QUARTIC SPLINE COLLOCATION METHODS FOR SECOND-ORDER TWO-POINT BOUNDARY VALUE ODE PROBLEMS

by

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Abstract

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Collocation methods based on quartic splines are presented for second-order two-point boundary value problems. In addition to the boundary conditions specified by the problem, extra boundary conditions are introduced in order to uniquely determine the quartic spline approximation. The standard quartic spline collocation method gives fourth order bounds. Two optimal methods, namely the extrapolated (one-step) and the deferred-correction (two-step) methods, are formulated based on appropriate extra boundary conditions and an appropriate perturbation of the operators of the differential equation, boundary conditions and extra boundary conditions. The convergence analysis and error bounds are developed using a Green's functions approach. The analysis shows that the maximum discrete error is of sixth order, and the maximum global error is of fifth order for the optimal methods.

The properties of the matrices arising from the optimal methods for a certain class of problems are studied. Non-optimal collocation methods based on different extra boundary conditions are also investigated. Problems with layers are also considered, and a grid refinement technique is presented. The theoretical behavior of the collocation methods is verified by numerical results.

Dedication

To my parents, Ruzhen Niu and Rende Liu.

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Chapter 1

Introduction

In this thesis, we consider spline collocation methods to obtain a numerical solution of linear second-order two-point boundary value problems (BVPs) which consist of the operator equation

$$Lu \equiv p(x)u''(x) + q(x)u'(x) + r(x)u(x) = f(x), \quad x \in [a, b] \equiv I \tag{1.1}$$

subject to boundary conditions

$$\begin{aligned} Bu(a) &= g_1, && (1.2)\\ Bu(b) &= g_2, && (1.3) \end{aligned}$$

where B is a linear differential operator, $u(x)$ is the unknown function, $p(x), q(x), r(x)$ and $f(x)$ are given functions, and g_1 and g_2 are given scalars. The methods presented can easily be extended to non-separable boundary conditions.

We first briefly present the generic spline collocation method, then focus on quartic spline collocation methods for second-order two-point BVPs. Let us denote a partition Δ of $[a, b]$ by

$$\Delta = \{x_0, x_1, \cdots, x_{N-1}, x_N\}, \;\; a = x_0 < x_1 \cdots x_{N-1} < x_N = b.$$

We choose an approximation space S of dimension K. A spline is a piecewise polynomial $y(x)$ of certain degree associated with the partition Δ which is continuous at the knots, and its derivatives are continuous up to certain order at the knots. The maximum continuity (smoothness) of a spline of degree D is $D-1$. One way to represent a spline that belongs to such a space S is by using a set of piecewise polynomial basis functions $\phi_1(x), \phi_2(x), \cdots, \phi_K(x)$ that span the space S. Then, any $u_\Delta \in S$ can be written as $u_\Delta = \sum_{i=1}^{K} c_i \phi_i(x)$, where the scalars c_i, $i = 1, \cdots, K$, are referred to as the *degrees of freedom*.

Spline collocation approximates the solution of (1.1)-(1.3) by a spline $u_\Delta \in S$, for some chosen space S. We can determine the approximation u_Δ by solving for the degrees of freedom c_i, $i = 1, \cdots, K$. According to the collocation methodology, we choose a set of data points, referred to as *collocation points* in the interval I, some of which lie in the interior and some on the boundary of I. Let T and T_B be the set of collocation points where the operators L and B, respectively are evaluated. Usually, T consists of all interior collocation points and possibly some boundary points, while T_B consists of boundary points, which, for the BVP (1.1)-(1.3) are $x = a$ and $x = b$. In the standard collocation method, we solve for the unknown defined u_Δ by satisfying the collocation conditions

$$Lu_\Delta(x) = f(x), \quad x \in T, \tag{1.4}$$

$$Bu_\Delta(x) = g(x), \quad x = a \quad \text{and} \quad x = b. \tag{1.5}$$

These collocation conditions give rise to a system of equations with respect to the unknown degrees of freedom c_i, $i = 1, \cdots, K$.

The choices of approximation space, basis functions and collocation points are important for the accuracy of the approximation u_Δ, and the efficiency of the resulting collocation methods.

We choose the space of quartic splines with respect to a partition Δ of I as the approximation space. The functions in this space have $C^3[I]$ continuity. In the literature [1], [5], [9] and [15], the collocation points are usually the grid points of the partition for the splines of odd degree, or the midpoints of the partition for the splines of even degree. We adopt this strategy and choose the midpoints of the partition Δ, and two boundary points, $x = a$ and $x = b$, as the collocation points for our quartic spline collocation methods.

The dimension of the approximation space S for the quartic splines we are considering is $K = N + 4$. Then any $u_\Delta(x) \in S$ can be represented by

$$u_\Delta(x) = \sum_{i=-1}^{N+2} c_i \phi_i(x),$$

for a given set of basis functions $\phi_{-1}(x)$, $\phi_0(x), \cdots, \phi_{N+2}(x)$. Therefore, we need $N+4$ linearly independent conditions to determine c_i, $i = -1, \cdots, N+2$, and thus uniquely specify $u_\Delta(x)$.

By applying (1.4) at the N midpoints and (1.5) at the two boundary collocation points, we only have $N + 2$ equations for the degrees of freedom c_i, $i = -1, \cdots, N + 2$. Another two collocation conditions are needed in order to uniquely determine the degrees of freedom. These two collocation conditions are referred to as *extra boundary conditions* or *auxiliary boundary conditions* [9] since they will generally involve the boundary or near-boundary points. These extra boundary conditions are necessary for any quartic spline collocation method for second-order two-point BVPs, and the choice of them can be critical to formulate an optimal collocation method. In developing a quintic spline collocation method for second-order two-point BVPs, Irodotou-Ellina, Houstis and Kim [9] present one choice of auxiliary boundary conditions

$$Au = h\frac{d}{dx}(Lu),$$

where A is an operator for the auxiliary boundary conditions, and $h = \frac{b-a}{N}$. In this thesis, we present two choices of extra boundary conditions which lead to optimal quartic spline collocation methods. They involve developing a perturbation of L and applying it to the two boundary or near-boundary grid points. We also present three choices of extra boundary conditions which form non-optimal methods.

Collocation can be viewed as a generalization of interpolation. The collocation method reduces to interpolation if the differential operators L and B are the identity operators. The order of convergence of a collocation method can be analyzed relative to that of the interpolant in the same approximation space. The order of convergence of a certain collocation method is called *optimal*, if the order of convergence of the collocation method is the same as that of the interpolant in the same approximation space.

Some researchers have shown that the standard formulation of spline collocation methods gives rise to sub-optimal order of convergence, when splines of maximum smoothness are used. Russell and Shampine [12] present a spline collocation method for solving m-th order two-point linear or non-linear BVPs, and analyze the order of convergence of certain spline collocation methods. It is shown in [12] that the standard formulation of spline collocation methods is sub-optimal. An optimal spline collocation method using Gauss points for collocation points for m-th order two-point BVPs is introduced in [2]. This collocation method uses piecewise polynomials of degree $D \geq m+1$ in $C^{m-1}[a,b]$ (i.e. continuity below maximum) and more than one collocation point per subinterval of the partition. Therefore, a smooth ($C^m[a,b]$) spline collocation method, which uses only one data point per subinterval of the partition, has an advantage over the collocation method at Gauss points in terms of the size and the complexity of the arising linear systems. However, the success and application of standard spline collocation methods can be affected by their sub-optimal order of convergence.

Relatively recently, a class of optimal spline collocation methods have been formulated and analyzed by researchers. These optimal methods are based on an appropriate perturbation of the differential operators L and B. Applying this perturbation technique to the second-order two-point BVP, Daniel and Swartz in [7] developed an optimal cubic spline collocation using grid points as collocation points, Houstis, Christara and Rice in [5] developed an optimal quadratic spline collocation using midpoints as collocation points, and Irodotou-Ellina, Houstis and Kim in [9] developed an optimal quintic spline collocation using grid points as collocation points. For the fourth-order two-point BVPs, Irodotou-Ellina and Houstis [8] developed an optimal quintic spline collocation using grid points as collocation points, and Ying Zhu [16] developed an optimal quartic spline collocation using midpoints as collocation points.

In this thesis, we formulate optimal quartic spline collocation methods for second-order two-point BVPs based on the same perturbation techniques as used in the optimal spline collocation methods of [7], [5], [9], [8] and [16]. There are two optimal quartic spline collocation methods arising by the perturbation technique, the *extrapolated method* (one-step method) and the *deferred-correction method* (two-step method). Let us denote by E the operator corresponding to the extra boundary conditions, by t_1 and t_2 the collocation points where the extra boundary conditions will be applied, and by e_1 and e_2 the scalars determined with respect to the choice of E and t_1, t_2. Also let us denote by $\bar{L}$, $\bar{B}$ and $\bar{E}$ the perturbations of the differential operators L, B and E, respectively. In the one-step method, we determine a quartic spline $u_\Delta(x)$ that satisfies the conditions

$$\begin{aligned}(L+\bar{L})u_\Delta(x) &= f(x), \quad x \in T,\\ (B+\bar{B})u_\Delta(x) &= g_i, \qquad (x,i)=(a,1) \ \text{ and } \ (x,i)=(b,2),\\ (E+\bar{E})u_\Delta(t_i) &= e_i, \qquad i=1,2.\end{aligned}$$

The two-step method consists of two steps. In the first step, we apply the standard non-optimal quartic spline collocation method in which the fourth order collocation approximation $v(x)$ is determined by the conditions

$$\begin{aligned}Lv(x) &= f(x), \quad x \in T,\\ Bv(x) &= g_i, \qquad (x,i)=(a,1) \ \text{ and } \ (x,i)=(b,2),\\ Ev(t_i) &= e_i, \qquad i=1,2.\end{aligned}$$

In the second step, a sixth order collocation approximation $u_\Delta(x)$ is determined by the conditions

$$\begin{aligned}Lu_\Delta(x) &= f(x)-\bar{L}v(x), \quad x \in T,\\ Bu_\Delta(x) &= g_i-\bar{B}v(x), \qquad (x,i)=(a,1) \ \text{ and } \ (x,i)=(b,2),\\ Eu_\Delta(t_i) &= e_i-\bar{E}v(t_i), \qquad i=1,2.\end{aligned}$$

We study the properties of the system of equations arising in each of the two optimal methods for certain problems. We present the convergence analysis and error bounds for both the one-step and the two-step methods by using a Green's function approach.

This thesis utilizes the definition of a quartic spline interpolant and the theorems on the error of this interpolant in [16] by Zhu to formulate and analyze the collocation methods. A number of relations which Zhu has developed independently and used also appear in papers by Usmani [11] and by Hoskins and Meek [14].

In this thesis, we also consider boundary value problems in which the solution $u(x)$ has special features. In some BVPs, $u(x)$ varies rapidly in certain narrow parts of the domain, referred to as *layers* [6]. The error of $u_\Delta(x)$ in the layers tends to be very large, and dominates the errors over the whole domain. In order to achieve errors in the layers of approximately the same order as those in the non-layer areas, we develop a technique of grid refinement in the layer area. More specifically, we first apply an optimal quartic spline collocation method to the whole domain. Then we apply an error estimation technique to identify the layer(s), and a grid size estimator to predict the number of additional grid points N' needed in the layer region to keep the error below a certain tolerance. Finally we apply the optimal quartic spline collocation method to the refined mesh of N' grid points.

The organization of this thesis is as follows. Chapter 2 presents some properties of a certain quartic spline interpolant introduced in [16]. In Chapter 3, we investigate some optimal and non-optimal quartic spline collocation methods, and study the properties of the matrices arising from the optimal methods. The convergence analysis and error bounds for the optimal methods are discussed in Chapter 4. In Chapter 5 we present the techniques used for layer problems. Chapter 6 presents the results of the numerical experiments that verify the theoretical convergence of the collocation methods. Finally we present our conclusions in Chapter 7.

Chapter 2

Quartic Spline Interpolation

In this chapter, we introduce the quartic spline space and basis functions, then review the properties of a certain quartic spline interpolant introduced in [16].

2.1 Basis functions for quartic splines

Let Δ be a partition of the interval $I = [a, b]$ into subintervals using uniform grid points $x_i = a + i \times h$, where $h = \frac{b-a}{N}$, $i = 0, \cdots, N$. Consider also the set of collocation points

$$T = \{\tau_0 = x_0, \tau_1 = \frac{x_0 + x_1}{2}, \cdots, \tau_i = \frac{x_{i-1} + x_i}{2}, \cdots, \tau_{N+1} = x_N\}. \tag{2.1}$$

Note that T includes the midpoints of the subintervals and the two boundary points. The allocation of grid points and midpoints on the interval I is shown in Figure 2.1.

Let S^4_Δ be the space of quartic splines with respect to Δ with smoothness $C^3[a, b]$. One way to represent splines with respect to Δ is to use standard piecewise polynomial basis functions. Denote the basis functions for quartic B-splines by $\phi_i(x), i = -1, 0, 1, \cdots, N, N+1, N+2$.

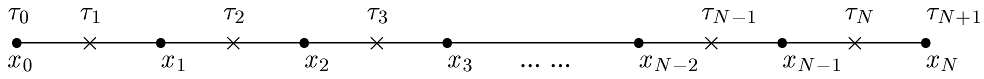

Figure 2.1: The uniform grid and midpoints

To define the standard quartic B-spline basis function $\phi_i(x)$, we first consider the model function $\phi(x)$

$$\phi(x) \equiv \frac{1}{24} \begin{cases} x^4 & \text{for } 0 \le x \le 1, \\ x^4 - 5(x-1)^4 & \text{for } 1 \le x \le 2, \\ x^4 - 5(x-1)^4 + 10(x-2)^4 & \text{for } 2 \le x \le 3, \\ x^4 - 5(x-1)^4 + 10(x-2)^4 - 10(x-3)^4 & \text{for } 3 \le x \le 4, \\ x^4 - 5(x-1)^4 + 10(x-2)^4 - 10(x-3)^4 + 5(x-4)^4 & \text{for } 4 \le x \le 5, \\ 0 & \text{elsewhere.} \end{cases} \tag{2.2}$$

We then define

$$\phi_i(x) = \phi(\frac{x-a}{h} - i + 3), \quad i = -1, \cdots, N+2 \tag{2.3}$$

as the basis functions for the quartic spline space. The basis functions $\phi_i(x)$ have the following features

- $\phi_i(x)$ is defined everywhere in I.
- $\phi_i(x)$ has $C^3[I]$ continuity.

