem (4.1)-(4.3) with f, g_1 and g_2 chosen so that the exact solution is $u(x,y) = e^{x+y}$ on $\Omega = (0,4)\times(0,4)$.

Problem 8.1.3 has non-homogeneous boundary conditions. Just as we mentioned before, our algorithm can deal with non-homogeneous boundary conditions equally well as with the homogeneous ones. Table 8.9 shows the results from Bi-Quartic SC and **BiHarmComp** applied to Problem 8.1.3.

Problem 8.1.4: Consider Problem (4.1)-(4.3) with g, g_1 and g_2 chosen so that the exact solution is $u(x,y) = x^3\ln(1+y) + \frac{y}{(1+x)}$ on $\Omega = (0,1)\times(0,1)$. The true solution $u(x,y)$

Table 8.4: Orders of convergence of S, w_Δ, v_Δ, u_Δ and derivatives at grid points for Problem 8.1.1

	$S-u$		$S-w_\Delta$		$w_\Delta-u$		$S-v_\Delta$		$v_\Delta-u$		$S-u_\Delta$		$u_\Delta-u$	
n	16	32	16	32	16	32	16	32	16	32	16	32	16	32
$D_x^0D_y^0$	6.0	6.0	2.1	2.0	2.1	2.0	4.3	4.0	4.5	4.0	6.0	6.0	5.4	5.5
D_x	4.0	4.0	2.1	2.0	2.0	2.0	4.0	4.0	3.8	3.8	5.7	5.3	3.8	3.9
D_y	4.0	4.0	2.0	2.0	1.9	2.0	4.0	4.0	3.8	3.8	5.7	5.1	3.8	3.9
D_x^2	4.0	4.0	2.1	2.0	2.0	2.0	2.9	2.9	3.0	3.0	4.4	2.8	3.9	3.7
D_xD_y	4.0	4.0	2.1	2.0	2.3	2.1	3.7	2.6	3.0	3.0	4.6	3.4	3.8	3.9
D_y^2	4.0	4.0	2.1	2.0	2.1	2.0	2.9	2.9	3.0	3.0	4.4	2.5	3.9	3.6
D_x^3	2.0	2.0	2.0	2.0	2.0	2.0	1.8	1.9	2.0	2.0	2.8	2.1	2.0	2.0
$D_x^2D_y$	4.0	4.0	2.2	2.1	2.2	2.1	2.9	2.1	3.0	2.2	2.9	1.9	2.8	1.9
$D_xD_y^2$	4.0	4.0	1.8	2.0	1.9	2.0	2.9	2.2	3.0	2.2	2.5	1.7	2.9	1.8
D_y^3	2.0	2.0	1.8	1.9	2.0	2.0	1.8	1.9	2.0	2.0	2.8	1.6	2.0	2.0
D_x^4	1.0	1.0	0.8	0.9	1.0	1.0	0.8	0.9	1.0	1.0	1.6	1.2	1.0	1.0
$D_x^3D_y$	2.0	2.0	2.0	1.8	2.0	1.8	1.7	0.6	2.0	1.8	1.8	0.6	1.9	1.9
$D_x^2D_y^2$	4.0	4.0	0.9	1.0	0.9	1.0	0.6	0.8	0.6	0.8	2.3	2.8	2.4	2.8
$D_xD_y^3$	2.0	2.0	1.4	1.2	2.0	1.7	1.3	0.7	2.0	1.8	1.3	0.8	1.9	1.9
D_y^4	1.0	1.0	0.8	0.9	1.0	1.0	0.8	0.9	1.0	1.0	1.6	0.7	1.0	1.0

is not separable and has some nonhomogeneous boundary conditions. The errors and orders of convergence are shown in Table 8.10. This problem has also been considered by Altas et al in [3] using the multigrid method. In Table 8.11, we present the results from [3] for comparison. While the authors of [3] get a maximum absolute error of 4.8×10^{-10} with a 64×64 mesh, we only need a 32×32 grid to obtain a 9.64×10^{-10} error.

Problem 8.1.5: Consider Problem (4.1)-(4.3) with f, g_1 and g_2 chosen so that the exact solution is $u(x, y) = [\cos(x) - 1][\cos(2y) - 1]$ on $\Omega = (0, 4) \times (0, 2)$.

Problem 8.1.5 has a rectangular (non-square) domain with non-homogeneous boundary conditions. The rectangular domain is partitioned by an $n_x \times n_y$ uniform mesh. Table 8.12 shows the results. Please note that our Bi-Quartic SC methods do not assume

Table 8.5: Orders of convergence of S, w_Δ, v_Δ, u_Δ and derivatives at midpoints for Problem 8.1.1

	$S-u$		$S-w_\Delta$		$w_\Delta-u$		$S-v_\Delta$		$v_\Delta-u$		$S-u_\Delta$		$u_\Delta-u$	
n	16	32	16	32	16	32	16	32	16	32	16	32	16	32
$D_x^0D_y^0$	0.0	0.0	2.1	2.0	2.1	2.0	4.3	4.0	4.3	4.0	5.2	5.7	5.2	5.7
D_x	3.8	3.9	1.9	2.0	2.0	2.0	3.7	3.8	3.7	3.8	4.3	4.2	3.8	3.9
D_y	3.8	3.9	2.1	2.0	2.2	2.0	3.7	3.8	3.7	3.8	4.5	4.2	3.8	3.9
D_x^2	3.8	3.9	1.1	1.6	1.2	1.7	2.9	3.0	2.3	2.7	3.4	3.3	3.6	3.8
D_xD_y	3.8	3.9	2.6	2.0	2.4	2.0	2.7	2.9	2.5	2.8	4.2	2.4	3.9	3.5
D_y^2	3.8	3.9	1.5	1.8	1.6	1.8	2.8	3.0	2.4	2.6	3.3	3.2	3.5	3.7
D_x^3	1.8	1.9	1.7	1.8	1.8	1.9	2.2	2.1	1.8	1.9	2.3	1.6	1.8	1.9
$D_x^2D_y$	3.8	3.9	1.5	1.9	1.5	1.9	1.7	1.9	1.7	1.9	3.4	1.6	3.2	2.1
$D_xD_y^2$	3.8	3.9	1.5	2.0	1.5	2.0	1.6	1.9	1.7	1.9	3.3	1.6	3.4	1.5
D_y^3	1.8	1.9	1.7	1.8	1.8	1.9	2.2	2.1	1.8	1.9	2.5	1.6	1.8	1.9
D_x^4	1.8	1.9	1.6	1.8	1.6	1.8	1.1	1.0	1.9	1.6	0.5	0.8	1.8	1.7
$D_x^3D_y$	1.8	1.9	1.1	1.1	0.8	0.9	0.8	0.9	0.5	0.7	2.8	0.2	1.9	1.8
$D_x^2D_y^2$	3.8	3.9	0.7	1.0	0.7	1.0	0.6	0.9	0.7	0.9	1.4	-0.1	1.0	0.0
$D_xD_y^3$	1.8	1.9	1.0	1.0	0.6	0.8	0.8	0.9	0.5	0.7	2.5	0.4	1.9	1.8
D_y^4	1.8	1.9	1.6	1.2	1.4	1.4	1.0	1.0	1.8	1.6	0.7	0.7	1.7	1.6

same stepsizes along both directions. However, when the stepsizes along x-direction and y-direction are different, the analysis of the fast solver becomes difficult.

In the previous chapters, we have assumed that u belongs to C^{10}. We emphasize that this assumption is only for the purpose of the theoretical analysis. In practice, even when $u \notin C^{10}$ as long as $u \in C^6$, we still expect to see the optimal order of convergence. We demonstrate this point with the following examples.

Problem 8.1.6: Consider (4.1)-(4.3) with f, g_1 and g_2 chosen so that the exact solution is $u(x,y) = (xy)^{19/2}$ on $\Omega = (0,1) \times (0,1)$.

Problem 8.1.7: Consider (4.1)-(4.3) with f, g_1 and g_2 chosen so that the exact solution is $u(x,y) = |x - \frac{2}{3}|^7 |y - \frac{1}{3}|^7$ on $\Omega = (0,1) \times (0,1)$.

