Penalty and Penalty-Like Methods for Nonlinear HJB PDEs

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Abstract

There are numerous financial problems that can be posed as optimal control problems, leading to Hamilton-Jacobi-Bellman or Hamilton-Jacobi-Bellman-Issacs equations. We reformulate these problems as nonlinear PDEs, involving max and/or min terms of the unknown function, and/or its first and second spatial derivatives. We suggest efficient numerical methods for handling the nonlinearity in the PDE through an adaptation of the discrete penalty method [12] that gives rise to tridiagonal penalty matrices. We formulate a penalty-like method for the use with European exercise rights, and extend this to American exercise rights resulting in a double-penalty method. We also use our findings to improve the policy iteration algorithms described in [11]. Numerical results are provided showing clear second-order convergence, and where applicable, we prove the convergence of our algorithms.

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1 Introduction

There are a number of nonlinear adjustments to the Black-Scholes model [5] that can be posed as optimal control problems. Some examples that we examine in this paper are stock borrowing fees [10], stock borrowing fees with American options [11], and the family of transaction cost models introduced in [21], which is a model that has been criticized and refined in [30, 23, 2, 31].

Optimal control problems can be formulated as nonlinear Hamilton-Jacobi-Bellman (HJB) Partial Differential Equations (PDEs). We are interested in the numerical solution of these equations. Previous work involves a kind of policy iteration with Crank-Nicolson timestepping [11] and with Fully Implicit timestepping [14]. Similar methods to solve HJB Variational Inequalities (VIs) have also used policy iteration with Fully Implicit timestepping [27]. In recent years, there is a trend towards the use of deep learning [15] and neural network approaches [7] for the numerical solution, especially for high-dimensional problems.

We express the HJB problems as nonlinear PDE problems without involvement of controls. We directly discretize the nonlinear PDEs. The nonlinear term in the PDE may involve the unknown function, and/or its first and second spatial derivatives. We use second-order finite differences for spatial discretization and Crank-Nicolson-Rannacher [26] timestepping. For handling the nonlinearity, we use penalty-like methods, which, for the spatial derivatives, give rise to tridiagonal matrices. These penalty methods can be combined with the penalty iteration [12] for American exercise rights, giving rise to double-penalty iteration methods. Motivated by the penalty methods, we revisit the policy iteration method for the controlled HJB problems and suggest improvements for the case of American exercise rights. The improved policy iteration algorithms are designed to be mathematically equivalent to the penalty iteration ones. Numerical experiments show the second-order convergence of our methods.

As a terminology clarification, we note that, while the terms penalty iteration and policy iteration may often be used interchangeably in the literature, in this paper, when we refer to penalty iteration methods, we assume there are no controls and we apply the methods to penalized PDEs, while, policy iteration methods are applied to the controlled HJB PDEs.

The main contributions of this paper are

1. We present novel techniques of handling nonlinearities involving first and second spatial derivatives of the unknown function, and the use of multiple penalty and penalty-like matrices, similar to [6]. However, in this work, we handle nonlinearities associated not just with the unknown value function as is done in [6], but also with its first and second spatial derivatives.
2. We discuss the advantages that these new penalty and penalty-like methods have over the policy iteration [11] algorithms.
3. We prove the monotonic convergence of our numerical methods where applicable.
4. We introduce a new type of policy iteration algorithms for problems with American-style exercise rights, that have improved order of convergence and allow the use of variable time stepsizes, leading to more accurate and efficient solutions, when compared to the methods in [11].
5. In our penalty and policy iteration algorithms, we introduce stopping criteria that avoid oscillatory iterate behaviour in certain degenerate cases.
6. We provide numerical results on a variety of problems, including transaction cost models developed in [21], and give results for the model introduced in [31].
7. We extend the penalty iteration algorithm to the passport option pricing problem, which is a continuous control problem.

The organization of the paper is as follows. In Section 2, we present the controlled and the penalized PDE expressions of the problems considered. In Section 3, after describing the time and space discretization for the PDEs considered, we introduce the policy and penalty iteration methods, as well as the penalty matrices for each nonlinear problem considered. In Section 4, we present numerical experiments on each problem, and compare the policy and penalty iteration methods. In Section 5, we extend the penalty iteration method to a continuous control problem. Section 6 summarizes our conclusions. Section 7 acts as an Appendix and presents the study of convergence of the penalty iteration methods introduced, and convergence to the viscosity solution.

2 Problem formulation

In this section, we show both the control and the nonlinear PDE representations of the problems we are interested in. The PDE models that we examine are nonlinear extensions of the original Black-Scholes model [5] for European options.

In the context of nonlinear PDEs, the problems we are solving are of the form

$$V_\tau = \frac{\sigma^2 S^2}{2} V_{SS} + rSV_S - rV + \text{nonlinear terms}, \quad (1)$$

where, in the maximization form,

$$\text{nonlinear terms} = \max\{\mathcal{L}_1 V, \mathcal{L}_2 V, 0\} + \rho \max\{V^* - V, 0\} \tag{2}$$

with $\mathcal{L}_i$ being linear second order differential operators, ρ given by

$$\rho = \begin{cases} 1/\epsilon & \text{for American options} \\ 0 & \text{for European options,} \end{cases} \tag{3}$$

V^* the initial condition, and $0 < \epsilon \ll 1$, while the minimization form replaces the first $\max$ in (2) with $\min$. In the specific problems we solve, there are up to two $\mathcal{L}_i$ operators, but the methods proposed are not restricted by the number of $\mathcal{L}_i$'s.

For convenience, we define the differential operator $\mathcal{L}$ to represent the Black–Scholes operator

$$\mathcal{L}(r,\sigma)V \equiv \frac{\sigma^2 S^2}{2} V_{SS} + rSV_S - rV. \tag{4}$$

In the context of control problems, the problem we are solving in the maximization form is

$$V_\tau = \sup_{\mu\in\{0,1\}} \sup_{Q\in\hat{Q}} \left\{a(S,\tau,Q)V_{SS} + b(S,\tau,Q)V_S + c(S,\tau,Q)V + d(V,\mu)\right\}, \tag{5}$$

where Q denotes the control variables which can be one or more of $\{q_i, i = 1,2,3\}$, μ is another control variable used only when American exercise rights are considered, and $d(V,\mu)$ is defined by

$$d(V,\mu) = \begin{cases} \mu(V^* - V)/\epsilon & \text{for American options} \\ 0 & \text{for European options,} \end{cases} \tag{6}$$

while the minimization form replaces the second $\sup$ in (5) with a $\inf$. Clearly, when European exercise rights are considered, the first sup disappears. Again for convenience, we define an operator $\mathcal{J}$ similar to $\mathcal{L}$

$$\mathcal{J}(Q)V \equiv a(S,\tau,Q)V_{SS} + b(S,\tau,Q)V_S + c(S,\tau,Q)V. \tag{7}$$

Next, we give a brief description of each specific problem.

2.1 Stock Borrowing Fees problem

We extend the borrowing/lending problem from [4] to include stock borrowing fees, denotes by r_f. We assume $r_b > r_l > r_f$. The problem is described in detail in [10] and [11]. We have

$$V_\tau = \sup_Q \left\{ \frac{\sigma^2 S^2}{2} V_{SS} + q_3 q_1 (SV_S - V) + (1-q_3)((r_l - r_f)SV_S - q_2 V) \right\} \tag{8}$$

with $Q = (q_1, q_2, q_3)$, $q_1 \in \{r_l, r_b\}$, $q_2 \in \{r_l, r_b\}$, $q_3 \in \{0,1\}$ for the short position.

The corresponding nonlinear PDE is given by

$$V_\tau = \frac{\sigma^2 S^2}{2} V_{SS} + r_l(SV_S - V) + \max\{(r_b - r_l)(SV_S - V), -r_f SV_S, 0\} \tag{9}$$

for the short position and

$$V_\tau = \frac{\sigma^2 S^2}{2} V_{SS} + r_b(SV_S - V) + \min\{(r_l - r_b)(SV_S - V), -(r_b - r_l + r_f)SV_S, 0\} \tag{10}$$

for the long position.

2.2 Stock Borrowing Fees problem with American exercise rights

When we add an American early exercise feature to the problem described in Section 2.1, we get

$$V_\tau = \sup_{\mu,Q}\left\{\frac{\sigma^2 S^2}{2}V_{SS} + q_3 q_1(SV_S - V) + (1-q_3)((r_l - r_f)SV_S - q_2 V) + \mu\frac{V^* - V}{\epsilon}\right\} \tag{11}$$

where V^* denotes the payoff, $0 < \epsilon \ll 1$, $Q = (q_1, q_2, q_3)$, $q_1 \in \{r_l, r_b\}$, $q_2 \in \{r_l, r_b\}$, $q_3 \in \{0, 1\}$ and $\mu \in \{0, 1\}$.

The long position, which is claimed to be more interesting in [11], involves a $\sup\inf$

$$V_\tau = \sup_{\mu}\inf_{Q}\left\{\frac{\sigma^2 S^2}{2}V_{SS} + q_3 q_1(SV_S - V) + (1-q_3)((r_l - r_f)SV_S - q_2 V) + \mu\frac{V^* - V}{\epsilon}\right\}, \tag{12}$$

which makes it an HJBI (Hamilton-Jacobi-Bellman-Issacs) equation, corresponding to a stochastic game, again with $0 < \epsilon \ll 1$, $Q = (q_1, q_2, q_3)$, $q_1 \in \{r_l, r_b\}$, $q_2 \in \{r_l, r_b\}$, $q_3 \in \{0, 1\}$ and $\mu \in \{0, 1\}$.

The corresponding nonlinear PDEs are given by

$$V_\tau = \frac{\sigma^2 S^2}{2}V_{SS} + r_l(SV_S - V) + \max\{(r_b - r_l)(SV_S - V), -r_f SV_S, 0\} + \rho\max\{V^* - V, 0\} \tag{13}$$

for the short position, and

$$V_\tau = \frac{\sigma^2 S^2}{2}V_{SS} + r_b(SV_S - V) + \min\{(r_l - r_b)(SV_S - V), -(r_b - r_l + r_f)SV_S, 0\} + \rho\max\{V^* - V, 0\} \tag{14}$$

for the long position, where $\rho = \epsilon^{-1}$.

2.3 Transaction Cost model

Transaction cost models have been introduced in [21], and have been extensively debated and refined in [17, 32, 22, 30]. It is beyond the scope of the paper to discuss the merits and drawbacks of these models; we use the model given in [31], which is based on [30]. The model is given by the nonlinear PDE

$$V_\tau = \frac{\sigma^2 S^2}{2}V_{SS} + rSV_S - rV - \kappa S^2|V_{SS}| \tag{15}$$

where κ is the transaction cost.