The ϕ_i are linearly independent for $x \in [a, b]$, and therefore the dimension of S^4_Δ is $N + 4$. We can therefore write any quartic spline $S \in S^4_\Delta$ as

$$S(x) = \sum_{i=-1}^{N+2} c_i \phi_i(x).$$

Table 2.1 shows the values of these standard quartic spline basis functions and their derivatives evaluated at the grid points and the collocation points. We use $\phi'_{i-1}(\tau_i)$ as an example to illustrate how to compute the values of the table. From (2.2) we have

$$\phi'(x) \equiv \frac{4}{24} \begin{cases} x^3 & \text{for } 0 \le x \le 1, \\ x^3 - 5(x-1)^3 & \text{for } 1 \le x \le 2, \\ x^3 - 5(x-1)^3 + 10(x-2)^3 & \text{for } 2 \le x \le 3, \\ x^3 - 5(x-1)^3 + 10(x-2)^3 - 10(x-3)^3 & \text{for } 3 \le x \le 4, \\ x^3 - 5(x-1)^3 + 10(x-2)^3 - 10(x-3)^3 + 5(x-4)^3 & \text{for } 4 \le x \le 5, \\ 0 & \text{elsewhere.} \end{cases} \tag{2.4}$$

From (2.3) we have the first derivative of the basis function $\phi_i(x)$

$$\phi'_i(x) = \frac{1}{h}\phi'(\frac{x-a}{h} - i + 3), \quad i = -1, \cdots, N+2,$$

which implies

$$\begin{aligned} \phi'_{i-1}(\tau_i) &= \frac{1}{h}\phi'(\frac{7}{2}), \\ &= -\frac{11}{24h}. \end{aligned}$$

Table 2.1: Table of the values of the quartic spline basis functions and their derivatives.

	x_{i-2}	τ_{i-1}	x_{i-1}	τ_i	x_i	τ_{i+1}	x_{i+1}
ϕ_{i-2}	$\frac{11}{24}$	$\frac{76}{24\times16}$	$\frac{1}{24}$	$\frac{1}{24\times16}$	0	0	0
ϕ_{i-1}	$\frac{11}{24}$	$\frac{230}{24\times16}$	$\frac{11}{24}$	$\frac{76}{24\times16}$	$\frac{1}{24}$	$\frac{1}{24\times16}$	0
ϕ_i	$\frac{1}{24}$	$\frac{76}{24\times16}$	$\frac{11}{24}$	$\frac{230}{24\times16}$	$\frac{11}{24}$	$\frac{76}{24\times16}$	$\frac{1}{24}$
ϕ_{i+1}	0	$\frac{1}{24\times16}$	$\frac{1}{24}$	$\frac{76}{24\times16}$	$\frac{11}{24}$	$\frac{230}{24\times16}$	$\frac{11}{24}$
ϕ_{i+2}	0	0	0	$\frac{1}{24\times16}$	$\frac{1}{24}$	$\frac{76}{24\times16}$	$\frac{11}{24}$

	x_{i-2}	τ_{i-1}	x_{i-1}	τ_i	x_i	τ_{i+1}	x_{i+1}
ϕ'_{i-2}	$-\frac{12}{24h}$	$-\frac{11}{24h}$	$-\frac{4}{24h}$	$-\frac{1}{24\times2h}$	0	0	0
ϕ'_{i-1}	$\frac{12}{24h}$	0	$-\frac{12}{24h}$	$-\frac{11}{24h}$	$-\frac{4}{24h}$	$-\frac{1}{24\times2h}$	0
ϕ'_i	$\frac{4}{24h}$	$\frac{11}{24h}$	$\frac{12}{24h}$	0	$-\frac{12}{24h}$	$-\frac{11}{24h}$	$-\frac{4}{24h}$
ϕ'_{i+1}	0	$\frac{1}{24\times2h}$	$\frac{4}{24h}$	$\frac{11}{24h}$	$\frac{12}{24h}$	0	$-\frac{12}{24h}$
ϕ'_{i+2}	0	0	0	$\frac{1}{24\times2h}$	$\frac{4}{24h}$	$\frac{11}{24h}$	$\frac{12}{24h}$

	x_{i-2}	τ_{i-1}	x_{i-1}	τ_i	x_i	τ_{i+1}	x_{i+1}
ϕ''_{i-2}	$-\frac{12}{24h^2}$	$\frac{12}{24h^2}$	$\frac{12}{24h^2}$	$\frac{3}{24h^2}$	0	0	0
ϕ''_{i-1}	$-\frac{12}{24h^2}$	$-\frac{30}{24h^2}$	$-\frac{12}{24h^2}$	$\frac{12}{24h^2}$	$\frac{12}{24h^2}$	$\frac{3}{24h^2}$	0
ϕ''_i	$\frac{12}{24h^2}$	$\frac{12}{24h^2}$	$-\frac{12}{24h^2}$	$-\frac{30}{24h^2}$	$-\frac{12}{24h^2}$	$\frac{12}{24h^2}$	$\frac{12}{24h^2}$
ϕ''_{i+1}	0	$\frac{3}{24h^2}$	$\frac{12}{24h^2}$	$\frac{12}{24h^2}$	$-\frac{12}{24h^2}$	$-\frac{30}{24h^2}$	$-\frac{12}{24h^2}$
ϕ''_{i+2}	0	0	0	$\frac{3}{24h^2}$	$\frac{12}{24h^2}$	$\frac{12}{24h^2}$	$-\frac{12}{24h^2}$

We give one example of a quartic spline interpolant $\tilde{S}(x)$. Suppose $u(\tau_i), i = 1, \cdots, N$, $u(x_0)$, $u(x_1)$, $u(x_{N-1})$ and $u(x_N)$ are given. Consider the equations

$$\tilde{S}(x_0) = u(x_0), \tag{2.5}$$

$$\tilde{S}(x_1) = u(x_1), \tag{2.6}$$

$$\tilde{S}(\tau_i) = u(\tau_i), \quad \text{for} \quad i = 1, \cdots, N, \tag{2.7}$$

$$\tilde{S}(x_{N-1}) = u(x_{N-1}), \tag{2.8}$$

$$\tilde{S}(x_N) = u(x_N). \tag{2.9}$$

These equations give rise to the system of equations

$$\begin{bmatrix} 1 & 11 & 11 & 1 & & & & & & \\ & 1 & 11 & 11 & 1 & & & & & \\ \frac{1}{16} & \frac{76}{16} & \frac{230}{16} & \frac{76}{16} & \frac{1}{16} & & & & & \\ & \frac{1}{16} & \frac{76}{16} & \frac{230}{16} & \frac{76}{16} & \frac{1}{16} & & & & \\ & & & \ddots & \ddots & \ddots & & & & \\ & & & & \ddots & \ddots & \ddots & & & \\ & & & & \frac{1}{16} & \frac{76}{16} & \frac{230}{16} & \frac{76}{16} & \frac{1}{16} \\ & & & & 1 & 11 & 11 & 1 & \\ & & & & & 1 & 11 & 11 & 1 \end{bmatrix} \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_N \\ c_{N+1} \\ c_{N+2} \end{bmatrix} = 24 \begin{bmatrix} u(x_0) \\ u(x_1) \\ u(\tau_1) \\ u(\tau_2) \\ \vdots \\ \vdots \\ u(\tau_N) \\ u(x_{N-1}) \\ u(x_N) \end{bmatrix}. \tag{2.10}$$

This system of equations is nonsingular and therefore there is a unique solution $\tilde{S}(x) \in S^4_\Delta$.

2.2 Error of quartic spline interpolation

In this section we utilize the definition of a quartic spline interpolant S and theorems on the error of S as given in [16]. The definition and theorems will be used in Chapter 3 to formulate the optimal quartic spline collocation methods.

We first introduce some notation which will be used throughout the thesis. We denote $u(\tau_i), i = 0, \cdots, N+1$ by u_i, the kth derivative $D^k u(x)$ of $u(x)$ by $u^{(k)}$. We utilize the definition of a particular quartic spline interpolant S of $u \in C^{10}[a, b]$ as given in [16]. The interpolant S satisfies the conditions

$$\begin{aligned} S_i &= u_i, && \text{for} \quad i = 1, \cdots, N, && (2.11) \\ S_i^{(4)} &= u_i^{(4)} - \frac{h^2}{24} u_i^{(6)} + \frac{7h^4}{5760} u_i^{(8)}, && \text{for} \quad i = 1, 2, N-1, N. && (2.12) \end{aligned}$$

From [16] we have the relations

$$S_i^{(1)} = u_i^{(1)} - \frac{7h^4}{5760} u_i^{(5)} + O(h^6), \tag{2.13}$$

$$S_i^{(2)} = u_i^{(2)} + \frac{7h^4}{1920} u_i^{(6)} + O(h^6), \tag{2.14}$$

for $i = 1, \cdots, N$, and

$$\begin{aligned} S(x_i) &= u(x_i) + O(h^6), && (2.15) \\ S^{(1)}(x_i) &= u^{(1)}(x_i) + \frac{h^4}{720} u^{(5)}(x_i) + O(h^6), && (2.16) \\ S^{(2)}(x_i) &= u^{(2)}(x_i) - \frac{h^4}{240} u^{(6)}(x_i) + O(h^6), && (2.17) \end{aligned}$$

for $i = 0, \cdots, N$.

Further we define two discrete difference operators, δg_i and $\delta^2 g_i$, which will be used to formulate the quartic spline collocation methods in Chapter 3

$$\begin{aligned}
\delta g_i &= g_{i-1} - 2g_i + g_{i+1}, && \text{for} \quad i = 2, \cdots, N-1, \\
\delta^2 g_i &= g_{i-2} - 4g_{i-1} + 6g_i - 4g_{i+1} + g_{i+2}, && \text{for} \quad i = 3, \cdots, N-2.
\end{aligned}$$

From (2.13)-(2.17), the following relations are shown in [16]

$$u_i^{(5)} = \frac{\delta^2 S_i^{(1)}}{h^4} + O(h^2), \tag{2.18}$$

$$u_i^{(6)} = \frac{\delta^2 S_i^{(2)}}{h^4} + O(h^2), \tag{2.19}$$

for $i = 3, \cdots, N-2$.

If $u \in C^{10}[a, b]$, then the following approximations to $u^{(5)}$ and $u^{(6)}$ are justified in [16] at boundary and near-boundary points $\{x_0, \tau_1, \tau_2, \tau_{N-1}, \tau_N, x_N\}$ for $k = 5, 6$

$$u^{(k)}(x_0) = \frac{1}{2h^4}(7\delta^2 S_3^{(k-4)} - 5\delta^2 S_4^{(k-4)}) + O(h^2), \tag{2.20}$$

$$u^{(k)}(\tau_1) = \frac{1}{h^4}(3\delta^2 S_3^{(k-4)} - 2\delta^2 S_4^{(k-4)}) + O(h^2), \tag{2.21}$$

$$u^{(k)}(\tau_2) = \frac{1}{h^4}(2\delta^2 S_3^{(k-4)} - \delta^2 S_4^{(k-4)}) + O(h^2), \tag{2.22}$$

$$u^{(k)}(\tau_{N-1}) = \frac{1}{h^4}(2\delta^2 S_{N-2}^{(k-4)} - \delta^2 S_{N-3}^{(k-4)}) + O(h^2), \tag{2.23}$$

$$u^{(k)}(\tau_N) = \frac{1}{h^4}(3\delta^2 S_{N-2}^{(k-4)} - 2\delta^2 S_{N-3}^{(k-4)}) + O(h^2), \tag{2.24}$$

$$u^{(k)}(x_N) = \frac{1}{2h^4}(7\delta^2 S_{N-2}^{(k-4)} - 5\delta^2 S_{N-3}^{(k-4)}) + O(h^2). \tag{2.25}$$

In a similar way, if $u \in C^{10}[a, b]$, we can show, for $k = 5, 6$

$$u^{(k)}(x_1) = \frac{1}{2h^4}(5\delta^2 S_3^{(k-4)} - 3\delta^2 S_4^{(k-4)}) + O(h^2), \tag{2.26}$$

$$u^{(k)}(x_{N-1}) = \frac{1}{2h^4}(5\delta^2 S_{N-2}^{(k-4)} - 3\delta^2 S_{N-3}^{(k-4)}) + O(h^2). \tag{2.27}$$

We present the theorem about the global error bounds from [16].

THEOREM 1 *Let S be the quartic spline interpolant of $u \in C^{10}[I]$ defined by (2.11)-(2.12). We have the global error bounds*

$$\| S^{(k)} - u^{(k)} \|_\infty = O(h^{5-k}),$$

for $k = 0, \cdots, 4$.

Chapter 3

Quartic Spline Collocation

In this chapter, we investigate some quartic spline collocation methods for the solution of second-order two-point boundary problems. Some of the proposed methods are of non-optimal order of convergence and some of optimal order.

3.1 Second-order two-point BVP

In this section, we describe the problem to be solved. A linear second-order two-point boundary value problem consists of the operator equation

$$Lu = p(x)u''(x) + q(x)u'(x) + r(x)u(x) = f(x), \quad x \in [a, b] \tag{3.1}$$

subject to boundary conditions

$$\begin{aligned} Bu \equiv \{ \;\; B_1 u &= \alpha_1 u(a) + \beta_1 u'(a) = g_1, \\ B_2 u &= \alpha_2 u(b) + \beta_2 u'(b) = g_2 \;\; \}, \end{aligned} \tag{3.2}$$

where $u(x)$ is an unknown function, $p(x), q(x), r(x)$ and $f(x)$ are given functions, and α_1, α_2, β_1, β_2, g_1 and g_2 are given scalars. Note that the boundary conditions (3.2) are separable. We can easily adjust the methods proposed to non-separable boundary conditions, but for simplicity, we choose not to. Throughout the rest of the thesis we assume the solution u to the problem is in $C^{10}[I]$ and S is the interpolant of u defined by (2.11) and (2.12).

3.2 Discretization

Let us denote by $u_\Delta(x)$ a quartic spline collocation approximation to $u(x)$. We have

$$\begin{aligned} u_\Delta(x) &= \sum_{i=-1}^{N+2} c_i \phi_i(x), \\ u_\Delta^{(k)}(x) &= \sum_{i=-1}^{N+2} c_i \phi_i^{(k)}(x), \quad k = 1, 2, \end{aligned}$$

where $\phi_i(x)$ and c_i, $i = -1, \cdots, N+2$, are the quartic spline basis functions and respective degrees of freedom.

It is clear that $N+4$ linearly independent conditions are required in order to uniquely determine a quartic spline. The straightforward formulation of collocation applies the operator and boundary operator equations (3.1)-(3.2) at a certain prescribed set of collocation points.

Consider the set of collocation points T, as defined in (2.1). By applying collocation to (3.1) and (3.2) at the points of T, we have $N+2$ linearly independent conditions. More specifically, the $N+2$ collocation conditions are

$$\begin{aligned} Lu_\Delta(x) &= f(x), \quad \text{for} \quad x = \tau_i, i = 1, \cdots, N, \\ B_1 u_\Delta(x) &= g_1, \quad \text{for} \quad x = x_0, \\ B_2 u_\Delta(x) &= g_2, \quad \text{for} \quad x = x_N. \end{aligned}$$

In order to uniquely define the quartic spline $u_\Delta(x)$, another two boundary conditions are required. We refer to these two conditions as *extra boundary conditions* in contrast to the boundary conditions arising from (3.2). The choice of the extra boundary conditions is critical in formulating any quartic spline collocation method and, in particular, an optimal quartic spline collocation method. In the next three sections, we discuss in detail possible choices for these extra boundary conditions.

Let E_1 and E_2 be the operators of the extra boundary conditions, and let t_1 and t_2 be the collocation points, where these extra boundary conditions will be applied. The complete set of collocation conditions is then

$$\begin{aligned} B_1 u_\Delta(x_0) &= g_1, \\ E_1 u_\Delta(t_1) &= e_1, \\ L u_\Delta(\tau_i) &= f(\tau_i), \quad i = 1, \cdots, N, \\ E_2 u_\Delta(t_2) &= e_2, \\ B_2 u_\Delta(x_N) &= g_2, \end{aligned}$$

where the scalars e_1 and e_2 are determined with respect to the choice of E_1 and E_2. These collocation conditions give rise to a system of $N+4$ equations with $N+4$ unknowns.

In the next sections, we discuss various choices for E_1, E_2, t_1 and t_2.