Table 8.6: Orders of convergence of S, w_Δ, v_Δ, u_Δ and derivatives at sampled points for Problem 8.1.1

	$S-u$		$S-w_\Delta$		$w_\Delta-u$		$S-v_\Delta$		$v_\Delta-u$		$S-u_\Delta$		$u_\Delta-u$	
n	16	32	16	32	16	32	16	32	16	32	16	32	16	32
$D_x^0D_y^0$	4.9	5.1	2.1	2.0	2.1	2.0	4.3	4.0	4.6	4.1	6.0	6.0	4.9	5.3
D_x	4.0	4.0	2.1	2.0	2.1	2.0	4.0	4.0	5.0	3.8	5.7	6.1	3.8	3.9
D_y	4.0	4.0	2.1	2.0	2.1	2.1	4.0	4.0	5.0	3.7	5.7	5.2	3.8	3.9
D_x^2	2.9	3.3	2.1	2.0	2.1	2.0	2.9	2.9	3.0	3.0	4.4	3.0	2.9	3.3
D_xD_y	4.0	4.0	2.5	2.0	2.4	2.0	5.1	3.3	4.6	3.9	4.8	3.1	3.8	3.9
D_y^2	2.9	3.3	2.1	2.0	2.1	2.0	2.9	2.9	3.0	3.0	4.4	3.0	2.9	3.3
D_x^3	2.0	2.0	2.0	2.0	2.0	2.0	1.8	1.9	2.0	2.0	2.8	2.3	2.0	2.0
$D_x^2D_y$	2.9	3.2	3.0	2.6	3.0	2.6	3.6	2.7	3.6	2.7	3.9	1.4	3.0	3.1
$D_xD_y^2$	2.9	3.2	3.1	2.8	3.1	2.8	3.6	2.6	3.6	2.6	3.5	1.4	3.1	3.1
D_y^3	2.0	2.0	1.8	1.9	2.0	2.0	1.8	1.9	2.0	2.0	2.8	2.3	2.0	2.0
D_x^4	1.0	1.0	0.8	0.9	1.0	1.0	0.8	0.9	1.0	1.0	1.6	1.3	1.0	1.0
$D_x^3D_y$	2.0	2.0	2.5	2.5	2.4	3.2	2.7	1.2	2.5	2.5	2.9	0.5	1.9	1.9
$D_x^2D_y^2$	2.7	3.3	0.9	1.0	0.9	1.0	0.6	0.8	0.6	0.8	2.3	2.8	2.4	2.8
$D_xD_y^3$	2.0	2.0	2.7	1.9	2.5	2.9	2.8	1.1	2.5	2.5	2.7	0.3	1.9	1.9
D_y^4	1.0	1.0	0.8	0.9	1.0	1.0	0.8	0.9	1.0	1.0	1.6	1.3	1.0	1.0

The true solutions u in Problem 8.1.6 and Problem 8.1.7 belong to $C^{9.5}$ and C^6, respectively. From Tables 8.13 and 8.14, we can see that the three-step method still achieves fifth order of convergence at the sampled global points.

After we have seen the problem with $u \in C^6$, a natural question is: how does the three-step method behave when the true solution u does not belong to C^6? To answer this question, we study two problems with $u \in C^{5.5}$ and $C^{4.5}$, respectively.

Problem 8.1.8: Consider (4.1)-(4.3) with f, g_1 and g_2 chosen so that the exact solution is $u(x,y) = (xy)^{11/2}$ on $\Omega = (0,1) \times (0,1)$. Table 8.15 shows that three-step method still achieves fifth order of convergence at the sampled points.

Problem 8.1.9: Consider (4.1)-(4.3) with f, g_1 and g_2 chosen so that the exact solution

Table 8.7: Errors and orders of convergence from the three-step method, number of PGMRES iterations and flops in thousands from the algorithm **BiHarmComp**, for Problem 8.1.2

$u(x,y) = [1-\cos(2\pi x)][1-\cos(2\pi y)]$, $\Omega = (0,1)\times(0,1)$								
	grid points		midpoints		sampled global		iterations	flops
n	error	order	error	order	error	order	T_1	
8	9.76e-03		9.07e-03		9.76e-03		(4, 4, 4)	17
16	2.27e-05	8.75	2.69e-05	8.40	2.89e-05	8.40	(6, 5, 5)	92
32	1.46e-07	7.28	1.59e-07	7.40	2.92e-07	6.63	(6, 6, 5)	416
64	1.68e-09	6.44	5.14e-10	8.27	4.71e-09	5.95	(7, 6, 5)	1843
Complexity: $30n^2\log_2 n + 15mn^2$, m: No. of PGMRES iterations								

Table 8.8: Errors, orders of convergence, number of PCG iterations, and flops in thousands from the method in [6] for Problem 8.1.2

$u(x,y) = [1-\cos(2\pi x)][1-\cos(2\pi y)]$, $\Omega = (0,1)\times(0,1)$						
	grid points		sampled global		iterations	flops
n	error	order	error	order	Schur compl.	
16	7.85e-05		1.64e-04		6	70
32	4.86e-06	4.015	1.05e-05	3.965	7	332
64	3.03e-07	4.004	6.05e-07	4.118	8	1540
128	1.89e-08	4.001	3.97e-08	3.930	9	7012
Complexity: $20n^2\log_2 n + 32mn^2$, m: No. of PCG iterations						

Table 8.9: Errors, orders of convergence, and number of PGMRES iterations from the three-step method for Problem 8.1.3

$u(x, y) = e^{x+y}$, $\Omega = (0, 4) \times (0, 4)$								
	grid points		midpoints		sampled global		iterations	
n	error	order	error	order	error	order	T_1	T_2
8	1.77e-02		1.54e-02		2.82e-02		$(7, 6, 6)$	$(7, 6, 5)$
12	1.62e-03	5.90	1.41e-03	5.89	4.13e-03	4.74	$(7, 6, 5)$	$(7, 6, 5)$
16	3.39e-04	5.44	3.00e-04	5.39	1.04e-03	4.81	$(8, 6, 5)$	$(8, 6, 5)$
24	3.65e-05	5.49	3.26e-05	5.47	1.40e-04	4.93	$(8, 6, 4)$	$(8, 5, 5)$
32	7.33e-06	5.58	6.52e-06	5.60	3.54e-05	4.78	$(8, 6, 4)$	$(8, 5, 4)$
48	7.56e-07	5.60	1.02e-06	4.57	4.55e-06	5.05	$(9, 5, 4)$	$(9, 4, 3)$

Table 8.10: Errors, orders of convergence, and number of PGMRES iterations from the three-step method for Problem 8.1.4

$u(x, y) = x^3 \ln(1 + y) + \frac{y}{1+x}$, $\Omega = (0, 1) \times (0, 1)$								
	grid points		midpoints		sampled global		iterations	
n	error	order	error	order	error	order	T_1	T_2
8	2.62e-06		2.55e-06		2.67e-06		(7, 6, 5)	(7, 5, 5)
12	2.74e-07	5.57	2.58e-07	5.64	2.86e-07	5.51	(8, 5, 5)	(7, 5, 5)
16	5.20e-08	5.77	5.03e-08	5.69	5.54e-08	5.70	(8, 5, 4)	(8, 5, 5)
24	4.94e-09	5.80	4.77e-09	5.81	5.47e-09	5.71	(8, 5, 4)	(8, 5, 4)
32	8.26e-10	6.22	7.91e-10	6.25	9.73e-10	6.00	(9, 4, 3)	(8, 4, 4)

Table 8.11: Errors and orders of convergence from the 3-cycle full-multigrid method using 17 W-cycles in [3], for Problem 8.1.4

$u(x,y) = x^3 \ln(1+y) + \frac{y}{1+x}$, $\Omega = (0,1) \times (0,1)$		
	grid points	
n	error	order
16	8.3e-08	
32	5.8e-09	3.84
64	4.8e-10	3.59
Complexity: $O(n^2)$		

Table 8.12: Errors, orders of convergence, and number of PGMRES iterations from the three-step method for Problem 8.1.5

$u(x,y) = [\cos(x) - 1][\cos(2y) - 1]$, $\Omega = (0,4) \times (0,2)$									
		grid points		midpoints		sampled global		iterations	
n_x	n_y	error	order	error	order	error	order	T_1	T_2
12	8	6.26e-05		5.08e-05		6.36e-05		(7, 6, 5)	(7, 6, 5)
16	12	5.10e-06	8.71	4.71e-06	8.27	6.46e-06	7.95	(7, 6, 5)	(7, 6, 5)
32	24	6.55e-08	6.28	7.35e-08	6.00	1.19e-07	5.76	(8, 6, 4)	(8, 6, 4)
64	48	1.20e-09	5.78	1.41e-09	5.70	2.72e-09	5.46	(9, 5, 3)	(9, 5, 3)

Table 8.13: Errors, orders of convergence, and number of PGMRES iterations from the three-step method for Problem 8.1.6

$u(x,y) = (xy)^{19/2}$, $\Omega = (0,1) \times (0,1)$								
	grid points		midpoints		sampled global		iterations	
n	error	order	error	order	error	order	T_1	T_2
8	8.50e-05		3.34e-05		2.16e-04		(7, 6, 6)	(7, 6, 6)
16	1.89e-06	5.49	7.57e-07	5.46	6.94e-06	4.96	(8, 7, 6)	(7, 6, 6)
24	1.90e-07	5.67	1.68e-07	3.72	9.50e-07	4.90	(8, 7, 6)	(8, 6, 5)
32	3.63e-08	5.75	4.89e-08	4.28	2.41e-07	4.76	(9, 6, 6)	(8, 6, 5)
48	4.40e-09	5.20	7.59e-09	4.59	3.16e-08	5.01	(9, 6, 5)	(8, 5, 4)

Table 8.14: Errors, orders of convergence, and number of PGMRES iterations from the three-step method for Problem 8.1.7

$u(x,y) = \|x - \frac{2}{3}\|^7 \|y - \frac{1}{3}\|^7$, $\Omega = (0,1) \times (0,1)$								
	grid points		midpoints		sampled global		iterations	
n	error	order	error	order	error	order	T_1	T_2
8	2.69e-07		2.49e-07		6.80e-07		(7, 6, 7)	(6, 6, 6)
12	2.44e-08	5.92	1.91e-08	6.34	8.82e-08	5.04	(8, 6, 6)	(7, 6, 6)
16	4.70e-09	5.73	4.85e-09	4.76	2.15e-08	4.90	(8, 7, 6)	(7, 6, 6)
24	5.20e-10	5.43	7.56e-10	4.58	2.93e-09	4.92	(8, 7, 6)	(8, 6, 6)
32	1.34e-10	4.71	1.90e-10	4.80	7.37e-10	4.80	(9, 7, 6)	(8, 6, 5)
48	1.97e-11	4.73	2.60e-11	4.91	9.63e-11	5.02	(9, 6, 5)	(9, 5, 4)