Equation (15) can equivalently be expressed as

$$V_\tau = \frac{\sigma^2 S^2}{2}V_{SS} + rSV_S - rV + \min\{\kappa S^2 V_{SS}, -\kappa S^2 V_{SS}\}. \tag{16}$$

For a Transaction Cost model with American exercise rights, Yousuf [31], following the logic in [12], gives

$$V_\tau = \frac{\sigma^2 S^2}{2}V_{SS} + rSV_S - rV + \min\{\kappa S^2 V_{SS}, -\kappa S^2 V_{SS}\} + \rho\max\{V^* - V, 0\}, \tag{17}$$

where ρ has the same functionality as the one used for American options in [12].

For the cases where the payoff V^* is convex, $V_{SS} \geq 0$, and Equation (16) becomes

$$V_\tau = \left(\frac{\sigma^2}{2} - \kappa\right) S^2 V_{SS} + rSV_S - rV, \tag{18}$$

a type of Black-Scholes PDE; and Equation (17) becomes

$$V_\tau = \left(\frac{\sigma^2}{2} - \kappa\right) S^2 V_{SS} + rSV_S - rV + \rho \max\{V^* - V, 0\}, \tag{19}$$

a penalized PDE for American options, as given in [12]. In both cases, the volatility is adjusted and the adjusted volatility σ' is given by

$$\sigma' = \sqrt{\sigma^2 - 2\kappa}, \tag{20}$$

where we also impose the requirement that $\kappa < \sigma^2/2$, to ensure a positive coefficient of the diffusion term. (On the other hand, if the payoff V^* is concave, $V_{SS} \leq 0$, and $\sigma' = \sqrt{\sigma^2 + 2\kappa}$.)

We also derive a control problem for the Transaction Cost model

$$V_\tau = \inf_{q \in \{-\kappa, \kappa\}} \left\{ \left(\frac{\sigma^2}{2} + q\right) S^2 V_{SS} + rSV_S - rV \right\} \tag{21}$$

and American exercise rights can be added in the same fashion as in the Stock Borrowing Fee problem

$$V_\tau = \sup_{\mu \in \{0,1\}} \inf_{q \in \{-\kappa, \kappa\}} \left\{ \left(\frac{\sigma^2}{2} + q\right) S^2 V_{SS} + rSV_S - rV + \mu \frac{V^* - V}{\epsilon} \right\}. \tag{22}$$

3 Numerical methods

In this section, we discuss specific numerical methods, focusing on the treatment of nonlinear terms.

3.1 Discretization

We describe the discretization of the various specific forms of Equations (1) and (5) here. The semi-infinite domain $[0, \infty)$ on S is truncated to $[0, S_{\max}]$ for some sufficiently large $S_{\max}$. We use standard second-order finite differences for the spatial discretization and Crank-Nicolson-Rannacher timestepping [26] for the temporal discretization. Let $\tau_j, j = 0, \ldots, M$, be the timesteps at which the solution is computed, with $\tau_0 = 0 < \tau_1 < \cdots < T$. When the timesteps are uniform, the stepsize is $\Delta\tau = T/M$, but we also present results with variable (adaptive) stepsizes. Let $S_i, i = 0, \ldots, N$, be the (uniform or nonuniform) spatial gridpoints in $[0, S_{\max}]$.

Where V^* is a put or straddle payoff (i.e. $\max\{K - S, 0\}$ or $\max\{S - K, K - S\}$, we use the nonuniform grid described by a smooth mapping function in [9]. This ensures that there is a node on the sole cusp point which is the strike price K, a requirement for smooth convergence [24]. Since the nonuniform gridpoints are generated by a smooth mapping of uniform points, the finite difference approximations of the second derivatives are second order. The mapping function $w(x)$ is defined by

$$S_i \equiv w(x_i) = \left(1 + \frac{\sinh(\beta(x_i/x_N - \alpha))}{\sinh(\beta\alpha)}\right) K \tag{23}$$

where $x_i, i = 0, \ldots, N$, are uniform gridpoints in $[0, S_{\max}]$, α is a parameter that controls the density of the gridpoints in the interval $[0, K]$, and β ensures that $w(S_{\max}) = S_{\max}$ and is numerically solved. In our examples we choose $\alpha = 0.37$.

Where V^* is a butterfly spread payoff, we define a nonuniform grid with N subintervals where $\gamma N + 1$ of the points are allocated in the interval $[K - 2a, K + 2a]$ uniformly. To ensure a gridpoint on the three strike prices $K, K \pm a$, we pick γ so that γN is a multiple of four. The remaining points are allocated uniformly on the intervals $[0, K - 2a)$ and $(K + 2a, S_{\max}]$ with the distribution of points proportional to the lengths of the intervals. For the butterfly spread payoff, the grid arising is not smooth at $K \pm 2a$, and there is no guarantee that the finite difference approximations of the second derivatives are second order. However, the points of non-smoothness are away from the "interesting area", and, as will be seen in the numerical experiments, the grid turns out to be effective for different problems with the butterfly spread payoff. In our examples, we choose $\gamma = 0.20$.

The spatial grid arising from the mapping (23) is often used in financial problems, and the one used for the butterfly spread payoff is simple and turns out to be effective in the numerical experiments. We remark that other nonuniform grids as well as adaptive grids, such as those in [8], could be used and potentially further improve the performance of the methods, but this is not the focus of the paper.

Let v^j denote the computed solution vector arising from the approximate values of V at the spatial gridpoints at time τ_j, with v^0 being the initial condition vector. Since we use an iteration method to handle the nonlinearity, let $v^{j,k}, k = 0, \ldots, maxit$, denote the computed solution vector at iteration k of timestep j, with $maxit$ the maximum number of iterations permitted. In the context of policy iteration algorithms, let $A(Q)$ be the matrix arising from the space discretization of $\mathcal{J}(Q)$, and, in the case of American exercise rights, let $R(\mu)$ be a diagonal matrix $R(\mu) = diag\{\mu_i/\epsilon, i = 1, \ldots, N\}$, where, in the discrete equations, μ is a vector of controls $\mu_i \in \{0, 1\}$. Note that $R = 0$, when the exercise rights are European. Where the control Q has already been computed, the notation A^j and R^j are understood to be $A(Q^j)$ and $R(\mu^j)$. In the context of penalty iteration algorithms, let A be the constant matrix arising from the space discretization of $\mathcal{L}(r, \sigma)$.

For the boundary conditions at $S = 0$ and $S = S_{\max}$, we impose Dirichlet boundary conditions at $S = 0$ and linear boundary conditions at $S = S_{\max}$, and use one-sided finite differences to discretize the first derivative at that endpoint. Thus, we have N unknowns per timestep. However, in Section 7, for the analysis of the matrix properties, we assume that Dirichlet boundary conditions are used on both endpoints; this ensures that the discretization matrix is diagonally dominant. Numerical tests have shown that the solution does not vary substantially based on the type of boundary condition used.

3.2 Policy iteration

The first policy iteration method we use is based on [11], with some small improvements resulting in Algorithm 1. Specifically, we use the discretization methods described in the previous section (which can be considered equivalent to "Q-independent discretization" in [11]), we use the nonuniform grids mentioned earlier, and we use the equivalence of controls as the stopping criterion, to avoid incurring excessive computational cost. In Section 3.6, we do further improvements to the policy iteration that result in substantially more efficient algorithm when American exercise rights are present.

The stopping criterion used is the equivalence of controls. When

$$Q^{j,k} = Q^{j,k-1} \text{ and } \mu^{j,k} = \mu^{j,k-1} \tag{24}$$

Algorithm 1 Policy iteration for HJB PDEs at step j with θ timestepping

Require: Solve $(I-\theta\Delta\tau(A^j-R^j))v^j = g^{j-1}+\theta\Delta\tau R^j v^*$
where $g^{j-1}=(I+(1-\theta)\Delta\tau(A^{j-1}-R^{j-1}))v^{j-1}+(1-\theta)\Delta\tau R^{j-1}v^*$
subject to $Q_i^j = \arg\sup_{Q\in\hat{Q}}[A(Q)v^j]_i$ and $\mu_i^j = \arg\sup_{\mu\in\{0,1\}}[R(\mu)(v^*-v)]_i$
Initialize $v^{j,0}=v^{j-1}$, $\mu^{j,0}=\mu^{j-1}$, and $Q^{j,0}=Q^{j-1}$
for $k=1,\ldots,maxit$ **do**
  Solve $(I-\theta\Delta\tau(A^{j,k-1}-R^{j,k-1}))v^{j,k} = g^{j-1}+\theta\Delta\tau R^{j,k-1}v^*$
  Compute $Q_i^{j,k}=\arg\sup_{Q\in\hat{Q}}[A(Q)v^{j,k}]_i$, $\mu_i^{j,k}=\arg\sup_{\mu\in\{0,1\}}[R(\mu)(v^*-v^{j,k})]_i$
  if stopping criterion satisfied **then**
    Break
  end if
end for
Set $v^j=v^{j,k}$, $\mu^j=\mu^{j,k}$, and $Q^j=Q^{j,k}$

we terminate the iteration.

A convergence proof is provided in [28] and an example is given that the number of iterations taken is proportional to the number of controls (which is essentially the worst case, based on brute force search). However, we note that in initial value problems where the solution is continuous (as is the case here) the computed solution at the current timestep is always a close estimate of the solution at the next timestep, we never have a cold start as assumed in [28], hence, we do not need to worry about worst-case convergence.

3.3 Penalty matrices

To address the discretization of nonlinear terms in Equations (9), (10), (13), (14), (16), and (17), we use penalty matrices defined in the spirit of those used in [6] and [12]. However, there are two important differences:

1. We do not introduce a large penalty parameter such as ρ in [12] when handling the $\max\{\mathcal{L}_1V,\mathcal{L}_2V,0\}$ and $\min\{\mathcal{L}_1V,\mathcal{L}_2V,0\}$ terms. The large penalty parameter ρ in [12] is introduced to capture the nonlinearity in the Linear Complementarity Problem of the American option valuation. So strictly speaking, the matrices we introduce when handling the $\max\{\mathcal{L}_1V,\mathcal{L}_2V,0\}$ and $\min\{\mathcal{L}_1V,\mathcal{L}_2V,0\}$ are not penalty matrices but rather penalty-like (herein referred to as simply penalty) matrices. In this respect, these penalty matrices are rather similar to those used in [6] to address nonlinearity arising from different types of valuation adjustments. However, we note that the penalty matrix in [6] addresses only the V nonlinearity, while in this paper we present penalty matrices that address the nonlinearity in any derivative term (V_{SS}, V_S) as well.

2. Our penalty matrices are not diagonal matrices but tridiagonal matrices; this allows us to handle the nonlinear terms $\max\{\mathcal{L}_1V,\mathcal{L}_2V,0\}$ and $\min\{\mathcal{L}_1V,\mathcal{L}_2V,0\}$ involving the partial derivatives V_S and V_{SS}, such as $\max\{V-SV_S,0\}$ in the Stock Borrowing Fee problem described in Section 2.1, and $\min\{\kappa S^2V_{SS},-\kappa S^2V_{SS}\}$ in the Transaction Cost model in Section 2.3. Since the discretization matrix A is tridiagonal, this doesn't change the sparsity structure of the linear system we are solving. We will also show under what conditions adding the penalty matrix to A doesn't change its diagonal dominance or monotonicity.