3.3 Extra boundary conditions $Lu = f$ at x_1 and x_{N-1}

One choice of extra boundary conditions which leads to a uniquely defined quartic spline collocation approximation as well as to an optimal quartic spline collocation method is to apply the operator L at the grid points, x_1 and x_{N-1}, that is, the nodes near the boundary points. This implies that the extra boundary condition operators E_1 and E_2 are in fact L, and the extra boundary conditions are

$$Lu = p(x)u''(x) + q(x)u'(x) + r(x)u(x) = f(x), \text{ for } x = x_1 = a + h \text{ and } x = x_{N-1} = b - h. \tag{3.3}$$

3.3.1 Quartic spline interpolant relations

By (2.13)-(2.17), the interpolant S satisfies

$$LS_i = f_i + \frac{7h^4}{1920} p_i u_i^{(6)} - \frac{7h^4}{5760} q_i u_i^{(5)} + O(h^6), \quad \text{for} \quad i = 1, \cdots, N, \tag{3.4}$$

and

$$\begin{aligned} B_1 S(x_0) &= g_1 + \frac{h^4}{720} \beta_1 u^{(5)}(x_0) + O(h^6), && (3.5) \\ B_2 S(x_N) &= g_2 + \frac{h^4}{720} \beta_2 u^{(5)}(x_N) + O(h^6). && (3.6) \end{aligned}$$

Taking into account the extra boundary conditions (3.3), S also satisfies by (2.15)-(2.17)

$$\begin{aligned} LS(x_1) &= f(x_1) - \frac{h^4}{240} p(x_1) u^{(6)}(x_1) \\ &+ \frac{h^4}{720} q(x_1) u^{(5)}(x_1) + O(h^6), && (3.7) \end{aligned}$$

$$\begin{aligned} LS(x_{N-1}) &= f(x_{N-1}) - \frac{h^4}{240} p(x_{N-1}) u^{(6)}(x_{N-1}) \\ &+ \frac{h^4}{720} q(x_{N-1}) u^{(5)}(x_{N-1}) + O(h^6). && (3.8) \end{aligned}$$

The approximation of derivatives stated in (2.18)-(2.25) allow (3.4) and (3.5)-(3.6) to be written as

$$\begin{aligned} LS_1 &- \frac{7}{1920}p_1(3\delta^2S_3^{(2)} - 2\delta^2S_4^{(2)}) + \frac{7}{5760}q_1(3\delta^2S_3^{(1)} - 2\delta^2S_4^{(1)}) \\ &= f_1 + O(h^6), \end{aligned} \tag{3.9}$$

$$\begin{aligned} LS_2 &- \frac{7}{1920}p_2(2\delta^2S_3^{(2)} - \delta^2S_4^{(2)}) + \frac{7}{5760}q_2(2\delta^2S_3^{(1)} - \delta^2S_4^{(1)}) \\ &= f_2 + O(h^6), \end{aligned} \tag{3.10}$$

$$\begin{aligned} LS_i &- \frac{7}{1920}p_i\delta^2S_i^{(2)} + \frac{7}{5760}q_i\delta^2S_i^{(1)} \\ &= f_i + O(h^6), \quad \text{for} \quad i = 3, \cdots, N-2, \end{aligned} \tag{3.11}$$

$$\begin{aligned} LS_{N-1} &- \frac{7}{1920}p_{N-1}(2\delta^2S_{N-2}^{(2)} - \delta^2S_{N-3}^{(2)}) + \frac{7}{5760}q_{N-1}(2\delta^2S_{N-2}^{(1)} - \delta^2S_{N-3}^{(1)}) \\ &= f_{N-1} + O(h^6), \end{aligned} \tag{3.12}$$

$$\begin{aligned} LS_N &- \frac{7}{1920}p_N(3\delta^2S_{N-2}^{(2)} - 2\delta^2S_{N-3}^{(2)}) + \frac{7}{5760}q_N(3\delta^2S_{N-2}^{(1)} - 2\delta^2S_{N-3}^{(1)}) \\ &= f_N + O(h^6), \end{aligned} \tag{3.13}$$

and

$$B_1S(x_0) - \frac{1}{1440}\beta_1(7\delta^2S_3^{(1)} - 5\delta^2S_4^{(1)}) = g_1 + O(h^6), \tag{3.14}$$

$$B_2S(x_N) - \frac{1}{1440}\beta_2(7\delta^2S_{N-2}^{(1)} - 5\delta^2S_{N-3}^{(1)}) = g_2 + O(h^6). \tag{3.15}$$

From (2.26)-(2.27), we can write (3.7) and (3.8) as

$$\begin{aligned} LS(x_1) &+ \frac{1}{480}p(x_1)(5\delta^2S_3^{(2)} - 3\delta^2S_4^{(2)}) - \frac{1}{1440}q(x_1)(5\delta^2S_3^{(1)} - 3\delta^2S_4^{(1)}) \\ &= f(x_1) + O(h^6), \end{aligned} \tag{3.16}$$

$$\begin{aligned} LS(x_{N-1}) &+ \frac{1}{480}p(x_{N-1})(5\delta^2S_{N-2}^{(2)} - 3\delta^2S_{N-3}^{(2)}) - \frac{1}{1440}q(x_{N-1})(5\delta^2S_{N-2}^{(1)} - 3\delta^2S_{N-3}^{(1)}) \\ &= f(x_{N-1}) + O(h^6). \end{aligned} \tag{3.17}$$

Denote by L' the discrete operator defined by the left side of equations (3.9)-(3.13) and (3.16)-(3.17). Similarly, denote by B' the discrete operator defined by the left side of equations (3.14)-(3.15). The operators L' and B' can be viewed as perturbed operators L and B, respectively, at the collocation points. Then it is clear that S satisfies the following relations

$$\begin{aligned} L'S_i &= f_i + O(h^6), \quad i = 1, \cdots, N, \\ B'_k S(x_j) &= g_k + O(h^6), \quad k = 1, 2 \quad \text{and} \quad j = 0, N, \\ L'S(x_m) &= f(x_m) + O(h^6), \quad m = 1, N-1. \end{aligned}$$

We summarize the above observations as the following lemma.

LEMMA 1 *Let S be the quartic spline interpolant of the solution $u \in C^{10}[a,b]$ to (3.1)-(3.2). Then S satisfies the relations*

$$\begin{aligned} [LS - f]_{\tau_i} &= O(h^4), \quad i = 1, \cdots, N, \\ [BS - g]_{x_0, x_N} &= O(h^4), \\ [LS - f]_{x_1, x_{N-1}} &= O(h^4), \end{aligned}$$

$$\begin{aligned} [L'S - f]_{\tau_i} &= O(h^6), \quad i = 1, \cdots, N, \\ [B'S - g]_{x_0, x_N} &= O(h^6), \\ [L'S - f]_{x_1, x_{N-1}} &= O(h^6). \end{aligned}$$

3.3.2 Formulation of the quartic spline collocation methods

Before we formulate the optimal quartic spline collocation methods, we present the straightforward and non-optimal quartic spline collocation method. In the straightforward and non-optimal method, we determine a quartic spline $v \in S^4_\Delta$ that satisfies the relations

$$\begin{aligned} [Lv - f]_{\tau_i} &= 0, \quad i = 1, \cdots, N, \\ [Bv - g]_{x_0, x_N} &= 0, \\ [Lv - f]_{x_1, x_{N-1}} &= 0. \end{aligned}$$

In one formulation of the optimal quartic spline collocation method, we determine a quartic spline $u_\Delta \in S^4_\Delta$ that satisfies the relations

$$[L'u_\Delta - f]_{\tau_i} = 0, \quad i = 1, \cdots, N, \tag{3.18}$$

$$[B'u_\Delta - g]_{x_0, x_N} = 0, \tag{3.19}$$

$$[L'u_\Delta - f]_{x_1, x_{N-1}} = 0. \tag{3.20}$$

We refer to this method as the *one-step* or *extrapolated* quartic spline collocation method.

We have an alternative formulation of this optimal method in which we determine $u_\Delta \in S^4_\Delta$ through a *two-step* quartic spline collocation method. The two-step collocation method is defined by the follows steps:

Step 1:

Determine $v \in S^4_\Delta$ such that it satisfies

$$[Lv - f]_{\tau_i} = 0, \quad i = 1, \cdots, N, \tag{3.21}$$

$$[Bv - g]_{x_0, x_N} = 0, \tag{3.22}$$

$$[Lv - f]_{x_1, x_{N-1}} = 0. \tag{3.23}$$

Step 2:

Determine $u_\Delta \in S^4_\Delta$ such that it satisfies

$$[Lu_\Delta - \bar{f}]_{\tau_i} = 0, \quad i = 1, \cdots, N, \tag{3.24}$$

$$[Bu_\Delta - \bar{g}]_{x_0, x_N} = 0, \tag{3.25}$$

$$[Lu_\Delta - \bar{f}]_{x_1, x_{N-1}} = 0, \tag{3.26}$$

where $\bar{f}_i, i = 1, \cdots, N$, $\bar{g}(x_j), j = 0, N$, and $\bar{f}(x_m), m = 1, N-1$, are given by

$$\begin{aligned}
\bar{f}_1 &= f_1 + \frac{7}{1920}p_1(3\delta^2 v_3^{(2)} - 2\delta^2 v_4^{(2)}) \\
&- \frac{7}{5760}q_1(3\delta^2 v_3^{(1)} - 2\delta^2 v_4^{(1)}), \qquad (3.27) \\
\bar{f}_2 &= f_2 + \frac{7}{1920}p_2(2\delta^2 v_3^{(2)} - \delta^2 v_4^{(2)}) \\
&- \frac{7}{5760}q_2(2\delta^2 v_3^{(1)} - \delta^2 v_4^{(1)}), \qquad (3.28)
\end{aligned}$$

$$\bar{f}_i = f_i + \frac{7}{1920}p_i\delta^2 v_i^{(2)} - \frac{7}{5760}q_i\delta^2 v_i^{(1)}, \qquad (3.29)$$

$$\begin{aligned}
\bar{f}_{N-1} &= f_{N-1} + \frac{7}{1920}p_{N-1}(2\delta^2 v_{N-2}^{(2)} - \delta^2 v_{N-3}^{(2)}) \\
&- \frac{7}{5760}q_{N-1}(2\delta^2 v_{N-2}^{(1)} - \delta^2 v_{N-3}^{(1)}), \qquad (3.30) \\
\bar{f}_N &= f_N + \frac{7}{1920}p_N(3\delta^2 v_{N-2}^{(2)} - 2\delta^2 v_{N-3}^{(2)}) \\
&- \frac{7}{5760}q_N(3\delta^2 v_{N-2}^{(1)} - 2\delta^2 v_{N-3}^{(1)}), \qquad (3.31)
\end{aligned}$$

$$\begin{aligned}
\bar{g}(x_0) &= g_1 + \frac{1}{1440}\beta_1(7\delta^2 v_3^{(1)} - 5\delta^2 v_4^{(1)}), \qquad (3.32) \\
\bar{g}(x_N) &= g_2 + \frac{1}{1440}\beta_2(7\delta^2 v_{N-2}^{(1)} - 5\delta^2 v_{N-3}^{(1)}), \qquad (3.33)
\end{aligned}$$

$$\begin{aligned}
\bar{f}(x_1) &= f(x_1) - \frac{1}{480}p(x_1)(5\delta^2 v_3^{(2)} - 3\delta^2 v_4^{(2)}) \\
&+ \frac{1}{1440}q(x_1)(5\delta^2 v_3^{(1)} - 3\delta^2 v_4^{(1)}), \qquad (3.34) \\
\bar{f}(x_{N-1}) &= f(x_{N-1}) - \frac{1}{480}p(x_{N-1})(5\delta^2 v_{N-2}^{(2)} - 3\delta^2 v_{N-3}^{(2)}) \\
&+ \frac{1}{1440}q(x_{N-1})(5\delta^2 v_{N-2}^{(1)} - 3\delta^2 v_{N-3}^{(1)}). \qquad (3.35)
\end{aligned}$$

For the above formulation, we assume that $u^{(5)}$ and $u^{(6)}$ can be approximated at the collocation points by

$$
\begin{aligned}
u^{(k)}(\tau_1) &\approx \frac{1}{h^4}(3\delta^2 v_3^{(k-4)} - 2\delta^2 v_4^{(k-4)}), && (3.36)\\
u^{(k)}(\tau_2) &\approx \frac{1}{h^4}(2\delta^2 v_3^{(k-4)} - \delta^2 v_4^{(k-4)}), && (3.37)\\
u_i^{(k)} &\approx \frac{\delta^2 v_i^{(k-4)}}{h^4}, \quad i = 3, \cdots, N-2, && (3.38)\\
u^{(k)}(\tau_{N-1}) &\approx \frac{1}{h^4}(2\delta^2 v_{N-2}^{(k-4)} - \delta^2 v_{N-3}^{(k-4)}), && (3.39)\\
u^{(k)}(\tau_N) &\approx \frac{1}{h^4}(3\delta^2 v_{N-2}^{(k-4)} - 2\delta^2 v_{N-3}^{(k-4)}), && (3.40)\\
u^{(k)}(x_0) &\approx \frac{1}{2h^4}(7\delta^2 v_3^{(k-4)} - 5\delta^2 v_4^{(k-4)}), && (3.41)\\
u^{(k)}(x_N) &\approx \frac{1}{2h^4}(7\delta^2 v_{N-2}^{(k-4)} - 5\delta^2 v_{N-3}^{(k-4)}), && (3.42)\\
u^{(k)}(x_1) &\approx \frac{1}{2h^4}(5\delta^2 v_3^{(k-4)} - 3\delta^2 v_4^{(k-4)}), && (3.43)\\
u^{(k)}(x_{N-1}) &\approx \frac{1}{2h^4}(5\delta^2 v_{N-2}^{(k-4)} - 3\delta^2 v_{N-3}^{(k-4)}), && (3.44)
\end{aligned}
$$

for $k = 5, 6$.

3.3.3 Matrix analysis

In this section, we verify the solvability of the system of equations arising from a particular case of our BVP. Consider the special case of (3.1)-(3.2) where $p(x) = 1, q(x) = r(x) = 0, \alpha_1 = \alpha_2 = 1, \beta_1 = \beta_2 = 0, g_1 = g_2 = 0$, resulting in the boundary value problem

$$Lu = u''(x) = f(x), \quad x \in [a, b] \tag{3.45}$$

subject to boundary conditions

$$u(a) = 0, \quad u(b) = 0. \tag{3.46}$$

We apply the one-step quartic spline collocation method to (3.45)-(3.46) and verify that the quartic spline $u_\Delta \in S^4_\Delta$ in (3.18)-(3.20) can be determined uniquely.

We determine a system of equations with respect to $u^{(2)}_\Delta(\tau_i)$, $i = 1, \cdots, N$, $u^{(2)}_\Delta(x_1)$ and $u^{(2)}_\Delta(x_{N-1})$. From (3.18) we have

$$L' u_\Delta(\tau_i) = f_i, \quad i = 1, \cdots, N. \tag{3.47}$$

By the definition of operator L', we have, for $i = 3, \cdots, N-2$, the relation

$$\begin{aligned} L' u_\Delta(\tau_i) &= L u_\Delta(\tau_i) - \frac{7}{1920}\delta^2 u^{(2)}_\Delta(\tau_i) \\ &= u^{(2)}_\Delta(\tau_i) - \frac{7}{1920}(u^{(2)}_\Delta(\tau_{i-2}) - 4u^{(2)}_\Delta(\tau_{i-1}) + 6u^{(2)}_\Delta(\tau_i) \\ &\quad -4u^{(2)}_\Delta(\tau_{i+1}) + u^{(2)}_\Delta(\tau_{i+2})). \end{aligned} \tag{3.48}$$

From (3.47) and (3.48), we can obtain the relation

$$\frac{1}{1920}(-7u^{(2)}_\Delta(\tau_{i-2}) + 28u^{(2)}_\Delta(\tau_{i-1}) + 1878u^{(2)}_\Delta(\tau_i) + 28u^{(2)}_\Delta(\tau_{i+1}) - 7u^{(2)}_\Delta(\tau_{i+2})) = f_i,$$

for $i = 3, \cdots, N-2$. Similarly we can obtain relations with respect to $u^{(2)}_\Delta(\tau_i)$, $i = 1, 2, N-1, N$, $u^{(2)}_\Delta(x_1)$ and $u^{(2)}_\Delta(x_{N-1})$. These relations give rise to a system of equations with respect to

$u_\Delta^{(2)}(\tau_i), i = 1, \cdots, N$, $u_\Delta^{(2)}(x_1)$ and $u_\Delta^{(2)}(x_{N-1})$

$$\begin{bmatrix}
1899 & & 98 & -182 & 168 & -77 & 14 & & \\
20 & 1920 & -92 & 168 & -152 & 68 & -12 & & \\
-14 & & 1983 & -112 & 98 & -42 & 7 & & \\
-7 & & 28 & 1878 & 28 & -7 & & & \\
 & & -7 & 28 & 1878 & 28 & -7 & & \\
 & & & & \ddots & \ddots & & & \\
 & & & & & \ddots & \ddots & & \\
 & & & -7 & 28 & 1878 & 28 & & -7 \\
 & & 7 & -42 & 98 & -112 & 1983 & & -14 \\
 & & -12 & 68 & -152 & 168 & -92 & 1920 & 20 \\
 & & 14 & -77 & 168 & -182 & 98 & & 1899
\end{bmatrix}
\begin{bmatrix}
u_\Delta^{(2)}(\tau_1) \\
u_\Delta^{(2)}(x_1) \\
u_\Delta^{(2)}(\tau_2) \\
u_\Delta^{(2)}(\tau_3) \\
u_\Delta^{(2)}(\tau_4) \\
\vdots \\
\vdots \\
u_\Delta^{(2)}(\tau_{N-2}) \\
u_\Delta^{(2)}(\tau_{N-1}) \\
u_\Delta^{(2)}(x_{N-1}) \\
u_\Delta^{(2)}(\tau_N)
\end{bmatrix}$$

$$= 1920 \begin{bmatrix} f_1 \\ f(x_1) \\ f_2 \\ f_3 \\ f_4 \\ \vdots \\ \vdots \\ f_{N-2} \\ f_{N-1} \\ f(x_{N-1}) \\ f_N \end{bmatrix}. \tag{3.49}$$

The $(N+2)\times(N+2)$ coefficient matrix of (3.49) is a strictly diagonally dominant matrix. Thus, (3.49) has a unique solution for $[u_\Delta^{(2)}(\tau_1), u_\Delta^{(2)}(x_1), u_\Delta^{(2)}(\tau_2), \cdots, u_\Delta^{(2)}(x_{N-1}), u_\Delta^{(2)}(\tau_N)]^T$.