Table 8.15: Errors, orders of convergence, and number of PGMRES iterations from the three-step method for Problem 8.1.8

$u(x, y) = (xy)^{11/2}$, $\Omega = (0, 1) \times (0, 1)$								
	grid points		midpoints		sampled global		iterations	
n	error	order	error	order	error	order	T_1	T_2
8	1.44e-06		1.26e-06		3.51e-06		(7, 6, 6)	(6, 6, 5)
12	1.30e-07	5.93	1.45e-07	5.32	4.85e-07	4.88	(8, 6, 5)	(7, 5, 5)
16	3.57e-08	4.48	3.38e-08	5.07	1.17e-07	4.93	(8, 6, 5)	(7, 5, 5)
24	6.12e-09	4.35	5.83e-09	4.33	1.61e-08	4.90	(8, 5, 5)	(8, 5, 4)
32	1.75e-09	4.36	1.70e-09	4.29	3.98e-09	4.86	(9, 5, 4)	(8, 4, 4)
48	2.75e-10	4.56	2.69e-10	4.54	4.97e-10	5.13	(9, 4, 3)	(8, 4, 3)

is $u(x, y) = (xy)^{9/2}$ on $\Omega = (0, 1) \times (0, 1)$. The true solution u belongs to $C^{4.5}$. The three-

Table 8.16: Errors, orders of convergence, and number of PGMRES iterations from the three-step method for Problem 8.1.9

$u(x, y) = (xy)^{9/2}$, $\Omega = (0, 1) \times (0, 1)$								
	grid points		midpoints		sampled global		iterations	
n	error	order	error	order	error	order	T_1	T_2
8	1.52e-06		1.48e-06		1.65e-06		$(7, 5, 5)$	$(6, 5, 5)$
12	4.42e-07	3.04	4.36e-07	3.02	4.57e-07	3.16	$(8, 6, 5)$	$(7, 5, 4)$
16	1.85e-07	3.04	1.83e-07	3.01	1.89e-07	3.07	$(8, 5, 5)$	$(7, 5, 4)$
24	5.07e-08	3.19	5.09e-08	3.16	5.17e-08	3.20	$(8, 5, 4)$	$(8, 4, 4)$
32	1.99e-08	3.26	1.99e-08	3.26	2.01e-08	3.29	$(8, 5, 4)$	$(8, 4, 3)$
48	5.15e-09	3.33	5.14e-09	3.34	5.17e-09	3.34	$(9, 4, 2)$	$(8, 3, 3)$

step method has some order of convergence which is slightly better than third order as shown in Table 8.16. Please note that the spline collocation approximation from Step 2 has almost the same errors and orders of convergence as in Table 8.16, i.e., there is negligible improvement in accuracy from Step 2 to Step 3 for Problem 8.1.9.

8.2 One-Step Method for Biharmonic Problems

This section demonstrates the performance of the one-step method for some biharmonic problems. After the test problems are discretized by the one-step Bi-Quartic SC method, to reduce the condition numbers of the coefficient matrices, we re-scale the resulting linear systems so that each row vector of the coefficient matrices has unit infinity norm. Then the re-scaled linear systems are solved by the "\" operator in MATLAB.

Problem 8.2.1: Consider (4.1)-(4.3) with f, g_1 and g_2 chosen so that the exact solution is $u(x,y) = x^5y^5$ on $\Omega = (0,1) \times (0,1)$.

From Table 8.17, we can see that the numerical solution from the one-step method matches the true solution exactly at the grid points and the mid points, and achieves fifth order of convergence at the sampled global points and the Gaussian points. Therefore, the one-step method has super convergence at the grid points and the mid points.

Tables 8.18-8.21 show errors and orders of convergence of the derivatives of the numerical solution at discrete points. From these tables, we can see that $D_x^2u_\Delta$ and $D_y^2u_\Delta$ have significantly smaller absolute errors at midpoints and grid points than at sampled points and Gaussian points, and that $D_x^2D_yu_\Delta$ and $D_xD_y^2u_\Delta$ have fourth order of convergence at midpoints and grid points and only third order at sampled points and Gaussian points, moreover, that $D_x^3u_\Delta$, $D_y^3u_\Delta$, $D_x^3D_yu_\Delta$ and $D_xD_y^3u_\Delta$ exhibit much higher orders of convergence at Gaussian points than at other points.

Table 8.17: Errors and orders of convergence at various discrete points from the one-step method for Problem 8.2.1

$u(x,y) = x^5y^5$,$\Omega = (0,1) \times (0,1)$								
	grid points		midpoints		sampled global		Gaussian points	
n	error	order	error	order	error	order	error	order
10	3.61e-14		1.37e-14		4.33e-07		4.32e-07	
12	5.75e-15	-	2.30e-15	-	1.78e-07	4.89	1.77e-07	4.90
16	7.64e-15	-	7.63e-15	-	4.29e-08	4.94	4.29e-08	4.92
24	3.86e-15	-	3.86e-15	-	5.62e-09	5.01	5.78e-09	4.95
32	2.00e-14	-	2.00e-14	-	1.39e-09	4.85	1.39e-09	4.96
48	5.35e-14	-	5.36e-14	-	1.79e-10	5.06	1.85e-10	4.97

Problem 8.2.2: Consider (4.1)-(4.3) with f, g_1 and g_2 chosen so that the exact solution is $u(x,y) = |x - \frac{2}{3}|^7|y - \frac{1}{3}|^7$ on $\Omega = (0,1) \times (0,1)$.

This is the same problem as in Problem 8.1.7. The results in Table 8.22 show that the one-step method achieves fifth order of convergence at the sampled global points and sixth order of convergence at the grid points. The absolute errors are comparable to those from the three-step method.

Problem 8.2.3: Consider Problem (4.1)-(4.3) with g, g_1 and g_2 chosen so that the exact solution is $u(x,y) = x^3 \ln(1+y) + \frac{y}{(1+x)}$ on $\Omega = (0,1) \times (0,1)$.

This problem has also been considered in Problem 8.1.4 and by Altas et al in [3]. From Table 8.23, we can see that the one-step method generates comparable results to those from the three-step method.

To further test the accuracy of the one-step method, we consider two practical problems arising from the engineering field.

Problem 8.2.4: Consider Problem (4.1)-(4.3) on $\Omega = (0,1) \times (0,1)$ with $f \equiv 1$, $g_1 = g_2 \equiv 0$. It describes the bending of a square clamped plate under a uniform load,

Table 8.18: Errors and orders of convergence of u_Δ (one-step method) and derivatives at grid points for Problem 8.2.1

	errors					convergence			
n	10	16	24	32	48	16	24	32	48
$\|u-u_\Delta\|_\infty$	3.61e-14	7.64e-15	3.86e-15	2.00e-14	5.35e-14	-	-	-	-
$\|D_x(u-u_\Delta)\|_\infty$	1.67e-05	2.54e-06	5.02e-07	1.59e-07	3.14e-08	4.00	4.00	4.00	4.00
$\|D_y(u-u_\Delta)\|_\infty$	1.67e-05	2.54e-06	5.02e-07	1.59e-07	3.14e-08	4.00	4.00	4.00	4.00
$\left\|D_x^2(u-u_\Delta)\right\|_\infty$	1.91e-10	4.69e-12	6.48e-12	1.28e-11	4.71e-11	-	-	-	-
$\|D_xD_y(u-u_\Delta)\|_\infty$	1.67e-04	2.54e-05	5.02e-06	1.59e-06	3.14e-07	4.00	4.00	4.00	4.00
$\left\|D_y^2(u-u_\Delta)\right\|_\infty$	1.91e-10	4.92e-12	6.20e-12	1.84e-11	5.28e-11	-	-	-	-
$\left\|D_x^3(u-u_\Delta)\right\|_\infty$	1.00e-01	3.91e-02	1.74e-02	9.77e-03	4.34e-03	2.00	2.00	2.00	2.00
$\left\|D_x^2D_y(u-u_\Delta)\right\|_\infty$	3.33e-04	5.09e-05	1.00e-05	3.18e-06	6.29e-07	4.00	4.00	4.00	4.00
$\left\|D_xD_y^2(u-u_\Delta)\right\|_\infty$	3.33e-04	5.09e-05	1.00e-05	3.18e-06	6.30e-07	4.00	4.00	4.00	3.99
$\left\|D_y^3(u-u_\Delta)\right\|_\infty$	1.00e-01	3.91e-02	1.74e-02	9.77e-03	4.34e-03	2.00	2.00	2.00	2.00
$\left\|D_x^4(u-u_\Delta)\right\|_\infty$	6.00e-00	3.75e-00	2.50e-00	1.88e-00	1.25e-00	1.00	1.00	1.00	1.00
$\left\|D_x^3D_y(u-u_\Delta)\right\|_\infty$	5.00e-01	1.95e-01	8.68e-02	4.88e-02	2.17e-02	2.00	2.00	2.00	2.00
$\left\|D_x^2D_y^2(u-u_\Delta)\right\|_\infty$	1.25e-06	7.75e-08	2.31e-08	1.34e-07	1.56e-06	5.92	2.99	-6.13	-6.04
$\left\|D_xD_y^3(u-u_\Delta)\right\|_\infty$	5.00e-01	1.95e-01	8.68e-02	4.88e-02	2.17e-02	2.00	2.00	2.00	2.00
$\left\|D_y^4(u-u_\Delta)\right\|_\infty$	6.00e-00	3.75e-00	2.50e-00	1.88e-00	1.25e-00	1.00	1.00	1.00	1.00

whose solution can be expressed as the sum of an infinite series [42]. We use the sum of the truncated series with the first 1000 items as the true solution to calculate the errors of various methods. From Table 8.24, we notice that when $n \leq 80$, the orders of convergence are consistent with the our theoretical analysis. When $n = 96$, the error for the collocation approximation becomes larger, which is most likely due to the dominance of rounding errors over discretization errors.