To make it easier to define the penalty matrices used in the nonlinear iteration, let T_1 and T_2 denote the $N \times N$ matrices that compute the first and second derivatives from centered finite differences, and D_S denote a diagonal matrix of size N with the entries being the gridpoints S_i. Accordingly, D_S^2 is a diagonal matrix of size $N+1$ with the entries being S_i^2. Thus, the matrices $D_S T_1$ and $D_S^2 T_2$ discretize SV_S and $S^2 V_{SS}$, respectively. Let also I denote the identity matrix of appropriate dimension.

3.3.1 *Stock Borrowing Fees*

For the PDE problems (9) and (10), we introduce penalty matrices to handle the nonlinear terms $\max\{(r_b - r_l)(SV_S - V), -r_f SV_S, 0\}$ and $\min\{(r_l - r_b)(SV_S - V), -(r_b - r_l + r_f)SV_S, 0\}$, respectively.

For the short position, let $P_1 = (r_b - r_l)(D_S T_1 - I)$ and $P_2 = -r_f D_S T_1$ be the tridiagonal matrices arising from the discretization of $(r_b - r_l)(SV_S - V)$ and $-r_f SV_S$, respectively. Define the tridiagonal penalty matrix $P = P(v^j)$ by

$$P_{i,:} = \begin{cases} 0 & \text{if } [P_1 v^j]_i \le 0 \text{ and } [P_2 v^j]_i \le 0 \\ [P_1]_{i,:} & \text{if } [P_1 v^j]_i > 0 \text{ and } [P_1 v^j]_i > [P_2 v^j]_i \\ [P_2]_{i,:} & \text{if } [P_2 v^j]_i > 0 \text{ and } [P_1 v^j]_i \le [P_2 v^j]_i. \end{cases} \tag{25}$$

where the colon notation $P_{i,:}$ means the entire i-th row of P and is borrowed from MATLAB. Thus, P is constructed so that it discretizes $\max\{(r_b - r_l)(SV_S - V), -r_f SV_S, 0\}$, with the first branch of (25) corresponding to the case that $\max\{(r_b - r_l)(SV_S - V), -r_f SV_S, 0\} = 0$, the second branch to the case that $\max\{(r_b - r_l)(SV_S - V), -r_f SV_S, 0\} = (r_b - r_l)(SV_S - V)$, and the third branch to the case that $\max\{(r_b - r_l)(SV_S - V), -r_f SV_S, 0\} = -r_f SV_S$.

For the long position, let $P_1 = (r_l - r_b)(D_S T_1 - I)$ and $P_2 = -(r_b - r_l + r_f)D_S T_1$ be the tridiagonal matrices arising from the discretization of $(r_l - r_b)(SV_S - V)$ and $-(r_b - r_l + r_f)SV_S$, respectively. Define the tridiagonal penalty matrix $P = P(v^j)$ by

$$P_{i,:} = \begin{cases} 0 & \text{if } [P_1 v^j]_i \ge 0 \text{ and } [P_2 v^j]_i \ge 0 \\ [P_1]_{i,:} & \text{if } [P_1 v^j]_i < 0 \text{ and } [P_1 v^j]_i < [P_2 v^j]_i \\ [P_2]_{i,:} & \text{if } [P_2 v^j]_i < 0 \text{ and } [P_1 v^j]_i \ge [P_2 v^j]_i. \end{cases} \tag{26}$$

Again, P is constructed so that it discretizes $\min\{(r_l - r_b)(SV_S - V), -(r_b - r_l + r_f)SV_S, 0\}$.

3.3.2 *American exercise rights*

When the American exercise rights are added, the penalty matrix $P_A = P_A(v^j)$ is borrowed from [12]. The penalty matrix P_A is diagonal and is defined by

$$[P_A(v^j)]_{i,i} = \begin{cases} \rho & \text{if } v_i^j < V_i^* \\ 0 & \text{otherwise.} \end{cases} \tag{27}$$

The use of this matrix, as we shall see in Algorithm 3, is the same as the use in [12], and ρ is a sufficiently large positive penalty parameter.

3.3.3 *Transaction Cost*

The tridiagonal penalty matrix $P = P(v^j)$ to handle the term $\min\{\kappa S^2 V_{SS}, -\kappa S^2 V_{SS}\}$ in (16) and (17) is defined by

$$P_{i,:} = \begin{cases} -\kappa[D_S^2 T_2]_{i,:} & \text{if } [D_S^2 T_2 v^j]_i > 0 \\ \kappa[D_S^2 T_2]_{i,:} & \text{otherwise,} \end{cases} \tag{28}$$

It is easily seen that Pv discretizes $-\kappa S^2 |V_{SS}|$, or, equivalently, $\min\{\kappa S^2 V_{SS}, -\kappa S^2 V_{SS}\}$. Although not explicitly referring to penalty matrices, the non-smooth Newton's method in [13] gives rise to the same matrice P presented in (28).

REMARK 1 *Although we presented the penalty matrices for two particular nonlinear problems, the definition of the penalty matrices can be extended in a straightforward way to all nonlinear PDEs of the form (1)-(2).*

3.4 Penalty iteration

Now, with the definition of the penalty and penalty-like matrices complete, we are ready to give a general algorithm encompassing Equations (9), (10), (13), (14), (16), and (17).

Let P denote the penalty-like matrix arising from the terms $\max\{\mathcal{L}_1 V, \mathcal{L}_2 V, 0\}$ or $\min\{\mathcal{L}_1 V, \mathcal{L}_2 V, 0\}$. Then, the general form of the penalty iteration algorithm becomes Algorithm 2.

Algorithm 2 Tridiagonal penalty iteration at step j, with θ-timestepping

Require: Solve $[(I - \theta\Delta\tau(A + P(v^j)))]v^j = g^j$
  where $g^j = (I + (1-\theta)\Delta\tau(A + P(v^{j-1})))v^{j-1}$

Initialize $v^{j,0} = v^{j-1}$ and $P^{j,0} = P(v^{j-1})$
for $k = 1, \ldots, maxit$ **do**
  Solve $[I - \theta\Delta\tau(A + P^{j,k-1})]v^{j,k} = g^j$
  if first stopping criterion satisfied **then**
    Break
  end if
  Compute $P^{j,k} = P(v^{j,k})$
  if second stopping criterion satisfied **then**
    Break
  end if
end for
Set $v^j = v^{j,k}$

The first stopping criterion is

$$\max_i\{\frac{|v_i^{j,k} - v_i^{j,k-1}|}{\max(scale, |v_i^{j,k}|)}\} < tol. \tag{29}$$

The second stopping criterion is

$$\max_i\{\frac{|[P^{j,k-1}v^{j,k} - P^{j,k}v^{j,k}]_i|}{\max(scale, |[P^{j,k}v^{j,k}]_i|)}\} < tol. \tag{30}$$

Typical values are $scale = 1$ and $tol = 10^{-6}$.

When either stopping criterion is satisfied, the iteration is terminated.

The second stopping criterion is analogous to the equivalence of controls for the policy iteration. The motivation behind the second stopping criterion is that with only testing equivalence of penalty matrices (i.e. $P^{j,k-1} = P^{j,k}$) the penalty matrices could be substantially different from each other while the values that they compute (Pv) do not differ by much. This could lead to oscillations when the penalty matrices are generated, causing the iterations to continue forever.

The first stopping criterion is borrowed from [11] and is used as a safeguard. In practice, we found that removing this criterion does not affect the number of iterations by much, but it may be useful to have, as we avoid an extra computation of the penalty matrix, whenever this criterion takes into effect.

Under certain conditions, Algorithm 2 monotonically converges to the unique solution in a finite number of iterations. For a proof of this, see Section 7.

3.5 Double penalty iteration for American exercise rights

In the case of American exercise rights, such as the Stock Borrowing Fees with American exercise rights problem represented by Equations (13) and (14), or the Transaction Cost model with American exercise rights represented by Equation (17), Algorithm 3 is used. This algorithm applies to all problems with American instead of European exercise rights, and all of the results for this problem, such as monotonicity of iterates leading to convergence and the use of variable timesteps to restore the convergence rate to 2.0, still apply. Therefore, Algorithm 3 is written in a general form to reflect this potential.

Algorithm 3 Tridiagonal double-penalty iteration at step j, with θ-timestepping

```
Require: Solve [(I − θΔτ(A + P(v^j))) + P_A(v^j)]v^j = g^j + P_A(v^j)v^*
    where g^j = (I + (1 − θ)Δτ(A + P(v^{j−1})))v^{j−1}
Initialize v^{j,0} = v^{j−1}, P^{j,0} = P(v^{j−1}), and P_A^{j,0} = P_A(v^{j−1})
for k = 1, ..., maxit do
    Solve [(I − θΔτ(A + P^{j,k−1})) + P_A^{j,k−1}]v^{j,k} = g^j + P_A^{j,k−1}v^*
    if first stopping criterion satisfied then
        Break
    end if
    Compute P^{j,k} = P(v^{j,k}), P_A^{j,k} = P_A(v^{j,k})
    if second stopping criterion satisfied then
        Break
    end if
end for
Set v^j = v^{j,k}
```

The first stopping criterion is identical to Condition (29). The second stopping criterion is

$$\max_i \{ \frac{|[P^{j,k}v^{j,k} - P^{j,k-1}v^{j,k}]_i|}{\max(scale, |[P^{j,k}v^{j,k}]_i|)} \} < tol \text{ and } P_A^{j,k} = P_A^{j,k-1}. \tag{31}$$

When either condition (29) or condition (31) is satisfied the iteration is terminated. The motivation for the two criteria remains the same as for Algorithm 2. Note that the discussion earlier about oscillations in the computed penalty matrix does not apply to P_A, since P_A is diagonal and thus we simply use equivalence of penalty matrices.

For the short position of the Stock Borrowing Fees problem, we can prove that the algorithm converges monotonically in a finite number of iterations to the unique solution. For a proof of this, see Section 7. The long position does not allow such a proof, because the penalty matrix P causes the values of the iterates to decrease while the American penalty matrix P_A causes the values of the iterates to increase on each iteration. The same reasoning applies to the American Transaction Cost model, where the differing sign between P_A and P ensures that there is no monotone convergence for positive transaction cost.

3.6 Improved policy iteration for American exercise rights

As we will see in the numerical experiments, when problems with American exercise rights are solved with the policy iteration, the solution exhibits slightly reduced orders of convergence. In the context of penalty methods for American options [12], this issue is resolved with variable timestepping. However, with Algorithm 1, the technique of variable timestepping did not work well. We present an improved policy iteration algorithm which, for European exercise rights, is equivalent to Algorithm 1, but for American exercise rights, results in stable second order convergence and allows for variable timestepping.