We next determine the degrees of freedom (DOFs) of $u_\Delta(x) = \sum_{i=-1}^{N+2} c_i\phi_i(x)$. From the relations $u_\Delta^{(2)}(\tau_i) = \sum_{j=-1}^{N+2} c_j\phi_j^{(2)}(\tau_i)$, $i = 1, \cdots, N$, we have

$$u_\Delta^{(2)}(\tau_i) = \frac{1}{8h^2}(c_{i-2} + 4c_{i-1} - 10c_i + 4c_{i+1} + c_{i+2}),$$

for $i = 1, \cdots, N$. By the same arguments, we can obtain relations with respect to c_i, $i = -1, \cdots, N+2$ from the relations $u_\Delta^{(2)}(x_i) = \sum_{j=-1}^{N+2} c_j\phi_j^{(2)}(x_i)$, $i = 1, N-1$, and from the operator B' in (3.19) for $x = x_0, x_N$. These relations give rise to a system of equations

$$\begin{bmatrix} 1 & 11 & 11 & 1 & & & & & \\ & 1 & -1 & -1 & 1 & & & & \\ 1 & 4 & -10 & 4 & 1 & & & & \\ & 1 & 4 & -10 & 4 & 1 & & & \\ & & & \ddots & \ddots & & & & \\ & & & \ddots & \ddots & & & & \\ & & & 1 & 4 & -10 & 4 & 1 \\ & & & 1 & -1 & -1 & 1 & \\ & & & & 1 & 11 & 11 & 1 \end{bmatrix} \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_N \\ c_{N+1} \\ c_{N+2} \end{bmatrix} = \begin{bmatrix} 0 \\ 2h^2 u_\Delta^{(2)}(x_1) \\ 8h^2 u_\Delta^{(2)}(\tau_1) \\ 8h^2 u_\Delta^{(2)}(\tau_2) \\ \vdots \\ \vdots \\ 8h^2 u_\Delta^{(2)}(\tau_N) \\ 2h^2 u_\Delta^{(2)}(x_{N-1}) \\ 0 \end{bmatrix}. \tag{3.50}$$

We can uniquely determine the solution for $[c_{-1}, c_0, \cdots, c_{N+2}]^T$ from (3.50) by the following steps. From the first and second row of the system of equations, we have

$$\begin{aligned} c_{-1} &= -11c_0 - 11c_1 - c_2, \\ c_0 &= 2h^2 u_\Delta^{(2)}(x_1) + c_1 + c_2 - c_3, \end{aligned}$$

which are used to substitute c_{-1} and c_0 in the third row, and c_0 in the fourth row to yield

$$\begin{aligned} -28c_1 - 4c_2 + 8c_3 &= 8h^2 u_\Delta^{(2)}(\tau_1) + 14h^2 u_\Delta^{(2)}(x_1), \\ 5c_1 - 9c_2 + 3c_3 + c_4 &= 8h^2 u_\Delta^{(2)}(\tau_2) - 2h^2 u_\Delta^{(2)}(x_1). \end{aligned}$$

By similar arguments applied to the last four rows of (3.50), we obtain the system of equations with respect to $[c_1, c_2, \cdots, c_N]^T$

$$
\begin{bmatrix}
-28 & -4 & 8 & & & & & & \\
5 & -9 & 3 & 1 & & & & & \\
1 & 4 & -10 & 4 & 1 & & & & \\
 & 1 & 4 & -10 & 4 & 1 & & & \\
 & & & \ddots & \ddots & & & & \\
 & & & \ddots & \ddots & & & & \\
 & & & 1 & 4 & -10 & 4 & 1 \\
 & & & & 1 & 3 & -9 & 5 \\
 & & & & & 8 & -4 & -28
\end{bmatrix}
\begin{bmatrix}
c_1 \\ c_2 \\ c_3 \\ c_4 \\ \vdots \\ \vdots \\ c_{N-2} \\ c_{N-1} \\ c_N
\end{bmatrix}
$$

$$
= \begin{bmatrix}
8h^2 u_\Delta^{(2)}(\tau_1) + 14h^2 u_\Delta^{(2)}(x_1) \\
8h^2 u_\Delta^{(2)}(\tau_2) - 2h^2 u_\Delta^{(2)}(x_1) \\
8h^2 u_\Delta^{(2)}(\tau_3) \\
8h^2 u_\Delta^{(2)}(\tau_4) \\
\vdots \\
\vdots \\
8h^2 u_\Delta^{(2)}(\tau_{N-2}) \\
8h^2 u_\Delta^{(2)}(\tau_{N-1}) - 2h^2 u_\Delta^{(2)}(x_{N-1}) \\
8h^2 u_\Delta^{(2)}(\tau_N) + 14h^2 u_\Delta^{(2)}(x_{N-1})
\end{bmatrix}. \tag{3.51}
$$

The $N \times N$ coefficient matrix of (3.51) is pentadiagonal and diagonally dominant, with some rows being strictly diagonally dominant. Therefore, we have a unique solution for $[c_1, c_2, \cdots, c_N]^T$. Then we can uniquely determine c_0, c_{-1}, c_{N+1} and c_{N+2}, in that order.

3.4 Extra boundary conditions $Lu = f$ at x_0 and x_N

A second choice of extra boundary conditions is to apply L at the boundary points x_0 and x_N:

$$Lu = p(x)u''(x) + q(x)u'(x) + r(x)u(x) = f(x), \text{ for } x = x_0 = a \text{ and } x = x_N = b. \quad (3.52)$$

3.4.1 Quartic spline interpolant relations

Using (2.15)-(2.17) at x_0 and x_N, we see that S satisfies

$$\begin{aligned} LS(x_0) &= f(x_0) - \frac{h^4}{240}p(x_0)u^{(6)}(x_0) + \frac{h^4}{720}q(x_0)u^{(5)}(x_0) + O(h^6), & (3.53) \\ LS(x_N) &= f(x_N) - \frac{h^4}{240}p(x_N)u^{(6)}(x_N) + \frac{h^4}{720}q(x_N)u^{(5)}(x_N) + O(h^6). & (3.54) \end{aligned}$$

By (2.20)-(2.25), relations (3.53)-(3.54) can be written as

$$\begin{aligned} LS(x_0) &+ \frac{1}{480}p(x_0)(7\delta^2 S_3^{(2)} - 5\delta^2 S_4^{(2)}) - \frac{1}{1440}q(x_0)(7\delta^2 S_3^{(1)} - 5\delta^2 S_4^{(1)}) \\ &= f(x_0) + O(h^6), & (3.55) \\ LS(x_N) &+ \frac{1}{480}p(x_N)(7\delta^2 S_{N-2}^{(2)} - 5\delta^2 S_{N-3}^{(2)}) - \frac{1}{1440}q(x_N)(7\delta^2 S_{N-2}^{(1)} - 5\delta^2 S_{N-3}^{(1)}) \\ &= f(x_N) + O(h^6). & (3.56) \end{aligned}$$

Recall the definition of L' and B' by (3.9)-(3.15) and (3.16)-(3.17), and extend the definition of L' at x_0 and x_N by (3.55)-(3.56). Then we have the following lemma.

LEMMA 2 *Let S be the quartic spline interpolant of the solution $u \in C^{10}[a, b]$ to (3.1)-(3.2).*

Then S satisfies the relations

$$\begin{aligned}
[LS - f]_{\tau_i} &= O(h^4), \quad i = 1, \cdots, N, \\
[BS - g]_{x_0, x_N} &= O(h^4), \\
[LS - f]_{x_0, x_N} &= O(h^4),
\end{aligned}$$

$$\begin{aligned}
[L'S - f]_{\tau_i} &= O(h^6), \quad i = 1, \cdots, N, \\
[B'S - g]_{x_0, x_N} &= O(h^6), \\
[L'S - f]_{x_0, x_N} &= O(h^6).
\end{aligned}$$

3.4.2 Formulation of the quartic spline collocation methods

Like in Section 3.3.2, we first present the straightforward and non-optimal quartic spline collocation method. In the straightforward and non-optimal method, we determine a quartic spline $v \in S^4_\Delta$ that satisfies the relations

$$\begin{aligned}
[Lv - f]_{\tau_i} &= 0, \quad i = 1, \cdots, N, \\
[Bv - g]_{x_0, x_N} &= 0, \\
[Lv - f]_{x_0, x_N} &= 0.
\end{aligned}$$

In the optimal quartic spline collocation method we determine a quartic spline $u_\Delta \in S^4_\Delta$ that satisfies the following relations

$$[L'u_\Delta - f]_{\tau_i} = 0, \quad i = 1, \cdots, N, \tag{3.57}$$

$$[B'u_\Delta - g]_{x_0, x_N} = 0, \tag{3.58}$$

$$[L'u_\Delta - f]_{x_0, x_N} = 0. \tag{3.59}$$

We refer to this method as the *one-step* quartic spline collocation method.

We have an alternative formulation of this optimal method in which we determine $u_\Delta \in S^4_\Delta$ through a *two-step* quartic collocation method. The two-step collocation method is defined by the follows steps:

Step 1:

Find a $v \in S^4_\Delta$ such that it satisfies

$$[Lv - f]_{\tau_i} = 0, \quad i = 1, \cdots, N, \tag{3.60}$$

$$[Bv - g]_{x_0, x_N} = 0, \tag{3.61}$$

$$[Lv - f]_{x_0, x_N} = 0. \tag{3.62}$$

Step 2:

Determine $u_\Delta \in S^4_\Delta$ such that it satisfies

$$[Lu_\Delta - \bar{f}]_{\tau_i} = 0, \quad i = 1, \cdots, N, \tag{3.63}$$

$$[Bu_\Delta - \bar{g}]_{x_0, x_N} = 0, \tag{3.64}$$

$$[Lu_\Delta - \bar{f}]_{x_0, x_N} = 0, \tag{3.65}$$

where $\bar{f}_i, i = 1, \cdots, N$ and $\bar{g}(x_j), j = 0, N$ are determined by (3.27)-(3.33), and $\bar{f}(x_m), m = 0, N$ are determined by

$$\begin{aligned}
\bar{f}(x_0) &= f(x_0) - \frac{1}{480} p(x_0)(7\delta^2 v_3^{(2)} - 5\delta^2 v_4^{(2)}) \\
&+ \frac{1}{1440} q(x_0)(7\delta^2 v_3^{(1)} - 5\delta^2 v_4^{(1)}), \\
\bar{f}(x_N) &= f(x_N) - \frac{1}{480} p(x_N)(7\delta^2 v_{N-2}^{(2)} - 5\delta^2 v_{N-3}^{(2)}) \\
&+ \frac{1}{1440} q(x_N)(7\delta^2 v_{N-2}^{(1)} - 5\delta^2 v_{N-3}^{(1)}).
\end{aligned}$$

For the above formulation, we have the same assumption as (3.36)-(3.44).

3.4.3 Matrix analysis

We verify the solvability of the system of equations arising from a special case of (3.1)-(3.2) by the same arguments as in Section 3.3.3. Consider the boundary value problem (3.45)-(3.46). By the same arguments as for the system of equations (3.49), we have the system of equations with respect to $[u_\Delta^{(2)}(x_0), u_\Delta^{(2)}(\tau_1), \cdots, u_\Delta^{(2)}(\tau_N), u_\Delta^{(2)}(x_N)]^T$

$$
\begin{bmatrix}
1920 & 28 & -132 & 248 & -232 & 108 & -20 & & \\
 & 1899 & 98 & -182 & 168 & -77 & 14 & & \\
 & -14 & 1983 & -112 & 98 & -42 & 7 & & \\
 & -7 & 28 & 1878 & 28 & -7 & & & \\
 & & -7 & 28 & 1878 & 28 & -7 & & \\
 & & & & \ddots & \ddots & & & \\
 & & & & \ddots & \ddots & & & \\
 & & & -7 & 28 & 1878 & 28 & -7 & \\
 & & 7 & -42 & 98 & -112 & 1983 & -14 & \\
 & & 14 & -77 & 168 & -182 & 98 & 1899 & \\
 & & -20 & 108 & -232 & 248 & -132 & 28 & 1920
\end{bmatrix}
\begin{bmatrix}
u_\Delta^{(2)}(x_0) \\
u_\Delta^{(2)}(\tau_1) \\
u_\Delta^{(2)}(\tau_2) \\
u_\Delta^{(2)}(\tau_3) \\
u_\Delta^{(2)}(\tau_4) \\
\vdots \\
\vdots \\
u_\Delta^{(2)}(\tau_{N-2}) \\
u_\Delta^{(2)}(\tau_{N-1}) \\
u_\Delta^{(2)}(\tau_N) \\
u_\Delta^{(2)}(x_N)
\end{bmatrix}
$$

$$= 1920 \begin{bmatrix} f(x_0) \\ f1 \\ f_2 \\ f_3 \\ f_4 \\ \vdots \\ \vdots \\ f_{N-2} \\ f_{N-1} \\ f_N \\ f(x_N) \end{bmatrix}. \tag{3.66}$$

The $(N+2)\times(N+2)$ coefficient matrix of (3.66) is a strictly diagonally dominant matrix. Thus, (3.66) has a unique solution for $[u_\Delta^{(2)}(x_0), u_\Delta^{(2)}(\tau_1), \cdots, u_\Delta^{(2)}(\tau_N), u_\Delta^{(2)}(x_N)]^T$.