Some other papers also discussed the numerical solution of this problem [6, 24]. In particular, they present the calculated values of the deflection $u(0.5, 0.5)$ and the bending moments $M_x(1, 0.5)$, $M_y(0.5, 1)$, where $M_x = -\Delta u$ and $M_y = -\Delta u$ on the vertical and horizontal sides of $\partial\Omega$, respectively.

Table 8.19: Errors and orders of convergence of u_Δ (one-step method) and derivatives at midpoints for Problem 8.2.1

	errors					convergence			
n	10	16	24	32	48	16	24	32	48
$\|u-u_\Delta\|_\infty$	1.37e-14	7.63e-15	3.86e-15	2.00e-14	5.36e-14	1.24	1.68	-5.73	-2.43
$\|D_x(u-u_\Delta)\|_\infty$	1.13e-05	1.90e-06	3.96e-07	1.29e-07	2.61e-08	3.79	3.87	3.91	3.93
$\|D_y(u-u_\Delta)\|_\infty$	1.13e-05	1.90e-06	3.96e-07	1.29e-07	2.61e-08	3.79	3.87	3.91	3.93
$\left\|D_x^2(u-u_\Delta)\right\|_\infty$	1.69e-12	6.47e-13	1.02e-12	2.26e-12	1.19e-11	2.04	-1.13	-2.75	-4.11
$\|D_xD_y(u-u_\Delta)\|_\infty$	1.19e-04	1.96e-05	4.04e-06	1.31e-06	2.63e-07	3.83	3.89	3.93	3.95
$\left\|D_y^2(u-u_\Delta)\right\|_\infty$	1.74e-12	6.50e-13	1.03e-12	4.53e-12	1.23e-11	2.09	-1.13	-5.16	-2.46
$\left\|D_x^3(u-u_\Delta)\right\|_\infty$	3.87e-02	1.67e-02	7.81e-03	4.51e-03	2.06e-03	1.79	1.87	1.91	1.93
$\left\|D_x^2D_y(u-u_\Delta)\right\|_\infty$	2.50e-04	4.05e-05	8.25e-06	2.65e-06	5.33e-07	3.88	3.92	3.94	3.96
$\left\|D_xD_y^2(u-u_\Delta)\right\|_\infty$	2.50e-04	4.05e-05	8.25e-06	2.65e-06	5.33e-07	3.88	3.92	3.94	3.96
$\left\|D_y^3(u-u_\Delta)\right\|_\infty$	3.87e-02	1.67e-02	7.81e-03	4.51e-03	2.06e-03	1.79	1.87	1.91	1.93
$\left\|D_x^4(u-u_\Delta)\right\|_\infty$	3.55e-09	7.28e-10	8.11e-09	2.78e-08	2.27e-07	3.37	-5.95	-4.28	-5.18
$\left\|D_x^3D_y(u-u_\Delta)\right\|_\infty$	2.04e-01	8.60e-02	3.99e-02	2.29e-02	1.04e-02	1.83	1.89	1.93	1.95
$\left\|D_x^2D_y^2(u-u_\Delta)\right\|_\infty$	2.77e-09	1.66e-10	2.32e-09	9.47e-09	9.41e-08	5.99	-6.51	-4.90	-5.66
$\left\|D_xD_y^3(u-u_\Delta)\right\|_\infty$	2.04e-01	8.60e-02	3.99e-02	2.29e-02	1.04e-02	1.83	1.89	1.93	1.95
$\left\|D_y^4(u-u_\Delta)\right\|_\infty$	3.54e-09	1.21e-09	9.86e-09	5.41e-08	3.76e-07	2.29	-5.18	-5.92	-4.78

Table 8.25 and Figure 8.1 indicate that the computed values for the deflection $u(0.5, 0.5)$ from the one-step method are more accurate than the results in [6]. From Table 8.26 and Figure 8.2, we see that, for the calculation of the bending moment $M_x(1, 0.5)$, our results and the results from [6] are comparable.

Problem 8.2.5: Consider Problem (4.1)-(4.3) on $\Omega = (0, 1) \times (0, 1)$ with $f \equiv 0$, $g_1 \equiv 0$, $g_2 = 1$ on the upper side of $\partial\Omega$, and $g_2 = 0$ on the remaining sides of $\partial\Omega$. This problem describes creeping flow of a viscous incompressible fluid in a square cavity. Notice that the derivative u_y is discontinuous at $(0, 1)$ and $(1, 1)$, and it is known that the vorticity $v = \Delta u$ is unbounded near the points $(0, 1)$ and $(1, 1)$ (see [6, 29]). Table 8.27 shows the values $v = \Delta u_\Delta$ at the four corners. Notice that at the corners $(0, 0)$ and $(1, 0)$, v

Table 8.20: Errors and orders of convergence of u_Δ (one-step method) and derivatives at sampled points for Problem 8.2.1

	errors					convergence			
n	10	16	24	32	48	16	24	32	48
$\|u-u_\Delta\|_\infty$	4.33e-07	4.29e-08	5.62e-09	1.39e-09	1.79e-10	4.92	5.01	4.85	5.06
$\|D_x(u-u_\Delta)\|_\infty$	1.67e-05	2.54e-06	5.02e-07	1.59e-07	3.14e-08	4.00	4.00	4.00	4.00
$\|D_y(u-u_\Delta)\|_\infty$	1.67e-05	2.54e-06	5.02e-07	1.59e-07	3.14e-08	4.00	4.00	4.00	4.00
$\left\|D_x^2(u-u_\Delta)\right\|_\infty$	9.62e-04	2.35e-04	6.96e-05	2.94e-05	8.70e-06	3.00	3.00	3.00	3.00
$\|D_xD_y(u-u_\Delta)\|_\infty$	1.67e-04	2.54e-05	5.02e-06	1.59e-06	3.14e-07	4.00	4.00	4.00	4.00
$\left\|D_y^2(u-u_\Delta)\right\|_\infty$	9.62e-04	2.35e-04	6.96e-05	2.94e-05	8.70e-06	3.00	3.00	3.00	3.00
$\left\|D_x^3(u-u_\Delta)\right\|_\infty$	1.00e-01	3.91e-02	1.74e-02	9.77e-03	4.34e-03	2.00	2.00	2.00	2.00
$\left\|D_x^2D_y(u-u_\Delta)\right\|_\infty$	5.11e-03	1.21e-03	3.55e-04	1.49e-04	4.40e-05	3.06	3.03	3.02	3.01
$\left\|D_xD_y^2(u-u_\Delta)\right\|_\infty$	5.11e-03	1.21e-03	3.55e-04	1.49e-04	4.40e-05	3.06	3.03	3.02	3.01
$\left\|D_y^3(u-u_\Delta)\right\|_\infty$	1.00e-01	3.91e-02	1.74e-02	9.77e-03	4.34e-03	2.00	2.00	2.00	2.00
$\left\|D_x^4(u-u_\Delta)\right\|_\infty$	6.00e+00	3.75e+00	2.50e+00	1.87e+00	1.25e+00	1.00	1.00	1.00	1.00
$\left\|D_x^3D_y(u-u_\Delta)\right\|_\infty$	5.00e-01	1.95e-01	8.68e-02	4.88e-02	2.17e-02	2.00	2.00	2.00	2.00
$\left\|D_x^2D_y^2(u-u_\Delta)\right\|_\infty$	3.61e-02	9.00e-03	2.70e-03	1.11e-03	3.42e-04	2.96	2.97	3.09	2.91
$\left\|D_xD_y^3(u-u_\Delta)\right\|_\infty$	5.00e-01	1.95e-01	8.68e-02	4.88e-02	2.17e-02	2.00	2.00	2.00	2.00
$\left\|D_y^4(u-u_\Delta)\right\|_\infty$	6.00e+00	3.75e+00	2.50e+00	1.87e+00	1.25e+00	1.00	1.00	1.00	1.00

converges to 0 (the true solution) very fast, and at the corners $(0,1)$ and $(1,1)$, v shows the unboundedness in value, which is consistent with the analysis in [29].