The motivation can be found by analyzing the difference between the penalty iteration algorithm [12] and the policy iteration algorithm in [11] for American options. It can be determined that the maximization step computing μ and, therefore, $R(\mu)$, in the policy algorithm, is equivalent to calculating the penalty matrix P_A in the penalty algorithm; however, the difference lies in the linear system solved at each iteration. It can be shown that solving for the next iterate $v^{j,k}$ in the policy iteration of Algorithm 1 is solving

$$(I-\theta\Delta\tau A+\theta\Delta\tau R^{j,k-1})v^{j,k} = (I+(1-\theta)\Delta\tau A)v^{j-1}+\theta\Delta\tau R^{j,k-1}v^*+(1-\theta)\Delta\tau R^{j-1}(v^*-v^{j-1}). \quad (32)$$

In contrast, the penalty iteration algorithm [12] solves

$$(I - \theta\Delta\tau A + P_A^{j,k-1})v^{j,k} = (I + (1-\theta)\Delta\tau A)v^{j-1} + P_A^{j,k-1}v^*. \quad (33)$$

Note that $P_A^j v^j$ in (33) is treated fully implicitly and does not involve a θ (nor a $\Delta\tau$) factor, in contrast to how R^j is treated in (32). We then modify the policy iteration to match the computation of the penalty iteration, and arrive to the following algorithm.

The first stopping criterion is the same as in Algorithm 2 which is Condition (29). For the second stopping criterion, if we use the same condition as the one used in Algorithm 1, that is,

$$Q^{j,k} = Q^{j,k-1} \text{ and } \mu^{j,k} = \mu^{j,k-1},$$

then we run into the same problems with oscillations as mentioned for Algorithm 2. Instead, for the Q controls, we use a stopping criterion similar to Condition (30), which is also the first part of (31), that is,

$$\max_i\{\frac{|A(Q^{j,k})v^{j,k} - A(Q^{j,k-1})v^{j,k}|}{\max(scale, A(Q^{j,k})v^{j,k})}\} < tol \text{ and } \mu^{j,k} = \mu^{j,k-1}, \quad (34)$$

where the motivation and purpose are the same as for Algorithms 2 and 3.

Algorithm 4 Improved policy iteration for HJB PDEs (including American rights) at step j with θ timestepping

Require: Solve $(I - \theta\Delta\tau A^j + R^j)v^j = g^{j-1} + R^j v^*$
where $g^{j-1} = (I + (1-\theta)\Delta\tau A^{j-1})v^{j-1}$
subject to $Q_i^j = \arg\sup_{Q\in\hat{Q}}[A(Q)v^j]_i$ and $\mu_i^j = \arg\sup_{\mu\in\{0,1\}}[R(\mu)(v^* - v)]_i$

Initialize $v^{j,0} = v^{j-1}$, $\mu^{j,0} = \mu^{j-1}$, and $Q^{j,0} = Q^{j-1}$
for $k = 1, \ldots, maxit$ **do**
  Solve $(I - \theta\Delta\tau A^{j,k-1} + R^{j,k-1})v^{j,k} = g^{j-1} + R^{j,k-1}v^*$
  if first stopping criterion satisfied **then**
    Break
  end if
  Compute $Q_i^{j,k} = \arg\sup_{Q\in\hat{Q}}[A(Q)v^{j,k}]_i$, $\mu_i^{j,k} = \arg\sup_{\mu\in\{0,1\}}[R(\mu)(v^* - v^{j,k})]_i$
  if second stopping criterion satisfied **then**
    Break
  end if
end for
Set $v^j = v^{j,k}$, $\mu^j = \mu^{j,k}$, and $Q^j = Q^{j,k}$

3.7 Comments on penalty and policy iterations

The double-penalty iteration algorithm (Algorithm 3) that we introduced for American exercise rights (and other nonlinearities) is an extension of the penalty iteration algorithm (Algorithm 2) introduced for non-linearities other than American exercise rights. Essentially, Algorithm 3 covers even the case of European exercise rights, as, if $P_A = 0$, it reduces to Algorithm 2.

In contrast, the improved policy iteration (Algorithm 4) is *not*, in general, an extension of the policy iteration Algorithm 1. More specifically, Algorithm 4 is equivalent to Algorithm 1 in the case of European exercise rights, i.e. the case $R = 0$, but not in the case of American exercise rights. Both Algorithms 4 and 1 are applicable to both European and American exercise rights, but in the case of American rights Algorithm 4 is more effective than Algorithm 1, as will be shown through numerical experiments in the next section.

It is also important to note that Algorithm 4 is mathematically (but not computationally) equivalent to Algorithm 3. This equivalence is reflected in the next section, where the results are numerically identical to a tolerance level.

4 Numerical results

We present numerical results from applying Algorithms 1, 2, 3, and 4 to the respective control and penalty PDE problems. For convenience, we use the abbreviations SBF for Stock Borrowing Fee problem, SBFA for Stock Borrowing Fee problem with American exercise rights, TC for Transaction Cost model problem, and TCA for Transaction Cost model problem with American exercise rights. Where the policy and penalty methods agree with each other to a numerical tolerance, the tables are used for a side-by-side comparison of policy iteration and penalty iteration in terms of the number of iterations taken.

The payoff for the first two problems (SBF, SBFA) is a straddle payoff, that is, the sum of a call and a put, while for TC and TCA, we consider put and butterfly spread payoffs.

For all the problems considered, the exact solution is unknown. In order to study the error and the order of convergence, we approximate the error by the difference (change) between two grid resolutions of size N and $2N$. We calculate the experimentally observed order of convergence by $\log_2(\text{change}_N/\text{change}_{2N})$.

4.1 Stock Borrowing Fees

We provide the numerical results of the SBF problem, solved with penalty and policy iteration in Table 1. The SBF problem is complex due to having more controls compared to the subsequent problems. However, in Table 2, for European options, we have a very small number of average iterations per timestep, and very stable second order of convergence.

4.2 Stock Borrowing Fees with American exercise rights

We provide the numerical results of the SBFA problem, solved with penalty and policy iteration.

This is the first problem in which we see the degradation of the solution quality as observed in the rate of convergence for solutions computed with policy iteration Algorithm 1 (note that the rate drops with each increase in the grid resolution). This phenomenon is expected for American options; see [12] for similar results with an American put option. The remedy, as suggested in the same paper [12], is to use variable timesteps based on the difference in the solution at the two previous timesteps.

However, when we introduced the same variable timestep selection method as in [12] in the policy Algorithm 1, the convergence rate did not improve, and we did not get results compatible with the variable timestep selection method in the penalty Algorithm 3. This is where the penalty Algorithm 3 has an advantage over the policy Algorithm 1: if we introduce the variable timestep selection method in Algorithm 3, we end up correcting the rate of convergence back to 2.0. However, introducing variable timesteps in Algorithm 4, gives compatible results with variable timesteps in Algorithm 3. Although the average number of iterations per timestep with variable timesteps is higher compared to that with constant timesteps, it would be the wrong metric to focus on, since the total number of iterations is less with variable timesteps. In fact, comparing Tables 2 and 3, with variable timesteps approximately only half of the number of iterations are required to obtain a solution that is eight times more accurate than the solution obtained with constant timesteps.

4.3 Transaction Cost model

For testing our methods on the Transaction Cost model problem, we consider two payoffs, the put, which is convex, and the butterfly spread, which is non-convex. The butterfly spread payoff is given by

$$V^*(S) = \begin{cases} 0 & \text{if } S \leq K - a \text{ or } S > K + a \\ S - (K - a) & \text{if } K - a < S \leq K \\ (K + a) - S & \text{if } K < S \leq K + a, \end{cases} \tag{35}$$

where K is the center strike price and a is a constant. In order to compare our results with existing results in the literature, we use different parameters, which are shown individually on each table of results.

Common Information					Penalty Iterations		Policy Iterations	
Nodes	Tstep	Value	Change	Rate	Total	Average	Total	Average
101	27	22.669427	—	—	36	1.33	37	1.37
201	52	22.680662	1.12e-02	—	71	1.37	71	1.37
401	102	22.683470	2.81e-03	2.00	140	1.37	140	1.37
801	202	22.684172	7.02e-04	2.00	278	1.38	278	1.38
1601	402	22.684347	1.75e-04	2.00	553	1.38	553	1.38
3201	802	22.684391	4.39e-05	2.00	1104	1.38	1104	1.38

Table 1: Long position of Stock Borrowing Fees problem with constant timesteps; value computed at K; straddle payoff. Parameters: $T = 1$, $S_{\max} = 1000$, $K = 100$, $\sigma = 0.3$, $r_b = 0.05$, $r_l = 0.03$, $r_f = 0.004$; Penalty results by Algorithm 2, Policy results by Algorithm 1.

Common Information					Penalty Iterations		Policy Iterations	
Nodes	Tstep	Value	Change	Rate	Total	Average	Total	Average
101	102	23.068723	—	—	138	1.35	131	1.28
201	202	23.079662	1.09e-02	—	272	1.35	261	1.29
401	402	23.082678	3.02e-03	1.86	544	1.35	526	1.31
801	802	23.083541	8.63e-04	1.80	1078	1.34	1050	1.31
1601	1602	23.083804	2.63e-04	1.71	2166	1.35	2123	1.33
3201	3202	23.083890	8.51e-05	1.63	4322	1.35	4254	1.33

Table 2: Long position of Stock Borrowing Fees problem with American exercise rights and constant timesteps; value computed at K; straddle payoff; Parameters: $T = 1$, $S_{\max} = 1000$, $K = 100$, $\sigma = 0.3$, $r_b = 0.05$, $r_l = 0.03$, $r_f = 0.004$; Penalty results by Algorithm 3, Policy results by Algorithm 1.

Common Information					Penalty Iterations		Policy Iterations	
Nodes	Tstep	Value	Change	Rate	Total	Average	Total	Average
101	40	23.076824	—	—	71	1.77	68	1.70
201	82	23.082631	5.81e-03	—	142	1.73	139	1.70
401	166	23.083667	1.04e-03	2.49	277	1.67	273	1.64
801	332	23.083875	2.09e-04	2.31	561	1.69	558	1.68
1601	664	23.083922	4.68e-05	2.16	1127	1.70	1126	1.70
3201	1327	23.083932	1.05e-05	2.16	2237	1.69	2296	1.73

Table 3: Long position of Stock Borrowing Fees problem with American exercise rights and variable timesteps; value computed at K; straddle payoff; Parameters: $T = 1$, $S_{\max} = 1000$, $K = 100$, $\sigma = 0.3$, $r_b = 0.05$, $r_l = 0.03$, $r_f = 0.004$; Penalty results by Algorithm 3, Policy results by Algorithm 4.