The system of equations with DOFs of $u_\Delta(x) = \sum_{i=-1}^{N+2} c_i\phi_i(x)$ is similar to (3.50) with the exception of the second and last second rows. The complete system is

$$\begin{bmatrix} 1 & 11 & 11 & 1 & & & & & \\ 1 & -1 & -1 & 1 & & & & & \\ 1 & 4 & -10 & 4 & 1 & & & & \\ & 1 & 4 & -10 & 4 & 1 & & & \\ & & & \ddots & \ddots & & & & \\ & & & \ddots & \ddots & & & & \\ & & & 1 & 4 & -10 & 4 & 1 \\ & & & & 1 & -1 & -1 & 1 \\ & & & & 1 & 11 & 11 & 1 \end{bmatrix} \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_N \\ c_{N+1} \\ c_{N+2} \end{bmatrix} = \begin{bmatrix} 0 \\ 2h^2u_\Delta^{(2)}(x_0) \\ 8h^2u_\Delta^{(2)}(\tau_1) \\ 8h^2u_\Delta^{(2)}(\tau_2) \\ \vdots \\ \vdots \\ 8h^2u_\Delta^{(2)}(\tau_N) \\ 2h^2u_\Delta^{(2)}(x_N) \\ 0 \end{bmatrix}. \tag{3.67}$$

We can have the unique solution for $[c_{-1}, c_0, \cdots, c_{N+2}]^T$ from (3.67) by the following steps. From the first row of the system of equations, we have

$$c_{-1} = -11c_0 - 11c_1 - c_2,$$

which is used to substitute c_{-1} in the second and third row to yield

$$\begin{aligned} -12c_0 - 12c_1 &= 2h^2u_\Delta^{(2)}(x_0), \\ -7c_0 - 21c_1 + 3c_2 + c_3 &= 8h^2u_\Delta^{(2)}(\tau_1). \end{aligned}$$

Adding the first row with 11 times of the second row, we have

$$12c_{-1} + 12c_2 = 22h^2u_\Delta^{(2)}(x_0).$$

Applying the same arguments to the last three rows of (3.67), we obtain a new system of equations

$$\begin{bmatrix} 12 & & & 12 & & & & & & \\ & -12 & -12 & & & & & & & \\ & -7 & -21 & 3 & 1 & & & & & \\ & 1 & 4 & -10 & 4 & 1 & & & & \\ & & & \ddots & \ddots & & & & & \\ & & & \ddots & \ddots & & & & & \\ & & 1 & 4 & -10 & 4 & 1 & & & \\ & & & 1 & 3 & -21 & -7 & & & \\ & & & & & -12 & -12 & & & \\ & & & & 12 & & & & & 12 \end{bmatrix} \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_{N-1} \\ c_N \\ c_{N+1} \\ c_{N+2} \end{bmatrix} = \begin{bmatrix} 22h^2u_\Delta^{(2)}(x_0) \\ 2h^2u_\Delta^{(2)}(x_0) \\ 8h^2u_\Delta^{(2)}(\tau_1) \\ 8h^2u_\Delta^{(2)}(\tau_2) \\ \vdots \\ \vdots \\ 8h^2u_\Delta^{(2)}(\tau_{N-1}) \\ 8h^2u_\Delta^{(2)}(\tau_N) \\ 2h^2u_\Delta^{(2)}(x_N) \\ 22h^2u_\Delta^{(2)}(x_N) \end{bmatrix}. \quad (3.68)$$

The coefficient matrix of (3.68) is a diagonally dominant matrix, with some rows being strictly diagonally dominant. Therefore, we have a unique solution for $[c_{-1}, c_0, \cdots, c_{N+2}]^T$.

3.5 Other extra boundary conditions

In this section we present three choices of extra boundary conditions which form non-optimal quartic spline collocation methods. In these methods, we will use approximations to $f^{(2)}(x)$ and

$f^{(3)}(x)$, which we obtain by finite differences

$$\begin{aligned} f^{(2)}(x) &= -\frac{110}{132h^2}(f(x+5h) - \frac{61}{10}f(x+4h) + \frac{858}{55}f(x+3h) \\ &- \frac{1177}{55}f(x+2h) + \frac{847}{55}f(x+h) - \frac{297}{66}f(x)) + O(h^6), \end{aligned} \tag{3.69}$$

$$\begin{aligned} f^{(3)}(x) &= \frac{7}{4h^3}(f(x+5h) - \frac{41}{7}f(x+4h) + \frac{98}{7}f(x+3h) \\ &- \frac{118}{7}f(x+2h) + \frac{71}{7}f(x+h) - \frac{17}{7}f(x)) + O(h^6). \end{aligned} \tag{3.70}$$

We next describe three quartic spline collocation methods which, as shown in the numerical results, are non-optimal.

3.5.1 Method 1

In this non-optimal quartic spline collocation method, the second derivative is utilized in the extra boundary conditions. These extra boundary conditions are similar to the natural cubic spline interpolant end-conditions.

Step 1:

Find a $v \in S_\Delta^4$ such that it satisfies

$$\begin{aligned} [Lv - f]_{\tau_i} &= 0, \quad i = 1, \cdots, N, \\ [Bv - g]_{x_0, x_N} &= 0, \\ [Ev]_{x_0, x_N} &= 0, \end{aligned}$$

where

$$Ev = v^{(2)}(x).$$

Step 2:

Determine $u_\Delta \in S^4_\Delta$ such that it satisfies

$$\begin{aligned} [Lu_\Delta - \bar{f}]_{\tau_i} &= 0, \quad i = 1, \cdots, N, \\ [Bu_\Delta - \bar{g}]_{x_0, x_N} &= 0, \\ [Eu_\Delta - \bar{e}]_{x_0, x_N} &= 0, \end{aligned}$$

where $\bar{f}_i, i = 1, \cdots, N$ and $\bar{g}(x_j), j = 0, N$ are determined by (3.27)-(3.33), and $\bar{e}(x_m), m = 0, N$ are determined by finite deference approximation from v by (3.69).

As shown in the numerical results, this method is at most second order.

3.5.2 Method 2

This quartic spline collocation method is a variation of that in Section 3.5.1. It is a *three step* method.

Step 1:

Find a $w \in S^4_\Delta$ such that it satisfies

$$\begin{aligned} [Lw - f]_{\tau_i} &= 0, \quad i = 1, \cdots, N, \\ [Bw - g]_{x_0, x_N} &= 0, \\ [Ew]_{x_0, x_N} &= 0, \end{aligned}$$

where

$$Ew = w^{(2)}(x).$$

Step 2:

Find a $v \in S^4_\Delta$ such that it satisfies

$$\begin{aligned}
[Lv - f]_{\tau_i} &= 0, \quad i = 1, \cdots, N, \\
[Bv - g]_{x_0, x_N} &= 0, \\
[Ev - e]_{x_0, x_N} &= 0,
\end{aligned}$$

where $e(x_m), m = 0, N$ are determined by finite deference approximation from w by (3.69).

Step 3:

Determine $u_\Delta \in S^4_\Delta$ such that it satisfies

$$\begin{aligned}
[Lu_\Delta - \bar{f}]_{\tau_i} &= 0, \quad i = 1, \cdots, N, \\
[Bu_\Delta - \bar{g}]_{x_0, x_N} &= 0, \\
[Eu_\Delta - \bar{e}]_{x_0, x_N} &= 0,
\end{aligned}$$

where $\bar{f}_i, i = 1, \cdots, N$ and $\bar{g}(x_j), j = 0, N$ are determined by (3.27)-(3.33), and $\bar{e}(x_m), m = 0, N$ are determined by finite deference approximation from v by (3.69).

As shown in the numerical results, this method is at most second order.

3.5.3 Method 3

This quartic spline collocation method is a variation of that in Section 3.5.2. This method uses the third derivative, instead of the second derivative, as extra boundary conditions.

Step 1:

Find a $w \in S^4_\Delta$ such that it satisfies

$$
\begin{aligned}
[Lw - f]_{\tau_i} &= 0, \quad i = 1, \cdots, N, \\
[Bw - g]_{x_0, x_N} &= 0, \\
[Ew]_{x_0, x_N} &= 0,
\end{aligned}
$$

where

$$Ew = w^{(3)}(x).$$

Step 2:

Find a $v \in S^4_\Delta$ such that it satisfies

$$
\begin{aligned}
[Lv - f]_{\tau_i} &= 0, \quad i = 1, \cdots, N, \\
[Bv - g]_{x_0, x_N} &= 0, \\
[Ev - e]_{x_0, x_N} &= 0,
\end{aligned}
$$

where $e(x_m), m = 0, N$ are determined by finite deference approximation from w by (3.70).

Step 3:

Determine $u_\Delta \in S^4_\Delta$ such that it satisfies

$$
\begin{aligned}
[Lu_\Delta - \bar{f}]_{\tau_i} &= 0, \quad i = 1, \cdots, N, \\
[Bu_\Delta - \bar{g}]_{x_0, x_N} &= 0, \\
[Eu_\Delta - \bar{e}]_{x_0, x_N} &= 0,
\end{aligned}
$$

where $\bar{f}_i, i = 1, \cdots, N$ and $\bar{g}(x_j), j = 0, N$ are determined by (3.27)-(3.33), and $\bar{e}(x_m), m = 0, N$ are determined by finite deference approximation from v by (3.70).

As shown in the numerical results, this method is at most third order.

Chapter 4

Convergence Analysis and Error Bounds

We follow similar arguments as in [5] and [16] to present the analysis of the convergence and error bounds for the quartic spline collocation methods. We present the analysis of the two-step method and the one-step method for the extra boundary conditions (3.52).

4.1 Green's function approach

We consider the boundary value problem (3.1)-(3.2) with $p(x) = 1$ and homogeneous Dirichlet boundary conditions $g_1 = g_2 = 0$. We assume that u, u_Δ and v satisfy the homogeneous boundary conditions.

If we assume that the boundary value problem $u^{(2)} = 0$, $Bu = 0$ has a unique solution, then it implies [12] that there is a Green's function $G(x, t)$ for that problem. Let $y \equiv u''$, $y_\Delta \equiv u''_\Delta$, and $s \equiv v''$. Then u, u_Δ, v and their first derivatives can be obtained by

$$\begin{aligned}
u(x) &= \int_a^b G(x,t)y(t)dt, \\
u'(x) &= \int_a^b G_x(x,t)y(t)dt, \\
u_\Delta(x) &= \int_a^b G(x,t)y_\Delta(t)dt, \\
u'_\Delta(x) &= \int_a^b G_x(x,t)y_\Delta(t)dt, \\
v(x) &= \int_a^b G(x,t)s(t)dt, \\
v'(x) &= \int_a^b G_x(x,t)s(t)dt.
\end{aligned}$$

We introduce the operator K that is defined by

$$Ky(x) = q(x)\int_a^b G_x(x,t)y(t)dt + r(x)\int_a^b G(x,t)y(t)dt, \tag{4.1}$$

and maps $C^1(I)$ to $C(I)$. We also introduce the linear projection P_Δ that maps $C^1(I)$ to S_Δ^2 by piecewise quadratic interpolation at the midpoints $\{\tau_i\}_1^N$ and grid points x_0, x_N, i.e., at points $\{\tau_i\}_0^{N+1}$ since $\tau_0 \equiv x_0$ and $\tau_{N+1} \equiv x_N$.

4.2 Convergence analysis of the two-step method

With the notations introduced, we can rewrite equations (3.60),(3.62) and (3.63),(3.65) respectively as

$$P_\Delta(s + Ks) = P_\Delta f, \tag{4.2}$$

$$P_\Delta(y_\Delta + Ky_\Delta) = P_\Delta \bar{f}. \tag{4.3}$$

Since $P_\Delta s = s$ and $P_\Delta y_\Delta = y_\Delta$, we can simplify (4.2) and (4.3) as

$$(I + P_\Delta K)s = P_\Delta f, \tag{4.4}$$

$$(I + P_\Delta K)y_\Delta = P_\Delta \bar{f}. \tag{4.5}$$

Equation (3.1) can be rewritten as

$$y + Ky = f. \tag{4.6}$$

By the definition of P_Δ, $\| P_\Delta y - y \|_\infty$ converges to zero as h approaches zero for continuous function y. By the complete continuity of K [12], this in turn implies that $\| P_\Delta K - K \|_\infty$ converges to zero as h converges to zero. Therefore, by Neumann's theorem [10], we conclude that the operators $(I + P_\Delta K)^{-1}$ exist, and are uniformly bounded for sufficient small h. We will show the convergence order of each step for the two-step method.

THEOREM 2 *We assume that*

(a1) the coefficients $q(x)$, $r(x)$ and $f(x)$ are continuous on $[a, b]$,

(a2) the boundary value problem $Lu = f$, $Bu = 0$ has a unique solution in $C^4(I)$,

and

(a3) the problem $u'' = 0$, $Bu = 0$ has a unique solution,

then we conclude that

(b1) the collocation approximation $v \in S^4_\Delta$ defined in (3.21)-(3.23) in step 1 exists,

(b2) the global error satisfies

$$\| u - v \|_\infty = O(h^4), \tag{4.7}$$

$$\| (u - v)^{(k)} \|_\infty = O(h^{5-k}), \quad k = 1, 2, \tag{4.8}$$

and

(b3) the error at the collocation points satisfies

$$| (u-v)^{(k)}(\tau_i) |= O(h^4), \quad k=0,1,2, \;\; i=1,\cdots,N. \tag{4.9}$$

Proof: From the existence and uniform boundedness of $(I+P_\Delta K)^{-1}$, the solvability of (4.4) and (4.5) follows, hence the unique existence of v follows.

Recall the quartic spline interpolant S of u in (2.11)-(2.12). By (3.5)-(3.6) we have shown that $BS = O(h^4)$. Note that there exists a linear function w such that $Bw = BS = O(h^4)$ because of assumption (a3). It can be further shown that $\| w \|_\infty = O(h^4)$ and $\| w' \|_\infty = O(h^4)$. It is clear that the problem $(S-w)'' = S''$, $B(S-w) = 0$ is solvable. Then (3.4) and (3.53)-(3.54) can be rewritten in the operator notations introduced as

$$(I+P_\Delta K)S'' = P_\Delta f + O(h^4),$$

from which S'' is substituted by $(S-w)''$ to yield

$$(I+P_\Delta K)(S''-w'') = P_\Delta f + O(h^4). \tag{4.10}$$

Subtracting (4.4) and (4.10) we have

$$(I+P_\Delta K)(S''-w''-v'') = O(h^4).$$

From the uniform boundedness of $(I+P_\Delta K)^{-1}$, we obtain

$$\| S''-w''-v'' \|_\infty = O(h^4). \tag{4.11}$$

Since the unique solvability of $(S-w-v)''=0$, $B(S-w-v)=0$ is ensured by assumption (a3), we can obtain using the Green's function

$$\begin{aligned}(S-w-v)'(x) &= \int_a^b G_x(x,t)(S''-w''-v'')(t)dt,\\ (S-w-v)(x) &= \int_a^b G(x,t)(S''-w''-v'')(t)dt.\end{aligned}$$

These imply that

$$\| S-w-v \|_\infty = O(h^4), \tag{4.12}$$

$$\| S'-w'-v' \|_\infty = O(h^4). \tag{4.13}$$

We can establish the error bounds (4.7)-(4.8) and (4.9) from equations (2.13)-(2.14), relations (4.11) and (4.12)-(4.13), and the definition of w. This completes the proof.

THEOREM 3 *Under the hypothesis of Theorem 2, we conclude that*

(b1) the collocation approximation $u_\Delta \in S^4_\Delta$ defined in (3.24)-(3.26) in step 2 exists,

(b2) the global error satisfies

$$\| (u-u_\Delta)^{(k)} \|_\infty = O(h^{5-k}), \quad k=0,1,2, \tag{4.14}$$

and

(b3) the error at the collocation points satisfies

$$| (u-u_\Delta)(x_i) | = O(h^6), \quad i=0,\cdots,N, \tag{4.15}$$

$$| (u-u_\Delta)(\tau_i) | = O(h^6), \quad i=1,\cdots,N, \tag{4.16}$$

$$| (u-u_\Delta)^{(k)}(\tau_i) | = O(h^4), \quad k=1,2, \ \ i=1,\cdots,N. \tag{4.17}$$

Proof: From (4.11), (4.12)-(4.13), and the definition of w, we utilize the triangle inequality to obtain

$$\| S^{(k)} - v^{(k)} \|_\infty \le \| S^{(k)} - w^{(k)} - v^{(k)} \|_\infty + \| w^{(k)} \|_\infty = O(h^4),$$

for $k = 0, 1, 2$. If we assume that the expansion of the error $S^{(k)} - v^{(k)}$, $k = 0, 1, 2$, at the collocation points is smooth enough, then we have the relation

$$\delta^2 S_i^{(k)} = \delta^2 v_i^{(k)} + O(h^6), \tag{4.18}$$

for $k = 0, 1, 2$ and $i = 2, \cdots, N-1$. This relation is verified in the numerical results of Table 6.7. Then, equations (3.9)-(3.15) and (3.55)-(3.56) can be rewritten as

$$\begin{aligned} LS_i &= \bar{f}_i + O(h^6), & i = 1, \cdots, N, \\ LS(x_i) &= \bar{f}(x_i) + O(h^6), & i = 0, N, \\ B_k S(x_i) &= \bar{g}_k + O(h^6), & i = 0, N, \ \ k = 1, 2. \end{aligned}$$

Therefore, for u_Δ defined in (3.63)-(3.65), we have

$$L(S - u_\Delta)(\tau_i) = O(h^6), \qquad i = 1, \cdots, N, \tag{4.19}$$

$$L(S - u_\Delta)(x_i) = O(h^6), \qquad i = 0, N, \tag{4.20}$$

$$B(S - u_\Delta) = O(h^6). \tag{4.21}$$

Again note that there exists a linear function w such that $Bw = B(S - u_\Delta) = O(h^6)$, and $\| w \|_\infty = O(h^6)$, $\| w' \|_\infty = O(h^6)$. It is clear that the problem $(S - w - u_\Delta)'' = (S - u_\Delta)''$, $B(S - w - u_\Delta) = 0$ is solvable due to the assumption (a3). Then we can rewrite equations (4.19)-(4.21) as

$$(I + P_\Delta K)(S'' - w'' - u''_\Delta) = O(h^6).$$

Using the same arguments in the proof of Theorem 2, we can obtain the bounds

$$\| S^{(k)} - w^{(k)} - u_\Delta^{(k)} \|_\infty = O(h^6), \qquad k = 0, 1, 2.$$

By the triangle inequality

$$\| S^{(k)} - u_\Delta^{(k)} \|_\infty \;\leq\; \| S^{(k)} - w^{(k)} - u_\Delta^{(k)} \|_\infty + \| w^{(k)} \|_\infty,$$

it follows that

$$\| S^{(k)} - u_\Delta^{(k)} \|_\infty = O(h^6), \qquad k = 0, 1, 2. \tag{4.22}$$

The error bounds (4.14) and (4.15)-(4.17) follow from equation (2.13)-(2.17), Theorem 1, the relation (4.22), and the definition of w. This completes the proof.