To our best knowledge, there is no analytic solution to this problem. We cannot directly measure the order of convergence of the one-step method. Instead, we use the results from three consecutive gridsizes to approximate the order of convergence. Suppose the order of convergence is k, and $u_{[n]}$ is the numerical solution obtained from an $n \times n$

Table 8.21: Errors and orders of convergence of u_Δ (one-step method) and derivatives at Gaussian points for Problem 8.2.1

	errors					convergence			
n	10	16	24	32	48	16	24	32	48
$\|u-u_\Delta\|_\infty$	4.32e-07	4.29e-08	5.78e-09	1.39e-09	1.85e-10	4.91	4.95	4.96	4.97
$\|D_x(u-u_\Delta)\|_\infty$	2.95e-06	4.38e-07	8.54e-08	2.69e-08	5.28e-09	4.06	4.03	4.02	4.01
$\|D_y(u-u_\Delta)\|_\infty$	2.95e-06	4.38e-07	8.54e-08	2.69e-08	5.28e-09	4.06	4.03	4.02	4.01
$\left\|D_x^2(u-u_\Delta)\right\|_\infty$	8.69e-04	2.20e-04	6.67e-05	2.84e-05	8.51e-06	2.92	2.95	2.96	2.97
$\|D_xD_y(u-u_\Delta)\|_\infty$	2.55e-05	4.02e-06	8.08e-07	2.58e-07	5.14e-08	3.93	3.96	3.97	3.98
$\left\|D_y^2(u-u_\Delta)\right\|_\infty$	8.69e-04	2.20e-04	6.67e-05	2.84e-05	8.51e-06	2.92	2.95	2.96	2.97
$\left\|D_x^3(u-u_\Delta)\right\|_\infty$	1.38e-05	1.34e-06	1.78e-07	4.32e-08	1.08e-08	4.97	4.97	4.93	3.42
$\left\|D_x^2D_y(u-u_\Delta)\right\|_\infty$	4.47e-03	1.12e-03	3.38e-04	1.44e-04	4.28e-05	2.94	2.96	2.97	2.98
$\left\|D_xD_y^2(u-u_\Delta)\right\|_\infty$	4.47e-03	1.12e-03	3.38e-04	1.44e-04	4.28e-05	2.94	2.96	2.97	2.98
$\left\|D_y^3(u-u_\Delta)\right\|_\infty$	1.38e-05	1.34e-06	1.78e-07	4.36e-08	1.22e-08	4.97	4.97	4.90	3.14
$\left\|D_x^4(u-u_\Delta)\right\|_\infty$	3.11e+00	2.03e+00	1.38e+00	1.05e+00	7.06e-01	0.91	0.95	0.96	0.97
$\left\|D_x^3D_y(u-u_\Delta)\right\|_\infty$	1.60e-04	2.48e-05	4.94e-06	1.62e-06	5.58e-07	3.97	3.98	3.87	2.63
$\left\|D_x^2D_y^2(u-u_\Delta)\right\|_\infty$	3.61e-02	9.03e-03	2.71e-03	1.15e-03	3.44e-04	2.95	2.97	2.98	2.98
$\left\|D_xD_y^3(u-u_\Delta)\right\|_\infty$	1.60e-04	2.48e-05	4.95e-06	1.63e-06	6.98e-07	3.97	3.97	3.86	2.09
$\left\|D_y^4(u-u_\Delta)\right\|_\infty$	3.11e+00	2.03e+00	1.38e+00	1.05e+00	7.06e-01	0.91	0.95	0.96	0.97

grid. Then we have the following approximations:

$$\begin{aligned}
\left\|u_{[n]}-u\right\|_\infty &\approx C\frac{1}{n^k},\\
\left\|u_{[2n]}-u\right\|_\infty &\approx C\frac{1}{(2n)^k},\\
\left\|u_{[4n]}-u\right\|_\infty &\approx C\frac{1}{(4n)^k},
\end{aligned}$$

where C is a unknown constant. Then the order of convergence k can be approximated by

$$k=\log_2\left\|u_{[n]}-u_{[2n]}\right\|_\infty-\log_2\left\|u_{[2n]}-u_{[4n]}\right\|_\infty.$$

The approximate orders of convergence in Table 8.28 show that the one-step method has roughly third order of convergence for Problem 8.2.5.

Table 8.22: Errors and orders of convergence at various discrete points from the one-step method for Problem 8.2.2

$u(x,y) = \lvert x-\frac{2}{3}\rvert^7 \lvert y-\frac{1}{3}\rvert^7$, $\Omega = (0,1)\times(0,1)$						
	grid points		midpoints		sampled global	
n	error	order	error	order	error	order
10	1.54e-07		1.31e-07		2.16e-07	
12	2.41e-08	10.18	1.96e-08	10.42	8.71e-08	4.98
16	4.61e-09	5.74	3.08e-09	6.43	2.12e-08	4.91
24	4.33e-10	5.83	9.91e-11	8.48	2.90e-09	4.91
32	7.97e-11	5.89	3.04e-11	4.11	7.26e-10	4.82
48	7.22e-12	5.92	6.71e-12	3.72	9.54e-11	5.01

Table 8.23: Errors and orders of convergence at various discrete points from the one-step method for Problem 8.2.3

$u(x,y) = x^3 \ln(1+y) + \frac{y}{1+x}$, $\Omega = (0,1)\times(0,1)$						
	grid points		midpoints		sampled global	
n	error	order	error	order	error	order
10	7.53e-07		7.38e-07		7.88e-07	
12	2.78e-07	5.47	2.64e-07	5.64	2.90e-07	5.47
16	5.27e-08	5.78	5.10e-08	5.72	5.61e-08	5.71
24	5.00e-09	5.81	4.82e-09	5.82	5.54e-09	5.71
32	9.42e-10	5.80	9.08e-10	5.81	1.08e-09	5.69
48	9.12e-11	5.76	8.83e-11	5.75	1.11e-10	5.61

Table 8.24: Errors and orders of convergence at the grid points from the one-step method for Problem 8.2.4

n	16	32	64	80	96
errors	1.568e-07	1.019e-08	2.100e-10	4.091e-11	4.278e-11
orders		3.944	5.600	7.330	-0.246

Table 8.25: Computed deflections, $u(0.5, 0.5)$, for Problem 8.2.4

n	$u(0.5, 0.5)$	our results		results from [6]	
	True Solution	Value	Error	Value	Error
16	1.26531908748e-03	1.26538938040e-03	7.029e-08	1.26528707e-03	3.202e-08
32	1.26531908748e-03	1.26532033290e-03	1.245e-09	1.26531700e-03	2.087e-09
64	1.26531908748e-03	1.26531909136e-03	3.880e-12	1.26531896e-03	1.275e-10
80	1.26531908748e-03	1.26531908820e-03	7.200e-13		
96	1.26531908748e-03	1.26531908787e-03	3.900e-13		
128	1.26531908748e-03			1.26531908e-03	7.480e-12

Table 8.26: Computed bending moments, $M_x(1, 0.5)$, for Problem 8.2.4

n	$M_x(1, 0.5)$	our results		results from [6]	
	True Solution	Value	Error	Value	Error
16	-5.13337648349e-02	-5.13352729268e-02	1.508e-06	-5.1333582e-02	1.828e-07
32	-5.13337648349e-02	-5.13337993593e-02	3.452e-08	-5.13337482e-02	1.663e-08
64	-5.13337648349e-02	-5.13337637103e-02	1.125e-09	-5.13337636e-02	1.235e-09
80	-5.13337648349e-02	-5.13337642512e-02	5.837e-10		
96	-5.13337648349e-02	-5.13337645004e-02	3.345e-10		
128	-5.13337648349e-02			-5.13337647e-02	1.349e-10