4.3.1 Put payoff

Since the put payoff is convex, the model linearizes and is reduced to Equation (18) for European options, or Equation (19) for American options. Table 4 and Tables 5 and 6 give the results which match the exact solution and the previously published results [12], respectively. In particular, on Table 4, besides the convergence to the exact solution 15.852055 (obtained through the Black-Scholes formula), we notice that we have very few excess iterations, something that matches our expectations for a linear problem. Also, on

Tables 5 and 6, we run a problem for which the adjusted volatility is $\sigma' = \sqrt{1 - 2 \times 0.18} = 0.8$, the same as for the problem in Table 11.1 of [12], and the values we obtain are very close to those in [12].

Common Information					Penalty Iterations		Policy Iterations	
Nodes	Tstep	Value	Change	Rate	Total	Average	Total	Average
101	102	15.843845	—	—	103	1.01	103	1.01
201	202	15.850002	6.16e-03	—	203	1.00	203	1.00
401	402	15.851542	1.54e-03	2.00	403	1.00	403	1.00
801	802	15.851927	3.85e-04	2.00	803	1.00	803	1.00
1601	1602	15.852023	9.63e-05	2.00	1603	1.00	1603	1.00
3201	3202	15.852047	2.41e-05	2.00	3203	1.00	3203	1.00

Table 4: European Transaction Cost model with put payoff (linear problem) and constant timesteps; value computed at K; Parameters: $T = 1$, $S_{\max} = 1000$, $K = 100$, $\sigma = 0.65$, $r = 0.05$, $\kappa = 0.1$. Penalty results by Algorithm 2, Policy results by Algorithm 1. Exact value is 15.852055.

Common Information					Penalty Iterations		Policy Iterations	
Nodes	Tstep	Value	Change	Rate	Total	Average	Total	Average
101	27	14.660480	—	—	40	1.48	42	1.56
201	52	14.672650	1.22e-02	—	85	1.63	85	1.63
401	102	14.676668	4.02e-03	1.60	173	1.70	175	1.72
801	202	14.678055	1.39e-03	1.53	359	1.78	365	1.81
1601	402	14.678561	5.06e-04	1.46	745	1.85	747	1.86
3201	802	14.678753	1.93e-04	1.39	1535	1.91	1549	1.93

Table 5: American Transaction Cost model with put payoff and constant timesteps; value computed at K. Parameters: $T = 0.25$, $S_{\max} = 1000$, $K = 100$, $\sigma = 1.0$, $r = 0.1$, $\kappa = 0.18$. Penalty results by Algorithm 3, Policy results by Algorithm 4.

Common Information					Penalty Iterations		Policy Iterations	
Nodes	Tstep	Value	Change	Rate	Total	Average	Total	Average
101	42	14.671527	—	—	66	1.57	66	1.57
201	85	14.677064	5.54e-03	—	136	1.60	134	1.58
401	171	14.678432	1.37e-03	2.02	278	1.63	281	1.64
801	344	14.678768	3.36e-04	2.03	565	1.64	577	1.68
1601	687	14.678851	8.29e-05	2.02	1144	1.67	1146	1.67
3201	1374	14.678872	2.06e-05	2.01	2272	1.65	2287	1.66

Table 6: American Transaction Cost model with put payoff and variable timesteps; value computed at K. Parameters: $T = 0.25$, $S_{\max} = 1000$, $K = 100$, $\sigma = 1.0$, $r = 0.1$, $\kappa = 0.18$. Penalty results by Algorithm 3, Policy results by Algorithm 4.

It should be noted that, for this problem, the use of stopping criterion (30) instead of $P^{j,k-1} = P^{j,k}$ is critical for the success of the method, as, if $P^{j,k-1} = P^{j,k}$ is used, some components of the solution vector oscillate by a negligible amount (most likely due to roundoff error), and force the penalty matrix to change at each oscillation, which results in large number of iterations.

4.3.2 *Butterfly spread payoff*

A butterfly spread payoff shows the nonlinearity in Equations (16) and (17). Tables 7 and 8 show the results obtained.

Common Information					Penalty Iterations		Policy Iterations	
Nodes	Tstep	Value	Change	Rate	Total	Average	Total	Average
101	102	0.126405	—	—	121	1.19	121	1.19
201	202	0.125742	-6.63e-04	—	236	1.17	236	1.17
401	402	0.125485	-2.57e-04	1.37	474	1.18	474	1.18
801	802	0.125361	-1.24e-04	1.05	936	1.17	935	1.17
1601	1602	0.125323	-3.83e-05	1.70	1879	1.17	1874	1.17
3201	3202	0.125311	-1.20e-05	1.68	3736	1.17	3719	1.16

Table 7: European Transaction Cost model with butterfly spread payoff and constant timesteps; value computed at K; Parameters: $T = 1$, $S_{\max} = 1000$, $K = 100$, $a = 10$, $\sigma = 0.65$, $r = 0.05$, $\kappa = 0.1$. Penalty results by Algorithm 2, Policy results by Algorithm 1.

Common Information					Penalty Iterations		Policy Iterations	
Nodes	Tstep	Value	Change	Rate	Total	Average	Total	Average
101	102	8.555324	—	—	109	1.07	109	1.07
201	202	8.558220	2.90e-03	—	216	1.07	217	1.07
401	402	8.558916	6.95e-04	2.06	429	1.07	429	1.07
801	802	8.559093	1.77e-04	1.97	856	1.07	856	1.07
1601	1602	8.559139	4.64e-05	1.93	1711	1.07	1710	1.07
3201	3202	8.559152	1.28e-05	1.85	3450	1.08	3498	1.09

Table 8: American Transaction Cost model with butterfly spread payoff and constant timesteps; value computed at $1.1K$; Parameters: $T = 1$, $S_{\max} = 1000$, $K = 100$, $a = 10$, $\sigma = 0.65$, $r = 0.05$, $\kappa = 0.1$. Penalty results by Algorithm 3, Policy results by Algorithm 1.

Common Information					Penalty Iterations		Policy Iterations	
Nodes	Tstep	Value	Change	Rate	Total	Average	Total	Average
101	42	8.556308	—	—	54	1.20	54	1.20
201	87	8.558431	2.12e-03	—	110	1.24	109	1.22
401	176	8.558946	5.15e-04	2.04	220	1.24	219	1.24
801	353	8.559073	1.27e-04	2.02	433	1.23	427	1.21
1601	704	8.559073	3.12e-05	2.03	868	1.23	853	1.21
3201	1407	8.559112	7.81e-06	2.00	1731	1.23	1720	1.22

Table 9: American Transaction Cost model with butterfly spread payoff and variable timesteps; value computed at $1.1K$; Parameters: $T = 1$, $S_{\max} = 1000$, $K = 100$, $a = 10$, $\sigma = 0.65$, $r = 0.05$, $\kappa = 0.1$. Penalty results by Algorithm 3, Policy results by Algorithm 4.

We do not compute the convergence rate for the American Transaction Cost model at the center strike price, because the constraint from the American early exercise right causes the price to stay at the same

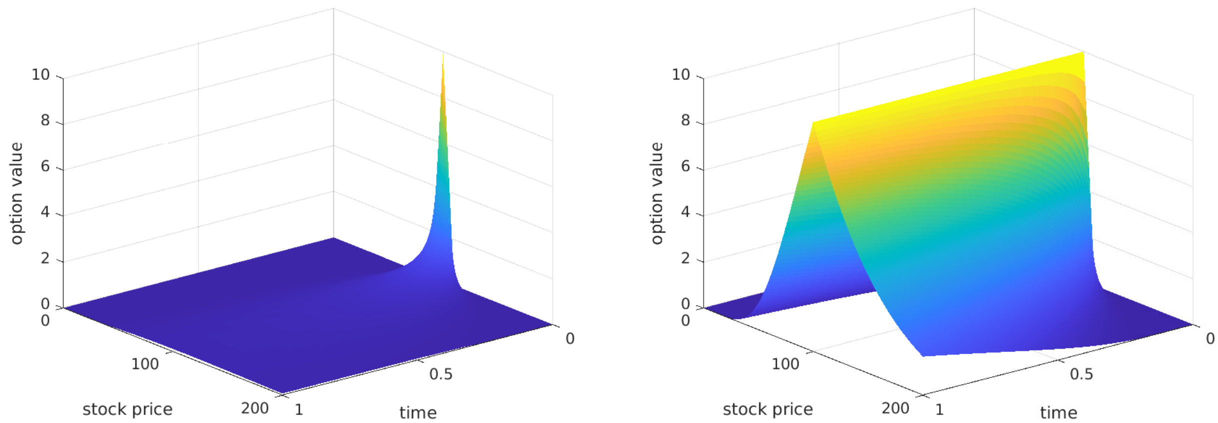

Figure 1: Plots of the three-dimensional mesh of values of the Transaction Cost model with European (left) and American (right) exercise rights. Parameters: $T = 1$, $S_{\max} = 1000$, $K = 100$, $\sigma = 0.65$, $r = 0.05$, $\kappa = 0.1$. Yellow represents larger values and blue represents smaller values.

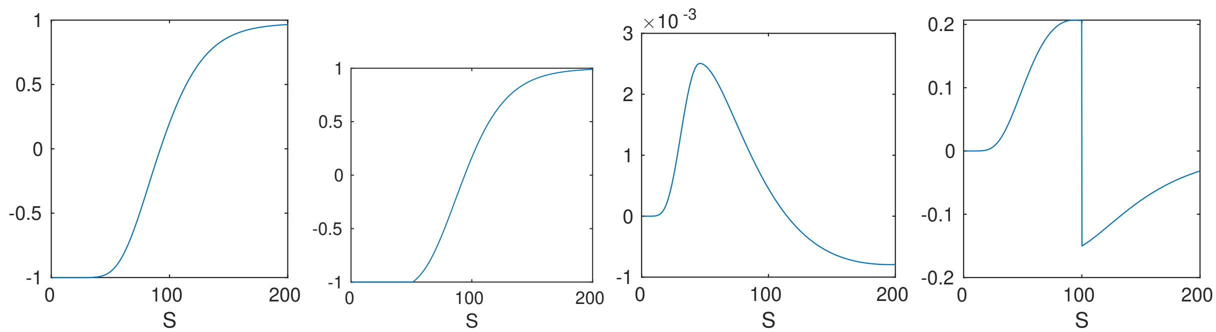

Figure 2: From left to right, values of Δ for Stock Borrowing Fees problem, Stock Borrowing Fees problem with American exercise rights, Transaction Cost problem, and Transaction Cost problem with American exercise rights. Long position/worst case considered.

value [16], within rounding errors and errors at the level of ρ^{-1}. Instead, we compute the convergence rate at $1.1K$, shown in Table 8. It is also possible to implement variable timesteps as in [12], to fix the very small decrease of quality in the convergence rate at $1.1K$. Results are included in Table 9.