4.3 Convergence analysis of the one-step method

We denote by Ω and Φ the coefficient matrices of $u_\Delta^{(1)}$ and $u_\Delta^{(2)}$ in (3.57) respectively. The matrix $\Phi = \frac{1}{1920}\Phi'$ where Φ' is the coefficient matrix in the left hand side of (3.66). The matrix Ω is

$$\frac{1}{5760}\begin{bmatrix}
5760 & -28 & 132 & -248 & 232 & -108 & 20 & & \\
 & 5781 & -98 & 182 & -168 & 77 & -14 & & \\
 & 14 & 5697 & 112 & -98 & 42 & -7 & & \\
 & 7 & -28 & 5802 & -28 & 7 & & & \\
 & & 7 & -28 & 5802 & -28 & 7 & & \\
 & & & & \ddots & \ddots & & & \\
 & & & & & \ddots & \ddots & & \\
 & & & 7 & -28 & 5802 & -28 & 7 & \\
 & & -7 & 42 & -98 & 112 & 5697 & 14 & \\
 & & -14 & 77 & -168 & 182 & -98 & 5781 & \\
 & & 20 & -108 & 232 & -248 & 132 & -28 & 5760
\end{bmatrix}$$

Let us denote by D_Δ the vector valued function $D_\Delta : C[I] \to R^{N+2}$ defined by

$$\begin{aligned}(D_\Delta m)_0 &= m(x_0),\\ (D_\Delta m)_i &= m(\tau_i), \quad i = 1, \cdots, N,\\ (D_\Delta m)_{N+1} &= m(x_N),\end{aligned}$$

for any $m \in C[I]$. Note that $D_\Delta m$ is a vector of size $N+2$ whose elements are the values of m at midpoints τ_i, $i = 1, \cdots, N$, and grid points x_0, x_N. We define $Q_\Delta : R^{N+2} \to S^2_\Delta$ to be the operator that maps vectors $[m_0, m_1, \cdots, m_{N+1}]^T$ to the associated quadratic interpolant m_Δ so that

$$\begin{aligned}m_\Delta(x_0) &= m_0,\\ m_\Delta(\tau_i) &= m_i, \quad i = 1, \cdots, N,\\ m_\Delta(x_N) &= m_{N+1}.\end{aligned}$$

Note that $P_\Delta = Q_\Delta D_\Delta$. Also let denote by E_q and E_r the $(N+2) \times (N+2)$ diagonal matrices such that

$$\begin{aligned}E_q &= \text{diag}(q(\tau_i)),\\ E_r &= \text{diag}(r(\tau_i)),\end{aligned}$$

for $i = 0, \cdots, N+1$. Then (3.57) and (3.59) can be rewritten as

$$\Phi D_\Delta u''_\Delta + E_q \Omega D_\Delta u'_\Delta + E_r D_\Delta u_\Delta = D_\Delta f. \tag{4.23}$$

We have shown that Φ is invertible. Multiplying both sides of (4.23) by Φ^{-1}, and then applying Q_Δ, we obtain

$$(I + R_\Delta K) y_\Delta = Q_\Delta \Phi^{-1} D_\Delta f, \tag{4.24}$$

where $R_\triangle K$ is the integral operator defined by

$$\begin{aligned} R_\triangle K y &= Q_\triangle \Phi^{-1} E_q \Omega D_\triangle \int_a^b G_x(x,t) y(t) dt \\ &+ Q_\triangle \Phi^{-1} E_r D_\triangle \int_a^b G(x,t) y(t) dt. \end{aligned} \tag{4.25}$$

Lemma 3 *The sequence of operators $R_\triangle K$ defined in (4.25) converges to the integral operator K as h approaches zero.*

Proof: Let $\bar{G}(x)$ denote $\int_a^b G(x,t)y(t)dt$, and $\bar{G}'(x)$ denote $\int_a^b G_x(x,t)y(t)dt$. By the triangle inequality and the relations

$$\begin{aligned} D_\triangle q(x) \bar{G}'(x) &= E_q D_\triangle \bar{G}'(x), \\ D_\triangle r(x) \bar{G}(x) &= E_r D_\triangle \bar{G}(x), \end{aligned}$$

we obtain

$$\begin{aligned} & \| R_\triangle K y - Q_\triangle D_\triangle K y \|_\infty \\ \leq & \| Q_\triangle \Phi^{-1} E_q \Omega D_\triangle \bar{G}'(x) - Q_\triangle \Phi^{-1} \Phi E_q D_\triangle \bar{G}'(x) \|_\infty \\ + & \| Q_\triangle \Phi^{-1} E_r D_\triangle \bar{G}(x) - Q_\triangle \Phi^{-1} \Phi E_r D_\triangle \bar{G}(x) \|_\infty . \end{aligned} \tag{4.26}$$

Since $\| \Phi^{-1} \|_\infty$ and $\| Q_\triangle \|_\infty$ are bounded independently of h, we can rewrite (4.26) as

$$\begin{aligned} & \| R_\triangle K y - Q_\triangle D_\triangle K y \|_\infty \\ \leq & \; C \| E_q \Omega D_\triangle \bar{G}'(x) - \Phi E_q D_\triangle \bar{G}'(x) \|_\infty \\ + & \; C \| E_r D_\triangle \bar{G}(x) - \Phi E_r D_\triangle \bar{G}(x) \|_\infty, \end{aligned} \tag{4.27}$$

where C is a constant independent of h. By the same arguments in [16], it can be easily shown that $\Phi E_q = E_q \Phi + O(h)$ using Taylor's series. This implies that (4.27) can be rewritten as

$$\begin{aligned} & \parallel R_\Delta Ky - Q_\Delta D_\Delta Ky \parallel_\infty \\ \leq \ & C \parallel E_q \parallel_\infty \parallel (\Omega - \Phi) D_\Delta \bar{G}'(x) \parallel_\infty \\ + \ & C \parallel E_r \parallel_\infty \parallel (I - \Phi) D_\Delta \bar{G}(x) \parallel_\infty + O(h). \end{aligned} \tag{4.28}$$

It is clear that E_q and E_r are bounded independently of h. Let denote by

$$\omega(\bar{G}, k) \equiv \text{sup}(\mid \bar{G}(x) - \bar{G}(y) \mid : \quad \mid x - y \mid \leq k)$$

the modulus of continuity of the continuous function $\bar{G}(x)$. Also denote by $[v]_i$ the i-th element of the vector v. Next we compute the entries of the matrices $(\Omega - \Phi)$ and $(I - \Phi)$ to obtain

$$\begin{aligned} & \parallel (\Omega - \Phi) D_\Delta \bar{G}'(x) \parallel_\infty \\ = \ & \max([(\Omega - \Phi) D_\Delta \bar{G}'(x)]_i : \quad i = 1, \cdots, N) \\ = \ & \frac{1}{5760} \max(\\ & \quad -112\bar{G}'_1 + 528\bar{G}'_2 - 992\bar{G}'_3 + 928\bar{G}'_4 - 432\bar{G}'_5 + 80\bar{G}'_6, \\ & \quad 84\bar{G}'_1 - 392\bar{G}'_2 + 728\bar{G}'_3 - 672\bar{G}'_4 + 308\bar{G}'_5 - 56\bar{G}'_6, \\ & \quad 56\bar{G}'_1 - 252\bar{G}'_2 + 448\bar{G}'_3 - 392\bar{G}'_4 + 168\bar{G}'_5 - 28\bar{G}'_6, \\ & \quad 28\bar{G}'_{i-2} - 112\bar{G}'_{i-1} + 168\bar{G}'_i - 112\bar{G}'_{i+1} + 28\bar{G}'_{i+2} : \quad i = 3, \cdots, N-2) \\ = \ & O(\omega(\bar{G}', 5h)), \end{aligned}$$

$$
\begin{aligned}
& \| (I-\Phi)D_\Delta \bar{G}(x) \|_\infty \\
= \;& \max([(I-\Phi)D_\Delta \bar{G}(x)]_i : \quad i=1,\cdots,N) \\
= \;& \frac{1}{1920}\max(\\
& \quad -28\bar{G}_1+132\bar{G}_2-248\bar{G}_3+232\bar{G}_4-108\bar{G}_5+20\bar{G}_6', \\
& \quad 21\bar{G}_1-98\bar{G}_2+182\bar{G}_3-168\bar{G}_4+77\bar{G}_5-14\bar{G}_6', \\
& \quad 14\bar{G}_1-63\bar{G}_2+112\bar{G}_3-98\bar{G}_4+42\bar{G}_5-7\bar{G}_6, \\
& \quad 7\bar{G}_{i-2}-28\bar{G}_{i-1}+42\bar{G}_i-28\bar{G}_{i+1}+7\bar{G}_{i+2} : \quad i=3,\cdots,N-2) \\
= \;& O(\omega(\bar{G},5h)).
\end{aligned}
$$

Therefore $(\Omega-\Phi)D_\Delta\bar{G}'(x)$ and $(I-\Phi)D_\Delta\bar{G}(x)$ are bounded by ω over a $5h$-interval. It is clear that $\omega(\bar{G}',5h)$ and $\omega(\bar{G},5h)$ converge to zero as h approaches zero. It follows that $\| R_\Delta Ky - Q_\Delta D_\Delta Ky \|_\infty$ approaches zero as h converges to zero. By the triangle inequality, we have

$$
\begin{aligned}
& \| R_\Delta K - K \|_\infty \\
\leq \;& \| R_\Delta K - Q_\Delta D_\Delta K \|_\infty + \| Q_\Delta D_\Delta K - K \|_\infty . \qquad (4.29)
\end{aligned}
$$

The two terms on the right hand side of (4.29) have been shown to converge to zero as h approaches zero. Therefore $\| R_\Delta K - K \|_\infty$ converges to zero as h converges to zero. This completes the proof.

THEOREM 4 *Under the hypothesis of Theorem 2, we conclude that*

(b1) the collocation approximation $u_\Delta \in S_\Delta^4$ defined in (3.57)-(3.59) in the one-step method exists,

(b2) the global error satisfies

$$\| (u-u_\Delta)^{(k)} \|_\infty = O(h^{5-k}), \quad k=0,1,2,$$

and

(b3) the error at the collocation points satisfies

$$
\begin{aligned}
\mid (u - u_\Delta)(x_i) \mid &= O(h^6), \quad i = 0, \cdots, N, \\
\mid (u - u_\Delta)(\tau_i) \mid &= O(h^6), \quad i = 1, \cdots, N, \\
\mid (u - u_\Delta)^{(k)}(\tau_i) \mid &= O(h^4), \quad k = 1, 2, \;\; i = 1, \cdots, N.
\end{aligned}
$$

Proof: We follow the same arguments in the proof of Theorem 2 and 3 for the two-step method. By Lemma 3 and Neumann's Theorem, it is concluded that $(I + R_\Delta K)^{-1}$ exists, and is uniformly bounded for sufficient small h. Therefore the system of collocation equations (4.24) has a unique solution, hence the unique existence of u_Δ follows.

There is a linear function w such that $Bw = BS = O(h^6)$, and $\| w \|_\infty = O(h^6)$, $\| w' \|_\infty = O(h^6)$. It is clear that the problem $(S - w)'' = S''$, $B(S - w) = 0$ is solvable due to the assumption (a3). From (3.57)-(3.59), we have

$$(I + R_\Delta K)(S'' - w'') = Q_\Delta f + O(h^6). \tag{4.31}$$

Subtracting (4.24) and (4.31), we have

$$(I + R_\Delta K)(S'' - w'' - u''_\Delta) = O(h^6).$$

Following the same arguments as in Theorem 3, we drive the error bounds. This completes the proof.

Chapter 5

Quartic Spline Collocation Method For Layer Problems

5.1 Layers

The quartic spline collocation methods presented in Chapter 3 can effectively apply to problems in which the solution $u(x)$ does not exhibit rough behavior, that is, it does not vary a lot within narrow parts of the domain. As will be shown by numerical examples in the next section, a quantity that indicates the magnitude of the quartic spline collocation method error is the 5th derivative, $u^{(5)}(x)$, of $u(x)$. In areas where $u^{(5)}(x)$ is very large the quartic spline collocation method error may be large, and where $u^{(5)}(x)$ is small the error will be small. For example, the 5th derivative of $u(x) = x^8$ for $x \in [0, 1]$, plotted in Figure 5.1, is bounded by 1.5×10^3, and does not vary rapidly.

However, in some problems, $u(x)$ varies rapidly in certain narrow parts of the domain, called

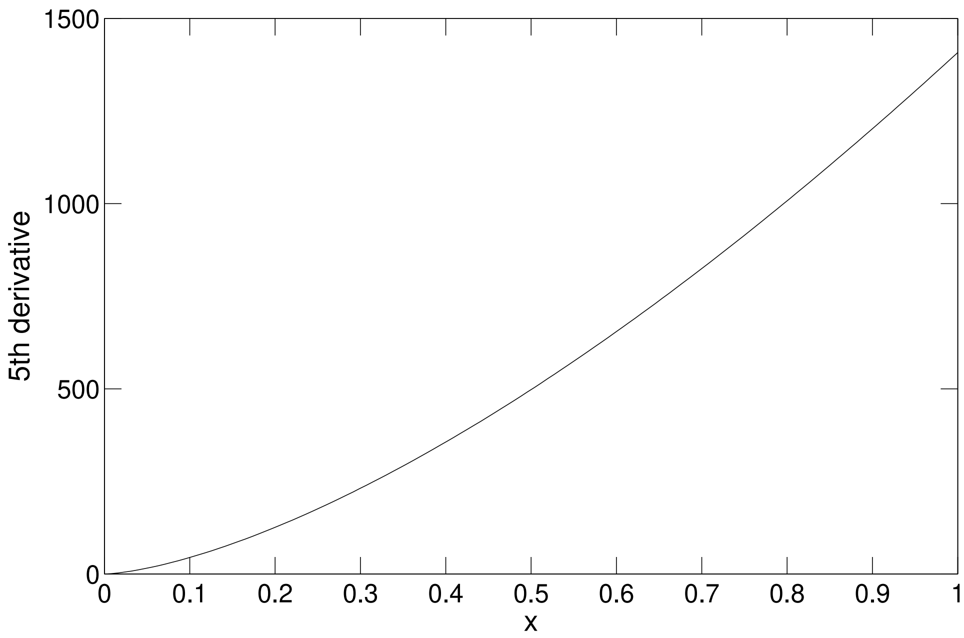

Figure 5.1: Plot of 5th derivative of $u(x) = x^8$

layers [6]. The part of the domain where $u(x)$ does not vary a lot is referred to as *area of little variation*. In the layers, the error and the 5th derivative of $u(x)$ tend to be very large. We give three examples of functions which exhibit layers, and later in the thesis we consider the problems whose solutions resemble these functions. The first function is

$$u(x) = \text{erf}(x/\sqrt{\epsilon}), \tag{5.1}$$

for some small positive number ϵ and x near 0. From Figures 5.2 and 5.3, it is clear that $u(x)$ of (5.1) has a layer near $x = 0$, and $u^{(5)}(x)$ gives a good indication of the region where the layer occurs. Note also that, the smaller the ϵ the sharper the peak and the narrower the layer. Another example is the function

$$u(x) = e^{-\frac{x+1}{\epsilon}} + 2e^{\frac{x-1}{\epsilon}}. \tag{5.2}$$

From Figures 5.4 and 5.5, we can see that layers occur at both ends of the domain. In the third example

$$u(x) = \cos(\pi x) - e^{-\frac{3x}{\epsilon}}. \tag{5.3}$$

Figures 5.6 and 5.7 demonstrate that the layer is located at the left end of the domain.