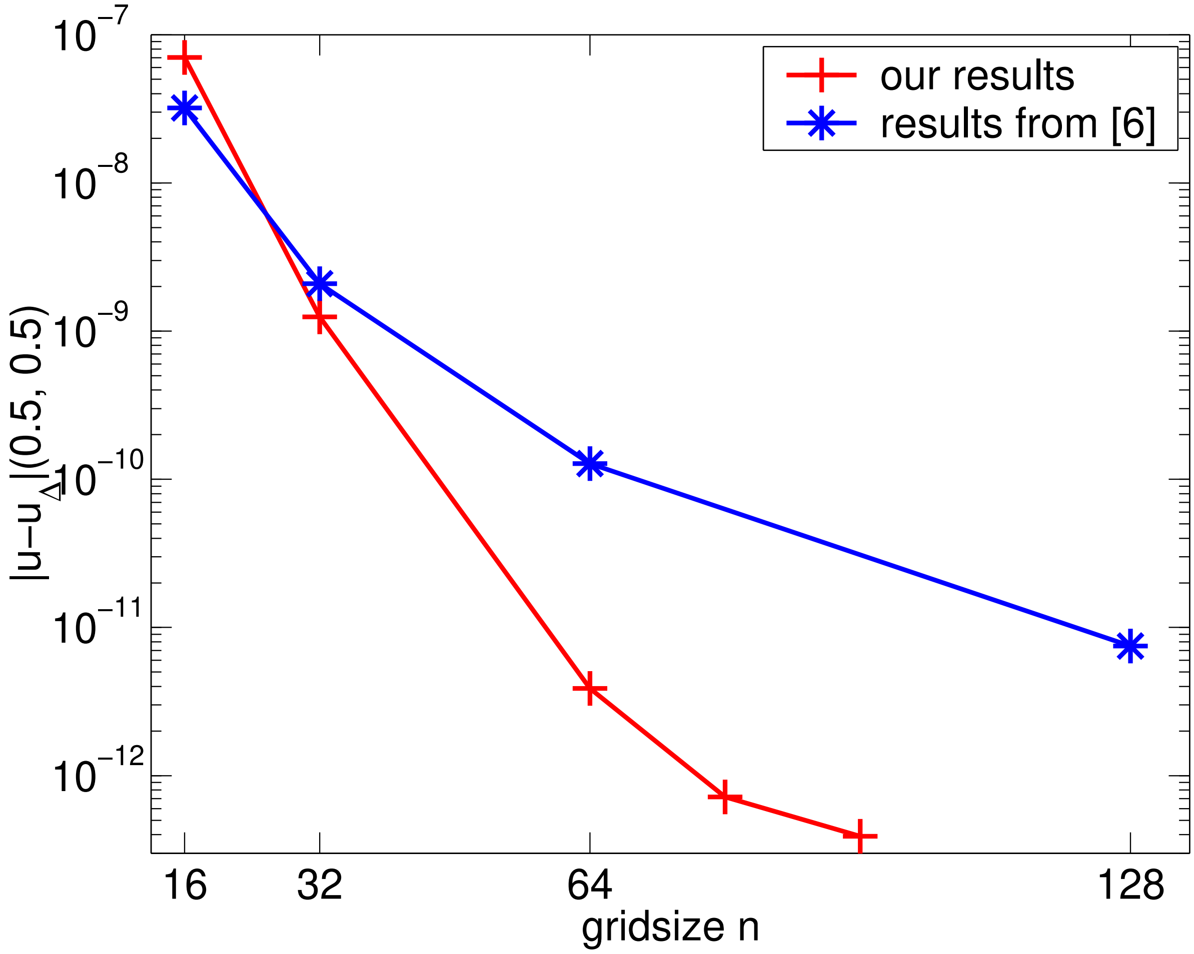


Figure 8.1: Errors of $u(0.5, 0.5)$ versus the gridsize n for Problem 8.2.4

Table 8.27: Vorticity $v = \Delta u_\Delta$ at corners for Problem 8.2.5

n	v at $(0,0)$, $(1,0)$	v at $(0,1)$, $(1,1)$
16	4.5157e-05	-107.5091
32	7.6285e-12	-215.0182
64	-7.0473e-19	-430.0363
128	3.2865e-19	-860.0727
256	5.2940e-20	-1720.1454

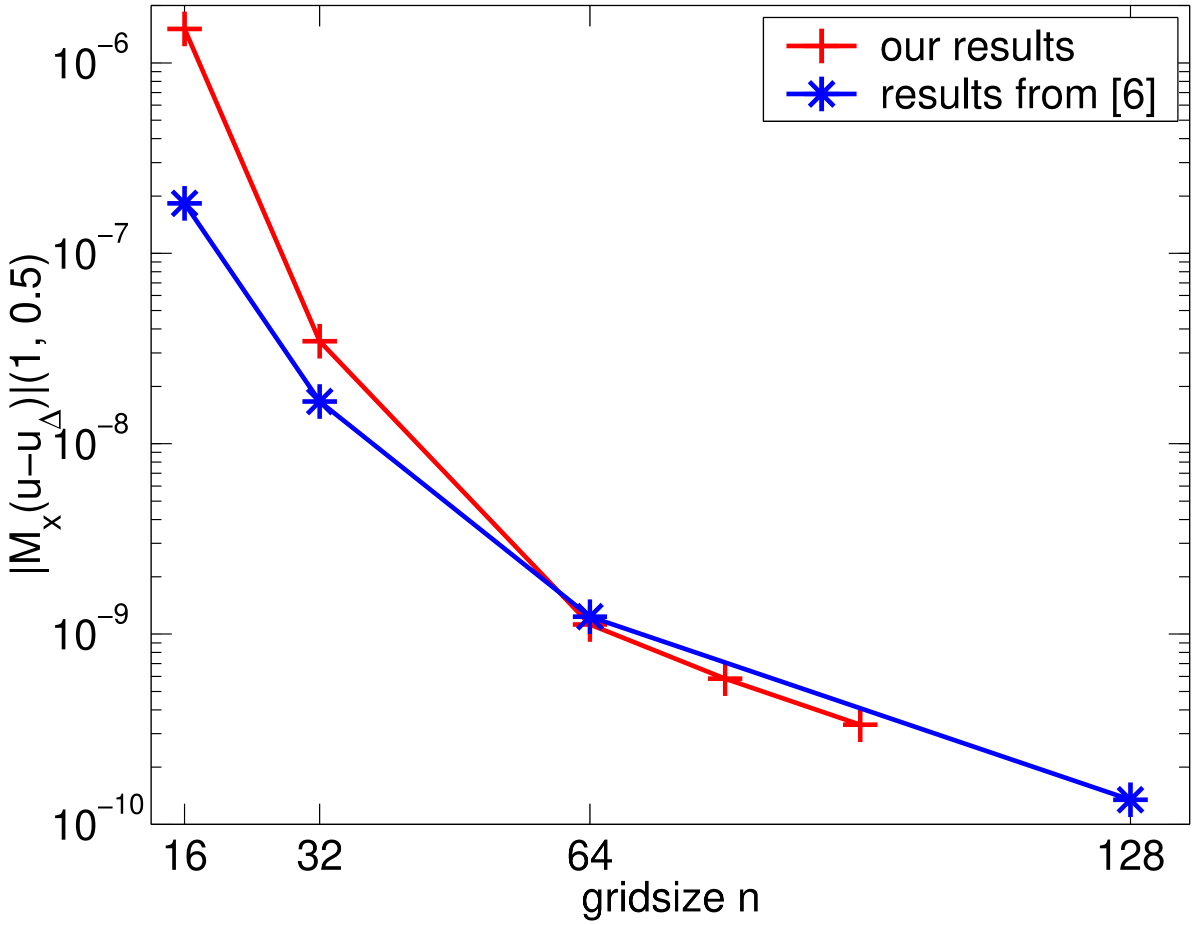


Figure 8.2: Errors of $M_x(1, 0.5)$ versus the gridsize n for Problem 8.2.4

Table 8.28: Approximate orders of convergence at the sampled points for the one-step method for Problem 8.2.5

n	64	128	256
orders	2.7603	3.3749	2.6050

Figure 8.3 shows the values of u_Δ (obtained with gridsize $n = 256$) from different view points on a 400×400 mesh over the domain $[0,1] \times [0,1]$, from which we can see the solution is quite smooth with no oscillations. The two plots from different view points also reflects the fact that first derivative of u is -1 along the boundary $y = 1$ and is 0 along the rest of the boundary.

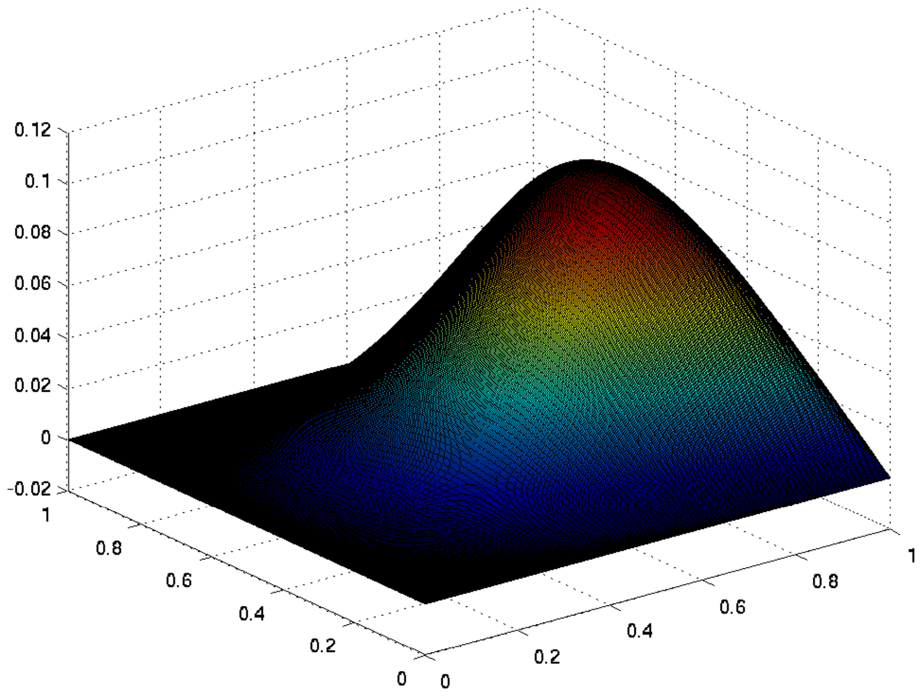
Several papers from literature discuss the numerical solutions to Problem 8.2.5 [29, 3, 6]. In particular, they calculate the coordinates of the primary vortex center (at which u attains its maximum modulus), the values of the stream function u and vorticity $v = \Delta u$ at the vortex. From Table 8.29, we can see that our calculations are in reasonable agreement with those results in literature.

8.3 Remarks

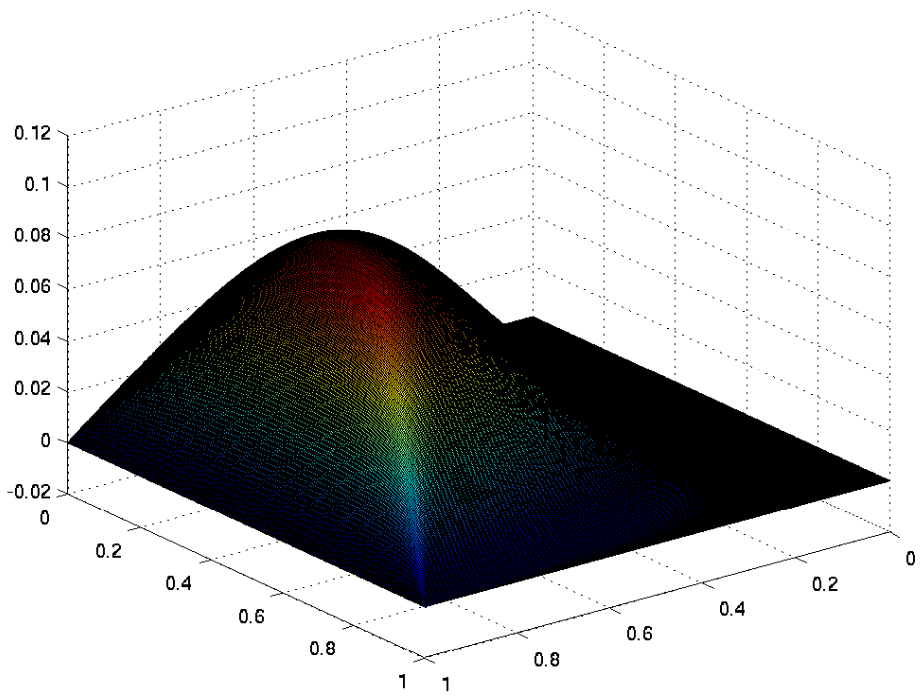
In this section, we make some remarks regarding the performance of our fast solver as compared to two other fast solvers, namely those in [3] and [6].