Although different behaviour at the central strike price is shown in [31], we have reason to believe in the correctness of our algorithm. The convergence rate is very clearly 2.0, as seen in Tables 8 and 9; in addition, [31] uses a Padé scheme [18], which imposes certain smoothness requirements on the solution which do not hold for American options. For one very well-known example, the solution of the American put option has a discontinuous second derivative [12], which makes precise computation of a solution difficult. In addition, our results are supported by analysis in [16], indicating that the solution at $S = K$ stays constant at a. A mesh plot of the solution is given in Figure 1 for European and American exercise rights.

4.4 Greeks

We discuss the approximation of Greeks, since their smoothness is important to practical applications.

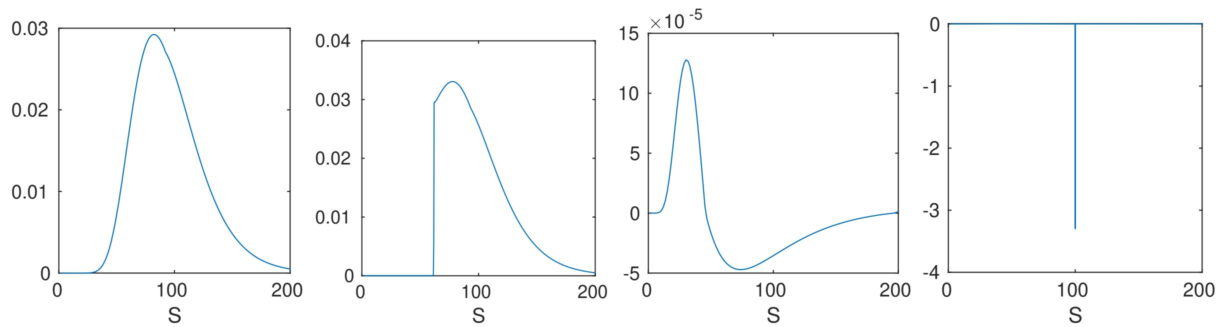

Figure 3: From left to right, values of Γ for Stock Borrowing Fees problem, Stock Borrowing Fees problem with American exercise rights, Transaction Cost problem, and Transaction Cost problem with American exercise rights. Long position/worst case considered.

As can be seen in Figure 2, the computed Deltas of the four problems (SBF, SBFA, TC, and TCA) are all smooth, with the exception of those problems with American options, which have cusps due to the constraint that enforces the American exercise right. We see similar behaviour for the computed Gammas in Figure 3, where the only jump is at the American exercise boundary.

Importantly, we do not see spurious oscillations in the plots, such as those found in [12] with constant timesteps in the pricing of American put options, nor do we see the discontinuities such as those in the Uncertain Volatility model, which is similar to the Transaction Cost problem, found in [25] or [14].

5 Extension of the penalty iteration method to continuous control problems

The penalty iteration method can easily be extended to problems with continuous control, by discretizing the control. As a demonstration, we consider the Passport Option (PO) pricing problem introduced in [1], which, after some transformations [29], is given by the PDE

$$V_\tau = \max_{|q|\le 1} \frac{\sigma^2}{2}(S-q)^2 V_{SS} + ((r-r_c-r_\gamma)q - (r-r_t-r_\gamma)S)V_S - r_\gamma V, \tag{36}$$

where q is the control and denotes the number of shares of S held by the investor, ranging in $[-1,1]$, and r_c, r_t and r_γ are various given rates. To apply the penalty iteration method, we discretize the control q on the interval $[-1,1]$. Following the methods of [19, 20], we use N_q evenly spaced points q_ℓ along $[-1,1]$. Then, for each of the points q_ℓ, $\ell = 1,\dots,N_q$, we define the matrix

$$P_\ell = \frac{\sigma^2}{2}(q_\ell^2 - 2q_\ell D_S)T_2 + (r-r_c-r_\gamma)q_\ell T_1, \tag{37}$$

arising from the terms in (36) involving the control q. To compute the penalty matrix P, we follow the same logic as for the Stock Borrowing Fee problem, and define the penalty matrix P by

$$[P]_{i,:} = [P_k]_{i,:}, \text{ where } k \text{ satisfies } [P_k v^j]_i \ge [P_\ell v^j]_i, \quad \forall \ell = 1,\dots,N_q. \tag{38}$$

For the PO problem, the finite difference discretization of the spatial derivatives, and the timestepping scheme remains the same as for all problems considered in the paper, with some details on the S-grid given next. As in [19, 20], the S domain is truncated to $[S_{\min}, S_{\max}]$. We use custom nonuniform S-grids, denser towards $S=0$.

To verify our results and demonstrate the correctness of our method, we compare our results with the problems in [29]. The authors of [29] attempted two types of problems: the first is a call option with payoff $\max(0, S)$ and the second is an option with digital call payoff $\mathbf{1}(S \geq 0)$. The first case is convex, which means that the control will always be either 1 or -1 and is determined by the sign of S. We also demonstrate our method on the capped call payoff $\min(1, \max(S, 0))$ (non-convex) which was used in [19]. The non-convex digital call and the capped call payoffs are more serious tests of our method. Note that compared to the results in [29], our results are not scaled by $S_{t=0} = 100$.

Table 10 presents results from pricing a call payoff passport option with the parameters chosen so that there is an analytic solution [1, 29]. Table 11 presents results from pricing a digital call payoff passport option with the parameters chosen as in Table 5 in [29]. Tables 12 and 13 present results from a capped call payoff passport option with the parameters chosen as in [19], with 9 and 101 points for the discretization of q, respectively. In all cases, the convergence rate is close to 2. In Tables 10 and 11, our penalty method reaches six and five, respectively, digits of accuracy with a reasonably low number of iterations. Though it is difficult to directly compare our method to the one in [19], we note that in Tables 12 and 13 fewer nodes and timesteps are used than in [19]. (In both [29] and [19], the number of timesteps is quadrupled in each refinement.) An additional note is that, even if the payoff is non-convex, a finer discretization of the control seems to have limited benefits, as the difference between Tables 12 ($N_q = 9$) and 13 ($N_q = 101$) is minimal. Finally, we remark that Algorithms 1 and 4 are also applicable to the PO problem.

Nodes	Tstep	Iter	Value	Change	Rate	avg iter
24	7	9	0.1291107	—	—	1.29
47	12	14	0.1308031	1.69e-03	—	1.17
93	22	23	0.1312359	4.33e-04	1.97	1.05
185	42	44	0.1313447	1.09e-04	1.99	1.05
369	82	100	0.1313719	2.72e-05	2.00	1.22
737	162	197	0.1313787	6.84e-06	1.99	1.22
1473	322	624	0.1313806	1.83e-06	1.91	1.94

Table 10: Passport Option problem with call payoff and constant timesteps; value computed at $S = K = 0$. Parameters: $T = 1$, S domain $[-3, 4]$, $\sigma = 0.30$, $r = 0$, $r_c = 0$, $r_t = 0$, $r_\gamma = 0$, $N_q = 9$. Penalty results by Algorithm 2. Exact value to six digits 0.131381 [1].

Nodes	Tstep	Iter	Value	Change	Rate	avg iter
21	7	11	0.2365808	—	—	1.57
41	12	21	0.2546439	1.81e-02	—	1.75
81	22	41	0.2596610	5.02e-03	1.85	1.86
161	42	82	0.2610057	1.34e-03	1.90	1.95
321	82	157	0.2613551	3.49e-04	1.94	1.91
641	162	303	0.2614405	8.54e-05	2.03	1.87
1281	322	638	0.2614715	3.10e-05	1.46	1.98
2561	642	1303	0.2614802	8.78e-06	1.82	2.03

Table 11: Passport Option problem with digital call payoff and constant timesteps; value computed at $S = -0.25$. Parameters: $T = 1$, S domain $[-3, 3]$, $\sigma = 0.20$, $r = 0.08$, $r_c = 0.1$, $r_t = 0.04$, $r_\gamma = 0.035$, $N_q = 21$. Penalty results by Algorithm 2.

Nodes	Tstep	Iter	Value	Change	Rate	avg iter
25	7	11	0.1147319	—	—	1.57
49	12	18	0.1159619	1.23e-03	—	1.50
97	22	36	0.1162905	3.29e-04	1.90	1.64
193	42	66	0.1163806	9.01e-05	1.87	1.57
385	82	135	0.1164050	2.44e-05	1.88	1.65
769	162	265	0.1164116	6.64e-06	1.88	1.64
1537	322	530	0.1164133	1.67e-06	1.99	1.65

Table 12: Passport Option problem with capped call payoff and constant timesteps; value computed at $S = 0$. Parameters: $T = 1$, S domain $[-4, 4]$, $\sigma = 0.20$, $r = 0.08$, $r_c = 0.12$, $r_t = 0.05$, $r_\gamma = 0.03$, $N_q = 9$. Penalty results by Algorithm 2.

Nodes	Tstep	Iter	Value	Change	Rate	avg iter
25	7	11	0.1147325	—	—	1.57
49	12	20	0.1159621	1.23e-03	—	1.67
97	22	44	0.1162908	3.29e-04	1.90	2.00
193	42	84	0.1163807	8.99e-05	1.87	2.00
385	82	164	0.1164051	2.44e-05	1.88	2.00
769	162	311	0.1164117	6.63e-06	1.88	1.92
1537	322	592	0.1164133	1.66e-06	2.00	1.84

Table 13: Passport Option problem with capped call payoff and constant timesteps; value computed at $S = 0$. Parameters: $T = 1$, S domain $[-4, 4]$, $\sigma = 0.20$, $r = 0.08$, $r_c = 0.12$, $r_t = 0.05$, $r_\gamma = 0.03$, $N_q = 101$. Penalty results by Algorithm 2.

6 Conclusions

We formulated and studied penalty and penalty-like methods for nonlinear HJB problems in computational finance, expressed as penalized nonlinear PDEs. We introduced the use of tridiagonal penalty-like matrices to account for the nonlinear terms involving the partial derivatives V_S and V_{SS}, and presented Algorithms 2 and 3, with the latter being an extension of the former, that can handle American (as well as European) exercise rights.

We modified the policy algorithm in [11, 14], to allow the effective use of variable timesteps, and to restore the stable second order of convergence (when Crank-Nicolson-Rannacher timestepping is used) for American exercise rights, and presented Algorithm 4.

Essentially, Algorithms 3 and 4 can handle all problems considered, including European and American exercise rights.

We have proved that the methods converge monotonically except for the HJBI case and the American Transaction Cost model, and we have shown that the average number of iterations taken is very small.

Comparing the penalty and policy iteration methods, more specifically, Algorithms 3 and 4, which are designed to be mathematically equivalent, and are shown to produce comparable numerical results, an important advantage of the penalty iteration methods is that they avoid enumeration of all combinations of values of control variables to pick the set that produces the minimum (or maximum) of the considered quantity, thus leading to less computational cost per iteration.

The penalty and policy methods described can be extended to certain continuous control problems.