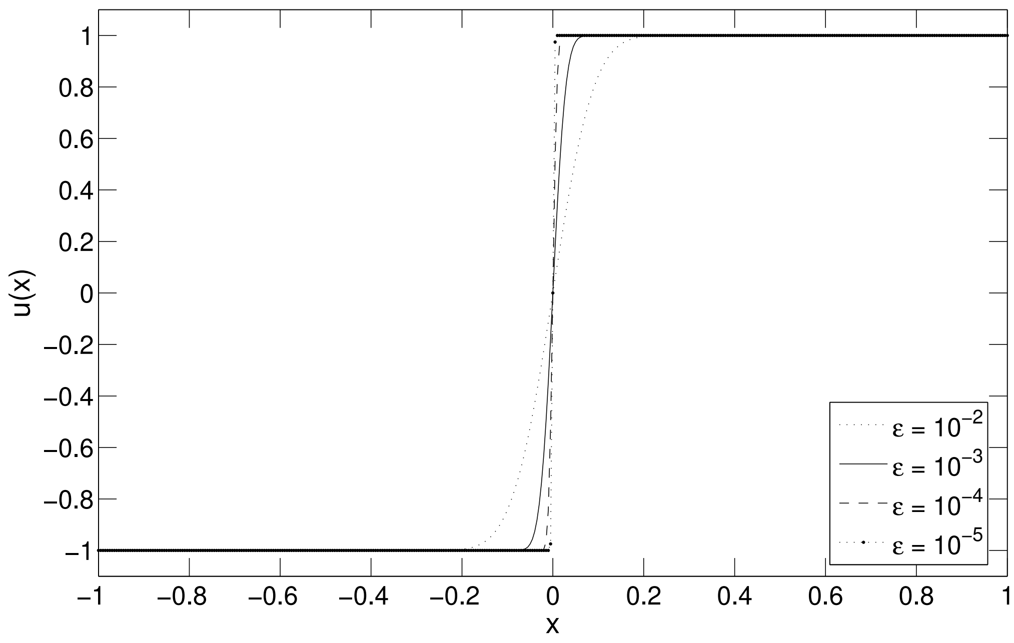

Figure 5.2: Plot of $u(x)$ of (5.1) for some values of ϵ

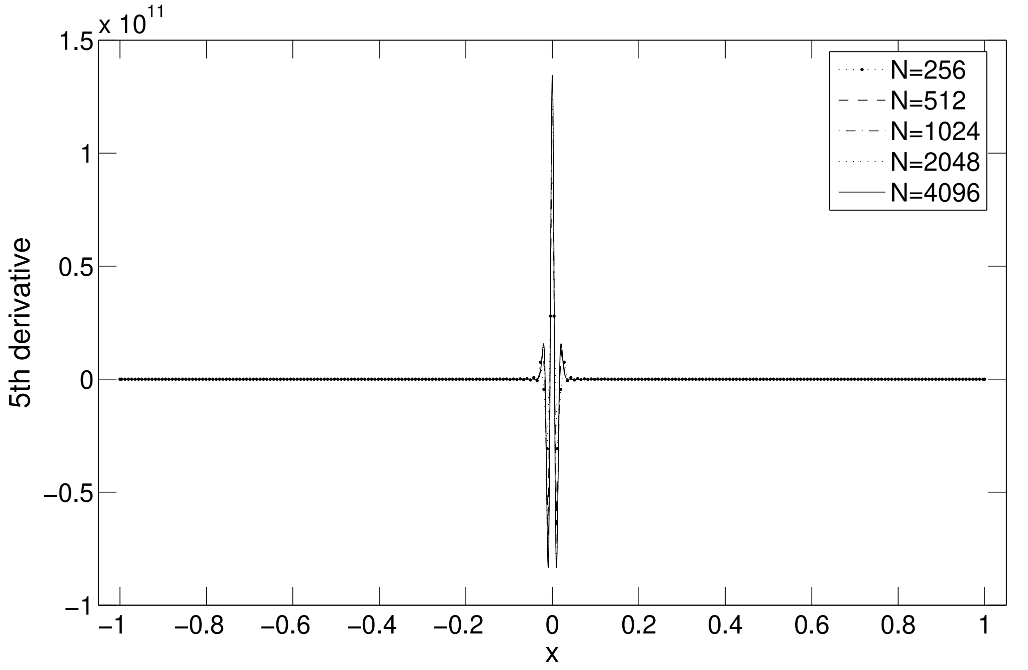

Figure 5.3: Plot of $u^{(5)}(x)$ corresponding to (5.1) where $\epsilon = 10^{-4}$

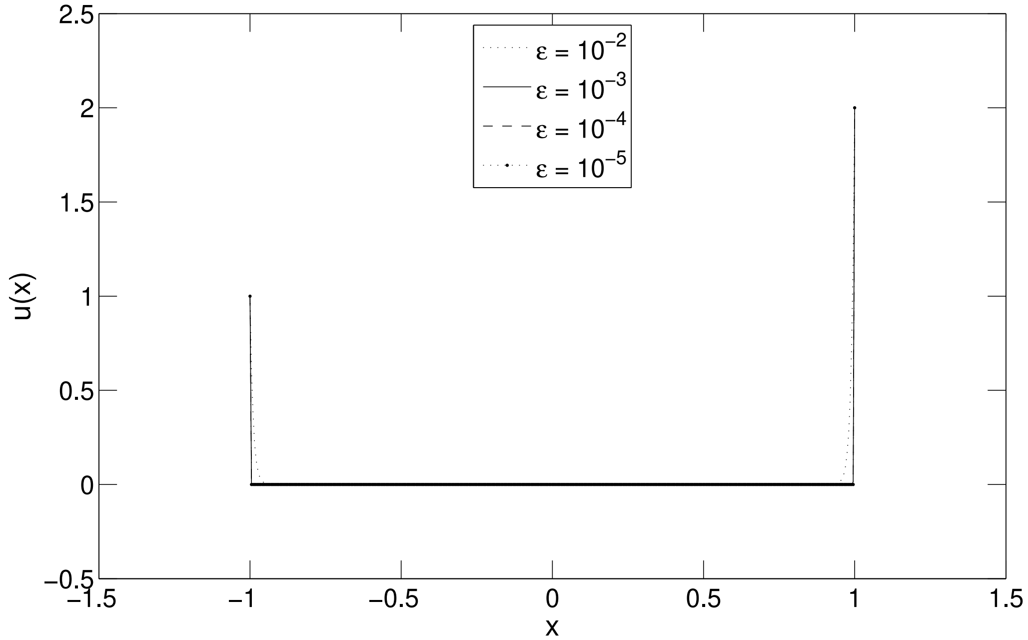

Figure 5.4: Plot of $u(x)$ of (5.2) for some values of ϵ

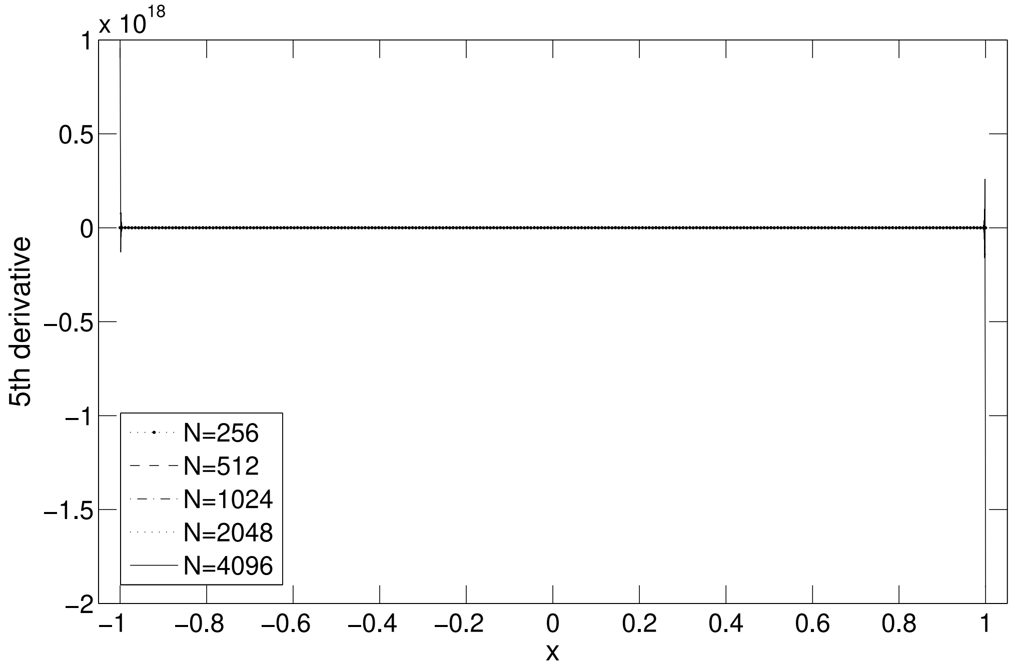

Figure 5.5: Plot of $u^{(5)}(x)$ corresponding to (5.2) where $\epsilon = 10^{-4}$

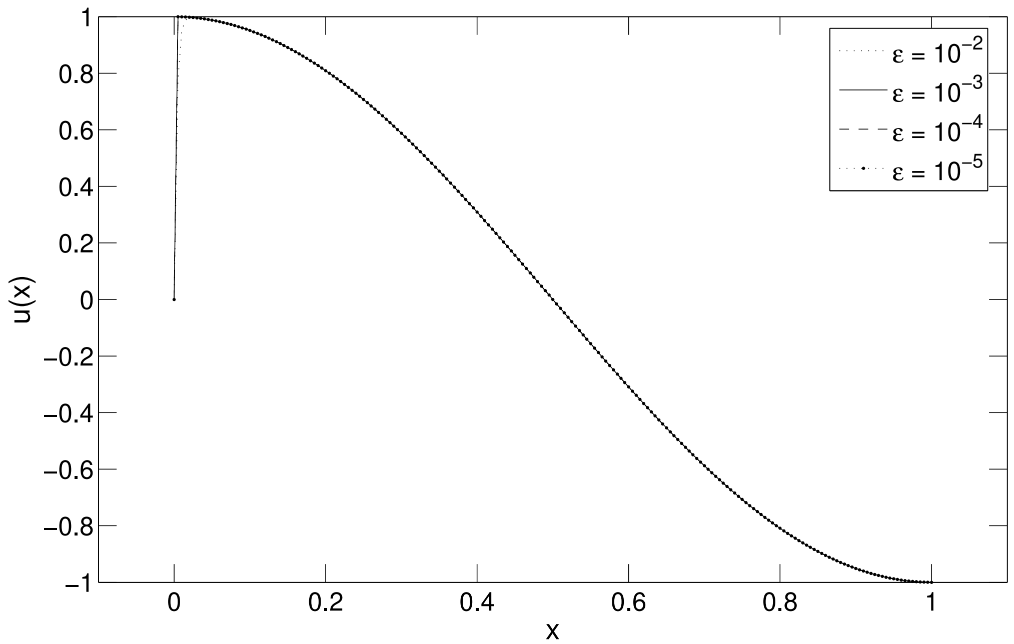

Figure 5.6: Plot of $u(x)$ of (5.3) for some values of ϵ

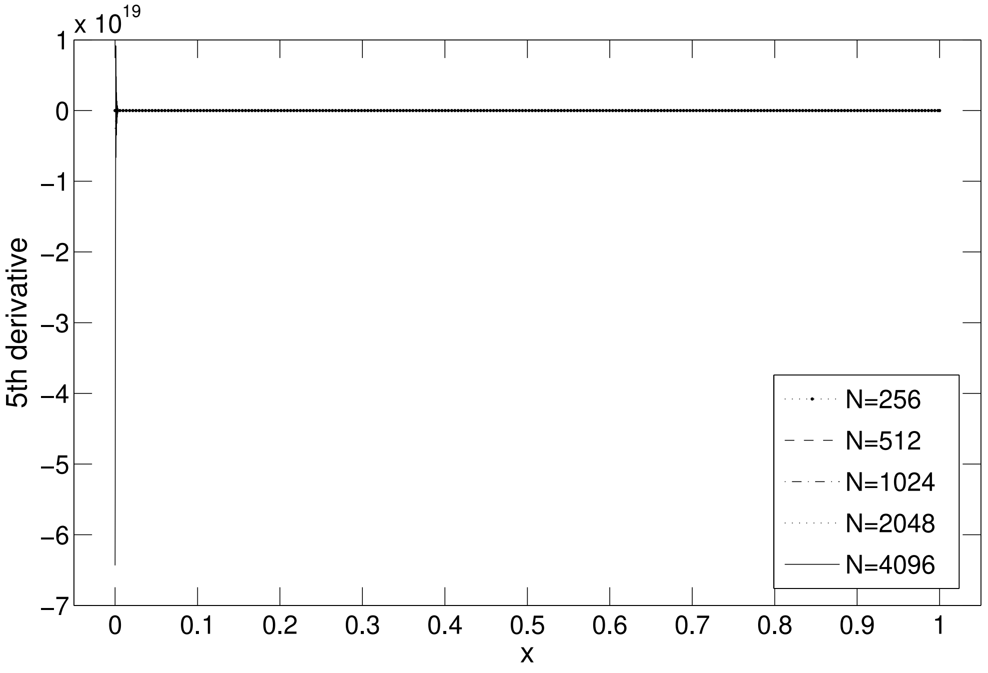

Figure 5.7: Plot of $u^{(5)}(x)$ corresponding to (5.3) where $\epsilon = 10^{-4}$

It turns our that the errors of the quartic spline collocation method over the whole domain for problems with layers are dominated by those in the layer(s). In order to achieve errors in the layer(s) close in magnitude to those in the area of little variation, we must apply a refinement, i.e. a smaller stepsize, to the layer region. However, this means that we would have to develop non-uniform grid quartic spline collocation methods, such as those for quadratic and cubic splines developed in [3] and [4]. Given the more complicated form of perturbations to obtain optimal order of convergence for quartic splines than for quadratic and cubic splines, such a development is quite cumbersome, and we leave it for future work. Therefore, in this chapter, we apply a simpler technique to deal with layer problems.

Assuming we have an approximation $u_\Delta(x)$ to $u(x)$ computed on some uniform grid of size N, we then apply error estimation and grid size estimation techniques to identify the layer(s), and predict the number of data points N' required in the layer region(s) to keep the error below a certain tolerance. These techniques are, to some extent, similar to those proposed for cubic and quadratic spline collocation methods in [3]. Then we refine the layer(s) with the predicted number of additional grid points, and apply the quartic spline collocation method once again, only in the layer region, with uniform grid (but smaller stepsize), and with the boundary conditions given by the approximation $u_\Delta(x)$ evaluated at the boundaries of the layers. In this way, we compute a new approximation $u_{\Delta'}(x)$ for the layer region. For the area of little variation, we don't apply further quartic spline collocation methods. The final approximation in the area of little variation is $u_\Delta(x)$, while in the layers it is $u_{\Delta'}(x)$. Thus, we maintain C^4 continuity everywhere in the domain except at the boundaries of the layers.