Our fast solver runs with complexity $O(n^2 \log n)$, that is, it solves the linear systems within $cn^2 \log n$ flops, where c is a positive number independent of n. Theoretically the multigrid algorithms in [3] have asymptotic complexity $O(n^2)$, that is, the algorithms solve the linear systems within κn^2 flops, where κ is a positive number independent of n. The constant κ (corresponding to the multigrid method) is usually quite larger than the constant c (corresponding to our fast solver). Therefore, when n is small, our solver is likely to require less flops than the multigrid algorithms. For example, if we assume $\kappa = 10c$, then as long as n is smaller than 1024, our solver out-performs the multigrid algorithms (see Table 8.30). Moreover, since our discretization scheme has sixth-order convergence, the gridsize n does not need to be large to obtain satisfactory results (see the previous numerical results in this chapter). Taking into account that the discretization method in [3] is a fourth-order method, we can see that our method

Figure 8.3: Plots of the values of u_Δ of Problem 8.2.5 from different view points



(a) Azimuth:$-37.5°$, Elevation: $30°$



(b) Azimuth:$142.5°$, Elevation: $30°$

Table 8.29: Primary vortex, stream function and vorticity for Problem 8.2.5

Our results			
n	Vortex	u	v
16	$(0.5, 0.7637696)$	0.10046440774305	-3.21080513403642
32	$(0.5, 0.7634016)$	0.10075465415085	-3.20839989288756
64	$(0.5, 0.7633632)$	0.10078249140947	-3.20753130446722
128	$(0.5, 0.7633632)$	0.10078250484808	-3.20743171740112
256	$(0.5, 0.7633632)$	0.10078190647967	-3.20741880163615
results from [6]			
n	Vortex	u	v
8	$(0.5, 0.7651)$	0.100019733	-3.21563572
16	$(0.5, 0.765)$	0.100100124	-3.21252713
32	$(0.5, 0.765)$	0.100081897	-3.21195846
64	$(0.5, 0.765)$	0.100076897	-3.21191707
128	$(0.5, 0.765)$	0.100076276	-3.21192238
Other results			
Results	Vortex	u	v
200 [29]	$(0.5, 0.76)$	0.0998	-3.2021
64 [3]	$(0.5, 0.766)$	0.10008	N/A

is expected to require less computational cost than the method in [3] to reach the same level of error, while it attains less error for the same computational cost. In Figure 8.4, we plot the computational cost of the two methods versus the error at the grid points for Problem 8.1.4. We assume that the computational cost of our method is $n^2 \log(n)$, while the cost of the method in [3] is $2n^2$, i.e. we make the rather optimistic assumption for the multigrid method that $\kappa = 2c$. Our method clearly out-performs the method in [3].

The fast solver in [6], which also runs with complexity $O(n^2 \log n)$, is based on the FFT technique and the preconditioned conjugate gradient (PCG) method, while our solver is based on the FFT technique and the PGMRES method. As far as computational effort is concerned, the main problem with the PGMRES method as compared to the PCG method is that in the mth iteration the PGMRES method needs to store all the previous $m-1$ approximation vectors and has to solve a linear least squares problem of size m. In practice, the number of iterations from the PGMRES method in our fast solver is fairly small (usually less than 9, see the previous numerical results in this chapter), so the extra cost from the PGMRES is negligible. Both PCG and PGMRES require one matrix-vector multiplication per iteration, therefore, we can say that the PCG and PGMRES iterations are of comparable cost. Moreover, the discretization scheme used in [6] has fourth-order convergence, while our discretization scheme has fifth-order convergence globally in the uniform norm and sixth-order on certain discrete points. Therefore our algorithm needs a smaller n to reach the same error than the algorithm in [6]. Hence, we believe that in general our algorithm should perform more efficiently than the algorithm in [6].

Figure 8.5 illustrates the relation between the approximation error at the grid points and the computation time for the algorithm in [6] and our fast solver, for Problem 8.1.2. Here we assume that our fast solver runs in $1.5n^2 \log n$ flops and the solver in [6] runs in $n^2 \log n$ flops. Under these assumptions, which preserve the relative performance of the methods as claimed in [6] and in this work, our method out-performs the one in [6].

Table 8.30: Numbers of flops from a multigrid solver and our solver under certain assumptions

gridsizes n	flops	
	multigrid algorithms [3] $(10cn^2)$	our solver $(cn^2 \log n)$
8	$640c$	$192c$
16	$2,560c$	$1,024c$
32	$10,240c$	$5,120c$
64	$40,960c$	$24,576c$
128	$163,840c$	$114,688c$
256	$655,360c$	$524,288c$
512	$2,621,440c$	$2,359,296c$
1024	$10,485,760c$	$10,485,760c$

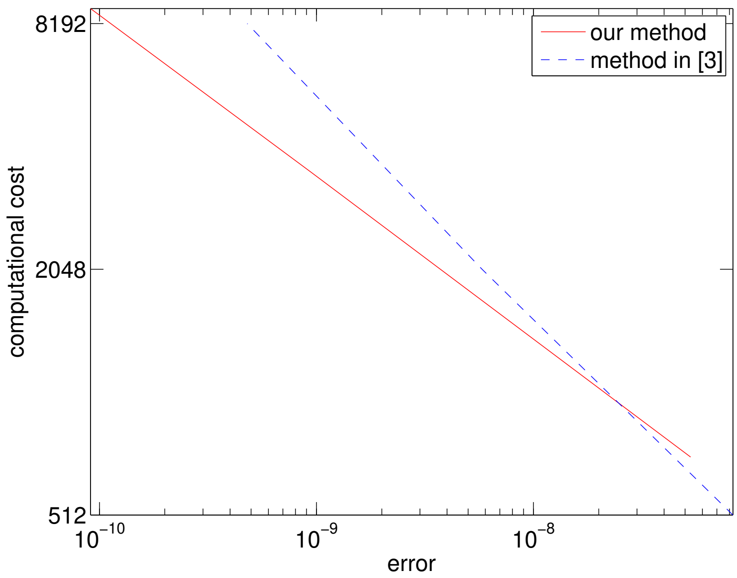


Figure 8.4: Computational cost versus error for the method in [3] and our method for Problem 8.1.4, under certain assumptions

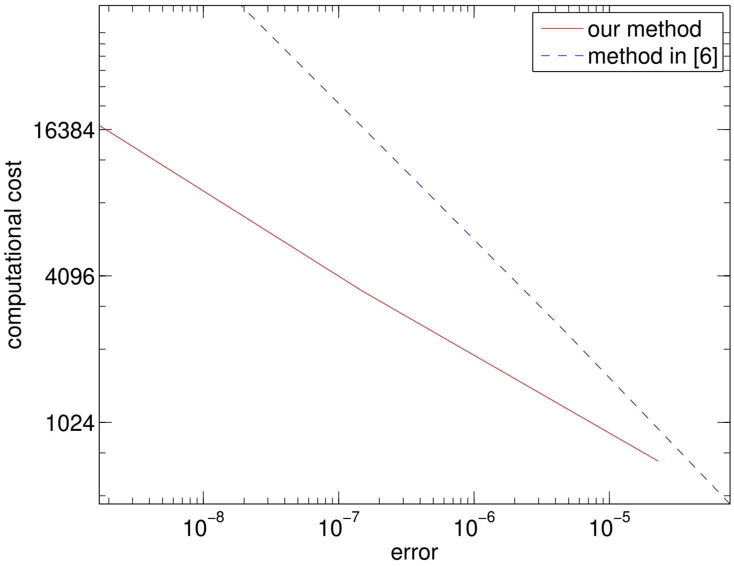


Figure 8.5: Computational cost versus error for the method in [6] and our method for Problem 8.1.2, under certain assumptions

Chapter 9

Summary and Future Work

This thesis considers the numerical solutions to the two-dimensional fourth order boundary value problems on a rectangular domain. Our approach falls in the spline collocation framework using bi-quartic splines.

We develop two optimal quartic spline collocation methods, namely, the *one-step* or *extrapolated* method, and the *three-step* or *deferred-correction* method. These methods achieve sixth order convergence at the grid points and midpoints, and fifth order convergence at other points. Some numerical results are presented to demonstrate the performance of the optimal methods for general two-dimensional fourth order boundary value problems. While a practical application is not identified yet at the moment, we believe that such practical problems solvable by the optimal methods exist. For example, certain biharmonic problem defined on certain non-rectangular domain could possibly be transformed to a more general fourth-order boundary value problem on a rectangular domain, in which case our optimal methods for general BVPs would be useful.