7 Appendix: proof of convergence

We first study the convergence of the penalty iteration at each timestep, then the convergence of the timestepping/discretization scheme to the viscosity solution. We carry the analysis based on monotonicity arguments. For the purpose of the analysis, we assume that the discrete equations are formed with Dirichlet boundary conditions on both endpoints, and the generic interest rate r is positive. We also assume that the spatial gridpoints satisfy conditions analogous to those mentioned in Lemma 4.1 of [8]. Under the above assumptions, the matrix $I - \theta\Delta\tau A$ arising from θ-timestepping and centered finite differences discretization of the original Black-Scholes operator (4) is strictly diagonally dominant with positive diagonal entries and nonpositive off-diagonal entries, therefore monotone. The problems considered involve (besides the original Black-Scholes) additional terms, and result in equations which, compared to the original Black-Scholes, have either adjusted interest rates for the rSV_S and rV terms, or adjusted volatility for the $\frac{\sigma^2}{2}S^2V_{SS}$ term. In the interest of brevity, we do not present all details, but mention, as two examples, that the SBF short position PDE involves either $r_lSV_S - r_lV$ or $r_bSV_S - r_bV$ or $(r_l - r_f)SV_S - r_lV$, and the TC problem involves $(\frac{\sigma^2}{2} \mp \kappa)S^2V_{SS}$. In order to make sure that the adjusted coefficients are positive, we assume that $r_b > r_l > r_f \geq 0$, and $\kappa < \sigma^2/2$. Under these assumptions, the matrix $I - \theta\Delta\tau(A + P^{j,k-1})$ arising from all problems considered is strictly diagonally dominant with positive diagonal entries and nonpositive off-diagonal entries, therefore monotone. Furthermore, note that the matrix $M^{j,k-1} = A + P^{j,k-1}$ is strictly diagonally dominant with negative diagonal entries and non-negative off-diagonal entries.

7.1 Monotonicity of the iterates

We first consider the case of European exercise rights, and prove the following statements:

1. When the penalty matrices compute the maximum with a positive coefficient (i.e. the right hand side of the PDE involves $+\max\{\}$), the iterates are monotone increasing. (The same can be proved when the penalty matrices compute the minimum with a negative coefficient, i.e. $-\min\{\}$, but we did not consider such PDE problems.)
2. Similarly, when the penalty matrices compute the minimum with a positive coefficient, (i.e. the right hand side of the PDE involves $+\min\{\}$), the iterates are monotone decreasing. (The same can be proved when the penalty matrices compute the maximum with a negative coefficient, but we did not consider such PDE problems.)

We will prove both statements simultaneously. The arguments follow the steps in [12] and [6], except that the penalty matrices here are tridiagonal, and we do not refer to particular matrix entries, but rather to the result of applying the penalty matrices to vectors. We note that, when the right hand side of the PDE involves $+\max\{\}$ (respectively $+\min\{\}$), $P^{j,k}v^{j,k}$ calculates the vector of maximum (respectively minimum) components. At each penalty iteration $k \geq 1$, we have to solve

$$[I - \theta\Delta\tau(A + P^{j,k-1})]v^{j,k} = g^j \qquad (39)$$

for $v^{j,k}$. Consider the next penalty iteration

$$[I - \theta\Delta\tau(A + P^{j,k})]v^{j,k+1} = g^j \tag{40}$$

and write the equation in the previous iteration (39) as

$$[I - \theta\Delta\tau(A + P^{j,k})]v^{j,k} = g^j + \theta\Delta\tau(P^{j,k-1} - P^{j,k})v^{j,k}. \tag{41}$$

Therefore,

$$[I - \theta\Delta\tau(A + P^{j,k})](v^{j,k+1} - v^{j,k}) = \theta\Delta\tau(P^{j,k} - P^{j,k-1})v^{j,k}. \tag{42}$$

When the right hand side of the PDE involves $+\max\{\}$, we have $P^{j,k}v^{j,k} \geq P^{j,k-1}v^{j,k}$. When the right hand side of the PDE involves $+\min\{\}$, we have $P^{j,k}v^{j,k} \leq P^{j,k-1}v^{j,k}$.

Since $I - \theta\Delta\tau(A + P^{j,k})$ is a monotone matrix, it follows from (42) that

$$\text{if} \quad \theta\Delta\tau(P^{j,k} - P^{j,k-1})v^{j,k} \geq 0, \quad \text{then} \quad v^{j,k+1} - v^{j,k} \geq 0 \tag{43}$$

$$\text{if} \quad \theta\Delta\tau(P^{j,k} - P^{j,k-1})v^{j,k} \leq 0, \quad \text{then} \quad v^{j,k+1} - v^{j,k} \leq 0. \tag{44}$$

Therefore, in (43) ($+\max\{\}$) the iterates are monotonically increasing, and in (44) ($+\min\{\}$) the iterates are monotonically decreasing. Thus, for both types of problems, we have monotone behaviour of the iterates.

We now consider American exercise rights. Since adding in the American penalty matrix causes the iterates to increase at each iteration [12], it wouldn't affect the monotonicity of the iterates if they are otherwise increasing, but if the iterates are monotonically decreasing otherwise, adding in the American penalty matrix breaks the monotone conditions. Thus, if the PDE right hand side involves $+\max\{\}$ or $-\min\{\}$, monotonicity (increasing direction) is guaranteed for both European and American exercise rights. On the other hand, if the PDE right hand side involves $+\min\{\}$ or $-\max\{\}$, monotonicity (decreasing direction) is guaranteed only for European exercise rights.

7.2 Proof of iteration convergence

Since, for European rights, the iterates are increasing/decreasing monotonically, we only need to show that they are bounded above/below. At each iteration, we solve $(I - \theta\Delta\tau M^{j,k-1})v^{j,k} = g^j$, where $M^{j,k-1} = A + P(v^{j,k-1})$. For a fixed grid, $||g^j||_\infty$ is bounded (i.e. g_i^j is bounded above and below). With the assumptions that $M_{i,i}^{j,k-1} < 0$, and $M_{i,i\pm1}^{j,k-1} > 0$, boundedness of $v^{j,k}$ above and below can be shown by standard maximum/minimum analysis. More specifically, we can show that

$$\frac{g_l^j}{1 + \theta\Delta\tau r} \leq v_i^{j,k} \leq \frac{g_u^j}{1 + \theta\Delta\tau r}, \text{ for } l = \arg\min_i v_i^{j,k},\ u = \arg\max_i v_i^{j,k}. \tag{45}$$

The monotonicity and boundedness of the iterates proves the convergence of the iteration.

For problems with an additional American penalty matrix, the proof of convergence doesn't change as long as the monotonicity arguments hold.

7.3 Uniqueness of solution

Recall that, for European rights, we have to solve the nonlinear system

$$[I - \theta\Delta\tau(A + P^{j,k})]v^{j,k} = g^j \tag{46}$$

with emphasis on the nonlinearity between the penalty matrix $P^{j,k}$ and $v^{j,k}$.

Suppose that there exist two solutions, (P_1, v_1) and (P_2, v_2). Then,

$$[I - \theta\Delta\tau(A + P_1)]v_1 = g^j \quad \text{and} \quad [I - \theta\Delta\tau(A + P_2)]v_2 = g^j. \tag{47}$$

Using the two relations in (47), we get

$$[I - \theta\Delta\tau(A + P_2)](v_2 - v_1) = \theta\Delta\tau(P_2 - P_1)v_1. \tag{48}$$

Here, we apply the results on monotonicity earlier to argue that in the case of the penalty matrix being based on a maximum, the RHS vector is always nonpositive, and if the penalty matrix is based on a minimum, the RHS vector is always non-negative. This leads us to conclude that $v_2 \geq v_1$ in the former case, and $v_1 \geq v_2$ in the latter case.

Since the choices of (P_1, v_1) and (P_2, v_2) are rather arbitrary, we can reverse the roles between them and get

$$[I - \theta\Delta\tau(A + P_1)](v_1 - v_2) = \theta\Delta\tau(P_1 - P_2)v_2, \tag{49}$$

which leads to the reverse conclusions, that is, $v_1 \geq v_2$ in the former case and $v_2 \geq v_1$ in the latter case. Therefore, $v_1 = v_2$ and $P_1 = P_2$, and the solution to the nonlinear problem (46) is unique.

For problems with an additional American penalty matrix, the proof of uniqueness follows the same process as in [12], assuming again that the results in Section 7.1 hold.

To conclude the penalty iteration convergence, we note for the tridiagonal treatment of the penalty-like matrix involving derivative terms, the family of penalty-like and double-penalty methods converge monotonically to the unique solution, with the exception of the double-penalty method for the long position of Stock Borrowing Fees with American rights, and the Transaction Cost with American rights, due to the differing sign between the penalty matrices.

7.4 Monotonicity of the discretization scheme

For a point i, after writing a discretization in the form

$$f_i(v_i^j, v_{i\pm1}^j, v_i^{j-1}, v_{i\pm1}^{j-1}) = 0, \tag{50}$$

a discretization is monotone if a positive perturbation to $v_{i\pm1}^j$, v_i^{j-1}, $v_{i\pm1}^{j-1}$ produces a positive perturbation to the v_i^j [25]. To simplify matters, it is sufficient to consider one single perturbation $\delta > 0$ to each of the four variables v_i^j, $v_{i\pm1}^j$, v_i^{j-1}, $v_{i\pm1}^{j-1}$, and, given relation (50), show that, for all i,

$$f_i(v_i^j + \delta, v_{i\pm1}^j, v_i^j, v_{i\pm1}^{j-1}) \geq 0 \tag{51a}$$
$$f_i(v_i^j, v_{i\pm1}^j + \delta, v_i^{j-1}, v_{i\pm1}^{j-1}) \leq 0 \tag{51b}$$
$$f_i(v_i^j, v_{i\pm1}^j, v_i^{j-1} + \delta, v_{i\pm1}^{j-1}) \leq 0 \tag{51c}$$
$$f_i(v_i^j, v_{i\pm1}^j, v_i^{j-1}, v_{i\pm1}^{j-1} + \delta) \leq 0. \tag{51d}$$

The θ-timestepping with spatial discretization for European exercise rights is given by

$$(I - \theta\Delta\tau^j(A + P(v^j)))v^j - (I + (1-\theta)\Delta\tau^j(A + P(v^{j-1})))v^{j-1} = 0. \tag{52}$$

The equation for a specific node v_i^j is

$$\begin{aligned} f_i(v_i^j, v_{i\pm1}^j, v_i^{j-1}, v_{i\pm1}^{j-1}) \equiv\ & v_i^j - \theta\Delta\tau^j(M_{i,i-1}^j v_{i-1}^j + M_{i,i}^j v_i^j + M_{i,i+1}^j v_{i+1}^j) \\ -\ & v_i^{j-1} - (1-\theta)\Delta\tau^j(M_{i,i-1}^{j-1} v_{i-1}^{j-1} + M_{i,i}^{j-1} v_i^{j-1} + M_{i,i+1}^{j-1} v_{i+1}^{j-1}) = 0 \end{aligned} \tag{53}$$

where $M^j = A + P(v^j)$ and $M^{j-1} = A + P(v^{j-1})$. Note that we have already assumed that M^j is strictly diagonally dominant, with negative diagonal entries and non-negative off-diagonal entries, for all timesteps j.