We then apply the optimal bi-quartic spline collocation methods to the biharmonic Dirichlet problem on a rectangular domain. A fast solver is developed for the resulting linear systems when the three-step method is applied to the biharmonic Dirichlet problem, which makes the biharmonic Dirichlet problem solvable in $O(n^2 \log n)$ flops with a gridsize

n. Moreover, a complete mathematical proof is provided for the performance of the fast solver.

A variety of numerical examples including some practical engineering problems are presented to demonstrate the accuracy of the bi-quartic spline collocation methods and the efficiency of the fast solver. In these numerical experiments, the bi-quartic spline collocation methods out-perform existing methods in accuracy with same gridsize n.

An interesting future problem is whether a fast solver can be developed for the linear system arising from the one-step method when applied to the biharmonic Dirichlet problem. Possible fast solvers may not only be an FFT-based solver, but also some preconditioned iterative solvers, such as multigrid, domain composition, etc.

Another possible piece of future research is to provide a complete proof for the conjectures used in the error analysis of the bi-quartic spline collocation methods.

Although we only consider the fourth-order PDEs with a rectangular domain, the bi-quartic spline collocation methods can be extended in two aspects. First, we can extend them to second-order PDEs. The difficulty here is that we need to make up additional conditions to uniquely determine a bi-quartic spline. In [32], optimal quartic spline collocation methods are developed for one-dimensional second order BVPs. Secondly, we can extend the bi-quartic spline collocation methods to problems on more general domains. Ng has studied quadratic and cubic spline collocation methods for problems with L- and T-shaped domains [37]. We expect we can similarly extend the current bi-quartic spline collocation methods to problems with more general domains, at least those with L- and T-shaped domains. Moreover, in [12, 11], non-uniform and adaptive collocation methods based on quadratic and cubic splines are developed and analyzed. It would be desirable to develop such methods for quartic splines, since quartic splines give higher order of convergence.

Bibliography

[1] A. Abushama and B. Bialecki. Modified nodal cubic spline collocation for Poisson's equations. *SIAM Journal on Numerical Analysis*. to appear.

[2] A. Abushama and B. Bialecki. Modified nodal cubic spline collocation for biharmonic equations. *Numerical Algorithms*, 43(4):331–353, 2006.

[3] I. Altas, J. Dym, M. M. Gupta, and R. P. Manohar. Multigrid solution of automatically generated high-order discretizations for the biharmonic equation. *SIAM Journal on Scientific Computing*, 19(5):1575–1585, 1998.

[4] M. Arad, A. Yakhot, and G. Ben-Dor. A highly accurate numerical solution of a biharmonic equation. *Numerical Methods for Partial Differential Equations*, 13:375–391, 1997.

[5] G. B. Arfken. *Mathematical methods for physicists*. Academic Press, 5th edition, 2001.

[6] B. Bialecki. A fast solver for the orthogonal spline collocation solution of the biharmonic Dirichlet problem on rectangles. *Journal of Computational Physics*, 191:601–621, 2003.

[7] P. Bjørstad. Fast numerical solution of the biharmonic Dirichlet problem on rectangles. *SIAM Journal on Numerical Analysis*, 20:59–71, 1983.

[8] B. L. Buzbee and F. W. Dorr. The direct solution of the biharmonic equation on rectangular regions and the Poisson equation on irregular regions. *SIAM Journal on Numerical Analysis*, 11:753–763, 1974.

[9] C. C. Christara. Quadratic spline collocation methods for elliptic partial differential equations. *BIT*, 34:33–61, 1994.

[10] C. C. Christara and K. S. Ng. Fast Fourier transform solvers and preconditioners for quadratic spline collocation. *BIT*, 42(4):702–739, 2002.

[11] C. C. Christara and K. S. Ng. Adaptive techniques for spline collocation. *Computing*, 76(3-4):259–277, 2006.

[12] C. C. Christara and K. S. Ng. Optimal quadratic and cubic spline collocation on nonuniform partitions. *Computing*, 76(3-4):227–257, 2006.

[13] C. C. Christara, J. Zhang, and Y. Zhu. Quartic spline collocation methods for fourth-order boundary value problems. In preparation, 2008.

[14] P. G. Ciarlet and R. Glowinski. Dual iterative techniques for solving a finite element approximation of the biharmonic equation. *Computer Methods in Applied Mechanics and Engineering*, 5:277–295, 1975.

[15] P. G. Ciarlet and P. A. Raviart. A mixed finite element method for the biharmonic equation. In C. de Boor, editor, *Proceedings of Symposium on Mathematical Aspects of Finite Elements in PDE*, pages 125–145. Academic Press, 1974.

[16] K. D. Cooper and P. M. Prenter. A coupled double slitting ADI scheme for first biharmonic using collocation. *Numerical Methods for Partial Differential Equations*, 6:321–333, 1990.

[17] J. W. Daniel and B. K. Swartz. Extrapolated collocation for two-point boundary-value problems using cubic splines. *Journal of the Institute of Mathematics and its Applications*, 16:161–174, 1975.

[18] L. W. Ehrlich. Solving the biharmonic equation as coupled finite difference equations. *SIAM Journal on Numerical Analysis*, 8:278–287, 1971.

[19] L. W. Ehrlich. Solving the biharmonic equation in a square: a direct versus semidirect method. *Communications of the ACM*, 16(11):711–714, 1973.

[20] R. Glowinski and O. Pironneau. Numerical methods for the first biharmonic equation and for the two-dimensional Stokes problem. *SIAM Review*, 21:167–212, 1979.

[21] G. H. Golub and C. F. Van Loan. *Matrix Computations*. The Johns Hopkins University Press, 3rd edition, 1996.

[22] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products*. Academic Press, Inc., 5th edition, 1994.

[23] G. H. Hardy. *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work*. New York: Chelsea, 3rd edition, 1999.

[24] W. Heinrichs. A spectral multigrid method for the stokes problem in streamfunction formulation. *Journal of Computational Physics*, 102:310–318, 1992.

[25] E. N. Houstis, C. C. Christara, and J. R. Rice. Quadratic spline collocation methods for two-point boundary value problems. *International Journal for Numerical Methods in Engineering*, 26(4):935–952, April 1988.

[26] E. N. Houstis, J. R. Rice, C. C. Christara, and E. A. Vavalis. Performance of scientific software. *The IMA Volumes in Mathematics and its applications*, 14:123–155, 1988. Mathematical Aspects of Scientific Software, J. R. Rice, ed.

[27] E. N. Houstis, E. A. Vavalis, and J. R. Rice. Convergence of an $O(h^4)$ cubic spline collocation method for elliptic partial differential equations. *SIAM Journal on Numerical Analysis*, 25(1):54–74, 1988.

[28] M. Irodotou-Ellina and E. N. Houstis. An $O(h^6)$ quintic spline collocation method for fourth order two-point boundary value problems. *BIT*, 28(2):288–301, 1988.

[29] M. A. Kelmanson. Modified integral equation solution of viscous flows near sharp corners. *Computers and Fluids*, 11(4):307–324, 1983.

[30] D. B. Knudson. *A piecewise Hermite bicubic finite element Galerkin method for the biharmonic Dirichlet problem.* PhD thesis, Colorado School of Mines, Golden, Colorado, U.S.A., 1997.

[31] Y. Kwon, R. Manohar, and J. W. Stephenson. Single cell fourth order methods for the biharmonic equation. *Congressus Numerantium*, 34:475–482, 1982.

[32] G. Liu. Quartic-spline collocation methods for second-order two-point boundary value problems. Master's thesis, Department of Computer Science, University of Toronto, August 2007.

[33] C. F. Van Loan. *Computational Frameworks for the Fast Fourier Transform.* Society for Industrial and Applied Mathematics, 1992.

[34] Z.-M. Lou, B. Bialecki, and G. Fairweather. Orthogonal spline collocation methods for biharmonic problems. *Numerische Mathematik*, 80:267–303, 1998.

[35] J. W. McLaurin. A general coupled equation approach for solving the biharmonic boundary value problem. *SIAM Journal on Numerical Analysis*, 11:14–33, 1974.

[36] V. V. Meleshko. Selected topics in the history of the two-dimensional biharmonic problem. *Applied Mechanics Reviews*, 56(1):33–85, 2003.

[37] K. S. Ng. *Spline Collocation on Adaptive Grids and Non-Rectangular Domains*. PhD thesis, Department of Computer Science, University of Toronto, June 2005.

[38] P. Peisker. On the numerical solution of the first biharmonic equation. *Mathematical Modelling and Numerical Analysis*, 22:655–676, 1988.

[39] Y. Saad. *Iterative methods for sparse linear systems*. PWS, 1996.

[40] J. W. Stephenson. Single cell discretizations of order two and four for biharmonic problems. *Journal of Computational Physics*, 55:65–80, 1984.

[41] W. Sun. Orthogonal collocation solution of biharmonic equations. *International Journal of Computer Mathematics*, 49:21–232, 1993.

[42] S. Timoshenko and S. Woinowsky-Krieger. *Theory of Plates and Shells*. McGraw-Hill, New York, 1959.

[43] Y. Zhu. Quartic-spline collocation methods for fourth-order two-point boundary value problems. Master's thesis, University of Toronto, Toronto, Ontario, Canada, 2001. `http://www.cs.toronto.edu/NA/reports.html#yz-01-msc`.