Recalling that relation (53) holds, It is straightforward to show (51a), (51b) and (51d). It is also easy to see that (51c) holds as long as $\Delta\tau^j < [-(1-\theta)M_{i,i}^{j-1}]^{-1}$. Thus, as long as $\Delta\tau^j < [-(1-\theta)M_{i,i}^{j-1}]^{-1}$, the discretization is monotone.

With American exercise rights, Equation (52) becomes

$$(I - \theta\Delta\tau^j(A + P(v^j)) + P_A(v^j))v^j - (I + (1-\theta)\Delta\tau^j(A + P(v^{j-1})))v^{j-1} - P_A(v^j)v^* = 0 \tag{54}$$

For the points i for which $[P_A]_{i,i} = 0$, the analysis is the same as for European exercise rights. For the points for which $[P_A]_{i,i} = \rho > 0$, Equation (53) becomes

$$\begin{aligned} f_i(v_i^j, v_{i\pm1}^j, v_i^{j-1}, v_{i\pm1}^{j-1}) \equiv\ & v_i^j - \theta\Delta\tau^j(M_{i,i-1}^j v_{i-1}^j + M_{i,i}^j v_i^j + M_{i,i+1}^j v_{i+1}^j) + \rho v_i^j \\ -\ & v_i^{j-1} - (1-\theta)\Delta\tau^j(M_{i,i-1}^{j-1} v_{i-1}^{j-1} + M_{i,i}^{j-1} v_i^{j-1} + M_{i,i+1}^{j-1} v_{i+1}^{j-1}) - \rho v_i^* = 0 \end{aligned} \tag{55}$$

Nothing changes in the analysis of the previous section with respect to perturbations on $v_{i\pm1}^j$, v_i^{j-1}, and $v_{i\pm1}^{j-1}$, that is, for showing (51b), (51c) and (51d). For showing (51a), since $\rho > 0$, we have

$$f_i(v_i^j + \delta, v_{i\pm1}^j, v_i^{j-1}, v_{i\pm1}^{j-1}) = \delta - \theta\Delta\tau^j M_{i,i}^j \delta + \rho\delta = (1 - \theta\Delta\tau^j M_{i,i}^j + \rho)\delta > 0.$$

Thus, the term arising from the American penalty matrix enhances the monotonicity inequality.

The restriction arising from case 3 above, written for every row i, results in

$$\Delta\tau^j < \min_i \left\{ -\frac{1}{(1-\theta)M_{i,i}^{j-1}} \right\}. \tag{56}$$

The above analysis shows that, under restriction (56), the discretization is monotone. For CN ($\theta = \frac{1}{2}$), relation (56) becomes

$$\Delta\tau^j < 2\min_i \left\{ -\frac{1}{M_{i,i}^{j-1}} \right\}, \tag{57}$$

while, for fully-implicit timestepping ($\theta = 1$), there is no restriction.

7.5 Stability of discretization

Under the restriction (56), as well as the conditions that ensure that M is strictly diagonally dominant with $M_{i,i}^{j-1} < 0$, and $M_{i,i\pm1}^{j-1} \geq 0$, and assuming Dirichlet and bounded by a fixed value conditions at each timestep, we can show by standard maximum analysis that

$$||v^j||_\infty \quad \leq \quad \max\{||v^{j-1}||_\infty, ||v^j||_{\infty,boundary}\} \tag{58}$$

which proves the stability of the scheme, under the indicated conditions.

To conclude the proof of convergence to the viscosity solution, we note that the monotonicity, stability and consistency (obvious from the way the penalty matrices are formed) of the scheme, ensure the convergence of the computed solution to the viscosity solution [3].

REMARK 2 *It is important to note that the conditions indicated are sufficient, but not necessary. We have, in fact, run many numerical experiments where the condition (56) is not satisfied, especially for fine discretizations, and we have used linear boundary condition at $S_{\max}$, but we did not notice any stability issues, as long as CN is used with Rannacher smoothing.*

REMARK 3 *Since $M_{i,i}^{j-1}$ involves a term of $O(\frac{1}{h_{i-1}h_i})$, it is clear that the restriction on the CN timestep size is quite severe, especially for fine spatial discretizations. Essentially, the restriction is of the same order as that of an explicit scheme. However, we note again, that, although in many experiments the timestep size chosen did not satisfy the above restriction, we did not notice any stability issues.*

REMARK 4 *Note that in Equation* (56) *there is no mention of ρ or of the American penalty matrix, hence, the restriction on the timestep size is independent of the type of exercise rights.*

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Declaration of Interest

The authors report no conflicts of interest. The authors alone are responsible for the content and writing of the paper.

References

[1] L. ANDERSEN, J. ANDREASEN, AND R. BROTHERTON-RATCLIFFE, *The passport option*, Journal of Computational Finance, 1 (1998), pp. 15–36.

[2] J. ANKUDINOVA AND M. EHRHARDT, *Fixed domain transformations and split-step finite difference schemes for nonlinear Black-Scholes equations for American options*, Nonlinear Models in Mathematical Finance: New Research Trends in Option Pricing, (2008), pp. 243–273.

[3] G. BARLES, *Convergence of numerical schemes for degenerate parabolic equations arising in finance theory*, Numerical methods in finance, 13 (1997), pp. 1–21.

[4] Y. BERGMAN, *Option pricing with differential interest rates*, The Review of Financial Studies, 8 (1995), pp. 475–500.

[5] F. BLACK AND M. SCHOLES, *The pricing of options and corporate liabilities*, Journal of political economy, 81 (1973), pp. 637–654.

[6] Y. CHEN AND C. CHRISTARA, *Penalty methods for bilateral XVA pricing in European and American contingent claims by a PDE model*, Journal of Computational Finance, 24 (2021), pp. 41–70.

[7] Y. CHEN AND J. W. WAN, *Deep neural network framework based on backward stochastic differential equations for pricing and hedging American options in high dimensions*, Quantitative Finance, (2020), pp. 1–23.

[8] C. CHRISTARA AND D.-M. DANG, *Adaptive and high-order methods for valuing American options*, Journal of Computational Finance, 14 (2011), pp. 73–113.

[9] N. CLARKE AND K. PARROTT, *Multigrid for American option pricing with stochastic volatility*, Applied Mathematical Finance, 6 (1999), pp. 177–195.

[10] D. DUFFIE, N. GARLEANU, AND L. H. PEDERSEN, *Securities lending, shorting, and pricing*, Journal of Financial Economics, 66 (2002), pp. 307–339.

[11] P. FORSYTH AND G. LABAHN, *Numerical methods for controlled Hamilton-Jacobi-Bellman PDEs in finance*, Journal of Computational Finance, 11 (2007), pp. 1–43.

[12] P. FORSYTH AND K. VETZAL, *Quadratic convergence for valuing American options using a penalty method*, SIAM J. Sci. Comput., 23 (2002), pp. 2095–2122.

[13] P. A. FORSYTH AND K. R. VETZAL, *Implicit solution of uncertain volatility/transaction cost option pricing models with discretely observed barriers*, Applied Numerical Mathematics, 36 (2001), pp. 427–445.

[14] ——, *Numerical Methods for Nonlinear PDEs in Finance*, Springer Berlin Heidelberg, Berlin, Heidelberg, 2012, pp. 503–528.

[15] J. HAN, A. JENTZEN, AND W. E, *Solving high-dimensional partial differential equations using deep learning*, Proceedings of the National Academy of Sciences, 115 (2018), pp. 8505–8510.

[16] S. D. HOWISON, C. REISINGER, AND J. H. WITTE, *The effect of nonsmooth payoffs on the penalty approximation of American options*, SIAM J. Financ Math., 4 (2013), pp. 539–574.

[17] Y. M. KABANOV AND M. M. SAFARIAN, *On Leland's strategy of option pricing with transactions costs*, Finance and Stochastics, 1 (1997), pp. 239–250.

[18] A. KHALIQ, D. VOSS, AND M. YOUSUF, *Pricing exotic options with L-stable Padé schemes*, Journal of Banking & Finance, 31 (2007), pp. 3438–3461.

[19] I. KOSSACZKỲ, M. EHRHARDT, AND M. GÜNTHER, *The tree-grid method with control-independent stencil*, in Proceedings of Equadiff 2017 Conference, 2017, pp. 79–88.

[20] ——, *A new convergent explicit tree-grid method for HJB equations in one space dimension*, Numer. Math. Theor. Meth. Appl., 11 (2018), pp. 1–29.

[21] H. E. Leland, *Option pricing and replication with transactions costs*, The Journal of Finance, 40 (1985), pp. 1283–1301.

[22] H. E. Leland et al., *Comments on "Hedging errors with Leland's option model in the presence of transactions costs"*, Finance Research Letters, 4 (2007), pp. 200–202.

[23] W. Liao and A. Q. Khaliq, *High-order compact scheme for solving nonlinear Black–Scholes equation with transaction cost*, International Journal of Computer Mathematics, 86 (2009), pp. 1009–1023.

[24] D. Pooley, K. Vetzal, and P. Forsyth, *Convergence remedies for non-smooth payoffs in option pricing*, Journal of Computational Finance, 6 (2003), pp. 25–40.

[25] D. M. Pooley, P. A. Forsyth, and K. R. Vetzal, *Numerical convergence properties of option pricing PDEs with uncertain volatility*, IMA J. Numer. Anal., 23 (2003), pp. 241–267.

[26] R. Rannacher, *Finite element solution of diffusion problems with irregular data*, Numerische Mathematik, 43 (1984), pp. 309–327.

[27] C. Reisinger and Y. Zhang, *A penalty scheme and policy iteration for nonlocal HJB variational inequalities with monotone drivers*, arXiv preprint arXiv:1805.06255, (2018).

[28] M. S. Santos and J. Rust, *Convergence properties of policy iteration*, SIAM J. Control Optim., 42 (2004), pp. 2094–2115.

[29] J. Wang and P. A. Forsyth, *Maximal use of central differencing for Hamilton–Jacobi–Bellman PDEs in finance*, SIAM Journal on Numerical Analysis, 46 (2008), pp. 1580–1601.

[30] P. Wilmott, T. Hoggard, and A. E. Whalley, *Hedging option portfolios in the presence of transaction costs*, Advances in Futures and Options Research, 7 (1994), pp. 21–35.

[31] M. Yousuf, A. Khaliq, and B. Kleefeld, *The numerical approximation of nonlinear Black–Scholes model for exotic path-dependent American options with transaction cost*, International Journal of Computer Mathematics, 89 (2012), pp. 1239–1254.

[32] Y. Zhao and W. T. Ziemba, *Hedging errors with Leland's option model in the presence of transaction costs*, Finance Research Letters, 4 (2007), pp. 49–58.