We realize that this is a very simple, and possibly crude technique of refinement, but, for some problems, it is effective. Moreover, our approach indicates that the error and grid size estimation techniques we develop for quartic spline collocation are reliable.

5.2 Error estimation

Let us denote by ϵ_i the maximum error in absolute value of quartic spline collocation approximation $u_\Delta(x)$ to $u(x)$ on subinterval $[x_i, x_{i+1})$. We make the assumption

$$\epsilon_i \approx Ch^5|u_i^{(5)}| + O(h^6), \tag{5.4}$$

where h is the mesh size of the grid partition, and C is a constant. This assumption is consistent with the ones in [3] for quadratic and cubic spline collocation methods. We obtain an approximation to the constant C by the following argument. Consider a simple BVP problem $u'' = 20x^3$ with $u(0) = 0$ and $u(1) = 1$. The solution to this BVP is $u(x) = x^5$. Then we have $u^{(5)} = 120$ which implies that (5.4) becomes $\epsilon_i \approx 120Ch^5$. By applying the quartic spline collocation method to this problem, and calculating its error, we obtain $C = \frac{1}{3840}$. Next we use two examples to demonstrate that (5.4) holds.

Example 1

The differential equation is

$$Lu = u''(x) + u'(x) + u(x), \quad x \in [0, 1]$$

and the boundary conditions

$$Bu = u(x) - u'(x), \quad \text{for} \quad x = 0 \quad \text{and} \quad x = 1.$$

The solution to this problem is

$$u(x) = e^x \sin(\pi x).$$

Example 2

The differential equation is

$$Lu = \epsilon u''(x) + xu'(x) = g, \quad x \in [-1, 1]$$

for $\epsilon = 10^{-4}$ and the boundary conditions

$$u(-1) = -2, \;\; u(1) = 0.$$

The solution to this problem is

$$u(x) = \cos(\pi x) + \frac{\text{erf}(x/\sqrt{2\epsilon})}{\text{erf}(1/\sqrt{2\epsilon})}.$$

We apply the quartic spline collocation method with extra boundary conditions (3.3) to these two BVPs. Throughout the thesis we define the i-th global point as $a+i(\frac{b-a}{N_g})$ where N_g is the number of global points in the interval I. In Tables 5.1 and 5.2, we present the errors at midpoints and global points respectively. The order of convergence of $\|u^{(5)} - u_\Delta^{(5)}\|_{\tau_i}$ is also given in the tables. In Figures 5.8 and 5.9, t_i denotes the i-th global point. We use 130 and 1022 global points for Examples 1 and 2 respectively, which also ensures that no global point is a mesh point. From Figures 5.8 and 5.9 we can see that $|Ch^5 u_i^{(5)}|$ can well match $|u(t_i) - u_\Delta(t_i)|$.

Table 5.1: Results for Example 1.

N	$\|u - u_\Delta\|_{\tau_i}$	$\|u - u_\Delta\|_\infty$	$\|u^{(5)}\|_{\tau_i}$	$\|u_\Delta^{(5)}\|_{\tau_i}$	$\|u^{(5)} - u_\Delta^{(5)}\|_{\tau_i}$	
8	1.3-04	1.3-04	6.7+02	6.5+02	3.6+02	
16	2.6-06	2.7-06	6.7+02	6.6+02	8.8+01	2.02
32	3.6-08	4.0-08	6.7+02	6.7+02	1.9+01	2.25
64	4.4-10	5.5-10	6.7+02	6.7+02	3.5+00	2.41
128	5.6-12	8.6-12	6.7+02	6.7+02	6.5-01	2.43

Table 5.2: Results for Example 2.

N	$\|u - u_\Delta\|_{\tau_i}$	$\|u - u_\Delta\|_\infty$	$\|u^{(5)}\|_{\tau_i}$	$\|u_\Delta^{(5)}\|_{\tau_i}$	$\|u^{(5)} - u_\Delta^{(5)}\|_{\tau_i}$	
256	2.4-04	6.3-04	1.6+10	1.0+10	7.2+09	
512	6.2-06	1.3-05	2.2+10	1.9+10	2.6+09	1.50
1024	1.1-07	2.5-07	2.3+10	2.3+10	7.3+08	1.81
2048	1.8-09	5.4-09	2.4+10	2.4+10	1.9+08	1.95
4096	2.8-11	1.4-10	2.4+10	2.4+10	4.7+07	1.99

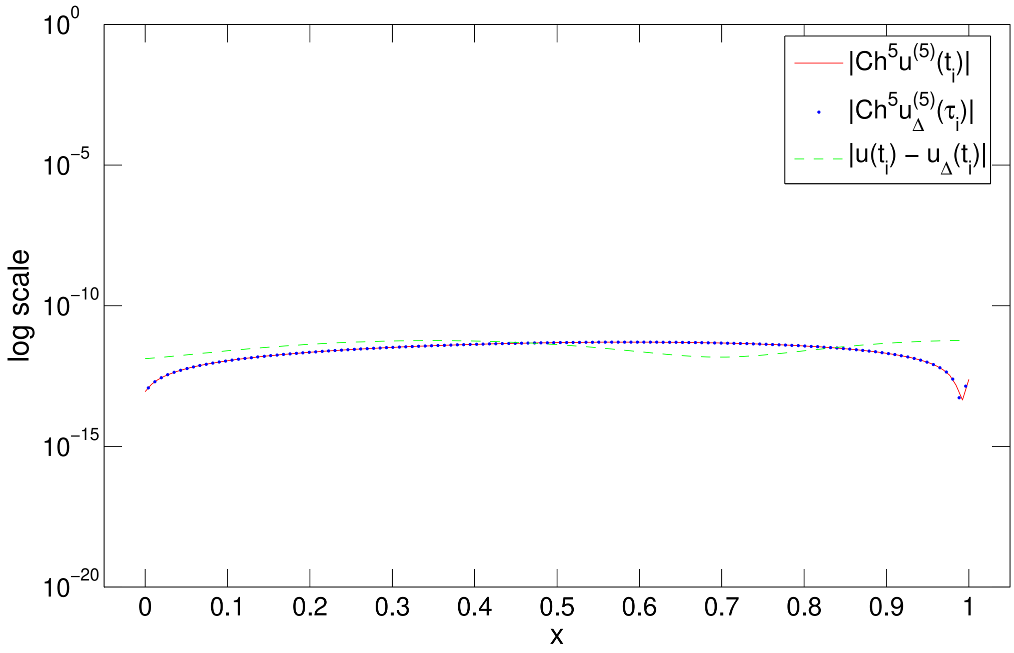

Figure 5.8: Plot for Example 1.

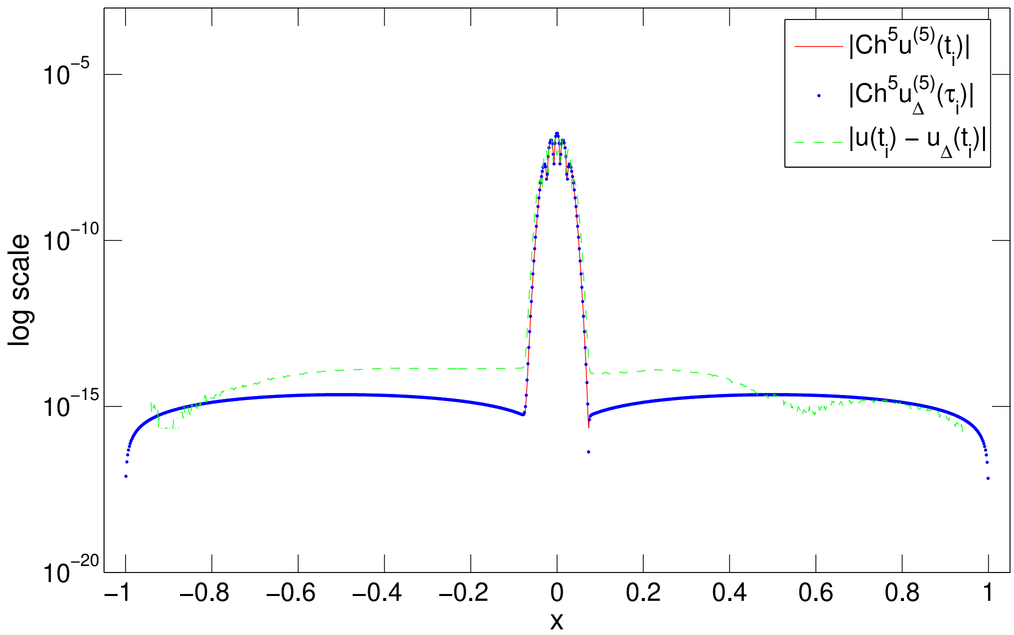

Figure 5.9: Plot for Example 2.

5.3 Refinement of the layer(s)

First we present the algorithm to identify the layer(s). We denote the domain by $I = [a, b]$, the layers by I_1 and I_2, and the area of little variation by $I_0 = I - I_1 \cup I_2$. (The algorithm can be easily adjusted for the case of one single layer, or even for more than two layers.) From (5.4) we have the error for the subinterval $[x_i, x_{i+1})$

$$\epsilon_i \approx Ch^5 |u_i^{(5)}|. \tag{5.5}$$

We consider the *monitor* function $\hat{u} = |u^{(5)}|^{\frac{1}{5}}$, and the form of the error

$$\hat{\epsilon}_i \equiv \epsilon_i^{\frac{1}{5}} = C^{\frac{1}{5}} h \hat{u}$$

which is linear in the stepsize. The ideal case is when the error is equidistributed, i.e. for each subinterval $[x_i, x_{i+1})$ to have

$$\hat{\epsilon}_i \approx C^{\frac{1}{5}} h \frac{\sum_{j=1}^{N} \hat{u}}{N},$$

or, equivalently,

$$\epsilon_i^{\frac{1}{5}} \approx C^{\frac{1}{5}} h \frac{\sum_{j=1}^{N} |u_j^{(5)}|^{\frac{1}{5}}}{N}. \tag{5.6}$$

In other words, the ideal case is when each $\hat{\epsilon}_i$ is equal to the average among the $\hat{\epsilon}_i$'s. We identify the layer(s) as the areas where $\hat{\epsilon}_i$ is far from the ideal (5.6), that is, we find the subintervals $[x_i, x_{i+1})$, where

$$\epsilon_i^{\frac{1}{5}} \geq \alpha C^{\frac{1}{5}} h \frac{\sum_{j=1}^{N} |u_j^{(5)}|^{\frac{1}{5}}}{N}, \tag{5.7}$$

where $\alpha = 2$. In other words, the error in these subintervals is $\alpha^5 = 32$ times larger than the 'average' error.

If we define θ in I as

$$\theta = \frac{b-a}{N} \sum_{j=1}^{N} \mid u_j^{(5)} \mid^{\frac{1}{5}} \approx \int_a^b |u^{(5)}|^{\frac{1}{5}} dx,$$

then (5.7) becomes

$$\epsilon_i^{\frac{1}{5}} \geq \alpha C^{\frac{1}{5}} h \frac{\theta}{b-a}, \tag{5.8}$$

or, taking into account (5.5),

$$|u_i^{(5)}|^{\frac{1}{5}} \geq \alpha \frac{\theta}{b-a}. \tag{5.9}$$

We use (5.9) as the criterion to identify which subintervals $[x_i, x_{i+1})$ form the layer regions.

We now justify the choice of the number of data points which are needed to refine the layer. Let us define ζ for I_0 and η for I_1 or I_2 respectively, as

$$\begin{aligned} \eta &= \|u_i^{(5)}\|_\infty \text{ where } \tau_i \in I_0, \\ \zeta &= \|u_i^{(5)}\|_\infty \text{ where } \tau_i \in I_1 \text{ or } I_2. \end{aligned}$$

The goal of the refinement is to set the worst error in the layer to be at most α^5 times that on area of little variation, that is,

$$Ch'^5\zeta \approx \alpha^5 Ch^5 \eta, \tag{5.10}$$

where h' is the mesh size of the refined grid partition in I_1 or I_2. Suppose $[x_j, x_k] \equiv I_1$ or I_2 is one layer, and N' subintervals in I_1 or I_2 are required to do the refinement. From (5.10) and $h' = \frac{x_k - x_j}{N'} = \frac{(k-j)h}{N'}$, we obtain

$$N' \approx \frac{N_r}{\alpha}(\frac{\zeta}{\eta})^{\frac{1}{5}}, \tag{5.11}$$

where $N_r = k - j$.

It is possible that N' in (5.11) comes up to be very large to target very small error. In order to prevent N' becoming too large, we impose the restriction

$$C(\frac{x_k - x_j}{N'})^5 \zeta \geq TOL, \tag{5.12}$$

where we set $TOL = 10^{-13}$. We avoid targeting errors smaller than the tolerance TOL, since, at this order of errors, the rounding errors would become dominant. Let $E = \frac{TOL}{C} \approx 5 \times 10^{-10}$. Then (5.12) can be written as

$$N' \leq (x_k - x_j)(\frac{\zeta}{E})^{\frac{1}{5}}. \tag{5.13}$$

We also need the restriction

$$N' \geq 2N_r \tag{5.14}$$

to ensure that N' is not too small.

Chapter 6

Numerical Results

In this chapter we present the numerical results to demonstrate the performance of the quartic spline collocation methods introduced in Chapters 3 and 5.

The numerical experiments are implemented in MATLAB in double precision. We present tables of the errors and analyze the convergence order of the methods. In the tables, $u_\Delta^{(i)}, i = 0, 1, \cdots, 4$, label the rows of the table reporting the corresponding errors $\|u^{(i)} - u_\Delta^{(i)}\|$. In most of the numerical experiments, we use 1002 points, to compute the so-called global errors. The global error is denoted by ∞ in the tables, and is considered to be an approximation to the infinity norm of the error. If we use another number of global points instead of 1002, we present the number explicitly. The number of global points is chosen to ensure that enough global points are taken and they do not fall on special locations relative to the mesh points. Since some researchers have achieved optimal order or superconvergence at Gauss points, we also calculate the errors at the two Gauss points in each subinterval. These two Gauss points have a distance of $\frac{3-\sqrt{3}}{6}h$ from the left and right grid point of the subinterval, respectively. In the tables, we denote by λ_i the errors and convergence at the Gauss points.

In Sections 6.1 and 6.2, we apply the two-step quartic spline collocation method with different choice of extra boundary conditions described in Sections 3.3 and 3.4. In Section 6.3, we present the numerical results for the non-optimal quartic spline collocation methods described in Section 3.5. The numerical results for layer problems are presented in Section 6.4. In Section 6.5, we apply the one-step quartic spline collocation method to the same problems as those in Section 6.1, and compare the numerical results with those obtained by the two-step collocation method. In Sections 6.1-6.3 and 6.5, we consider two sets of boundary conditions

$$u(a) = g_1, \quad u(b) = g_2, \tag{6.1}$$

$$u(a) - u'(a) = g_1, \quad u(b) - u'(b) = g_2. \tag{6.2}$$

6.1 Numerical results with extra boundary conditions $Lu = f$ at x_1 and x_{N-1}

We apply the two-step quartic spline collocation method to Problems 1-4 with the extra boundary conditions (3.3).

Problem 1

Consider the differential equation

$$Lu = u''(x) = f(x), \quad x \in [0, 1]$$

with boundary conditions (6.1). The functions $f(x)$, g_1 and g_2 are chosen so that $u(x) = x^5$ and $u(x) = x^6$ respectively. Tables 6.1 and 6.2 show the errors and the orders of convergence for $u(x) = x^5$ and $u(x) = x^6$ respectively.

In Table 6.1, the round-off dominates the approximation to $u(x)$, $u^{(2)}(x)$ and $u^{(4)}(x)$ at the midpoints, the approximation to $u(x)$ and $u^{(2)}(x)$ at the grid points, and the approximation to $u^{(3)}$ at the Gauss points. Except those round-off effect data, the orders of convergence of the quartic spline collocation method are 5, 4, 3, 2, and 1 for the approximation to $u(x)$, $u^{(1)}(x)$, $u^{(2)}(x)$, $u^{(3)}(x)$ and $u^{(4)}(x)$ respectively. This shows that the quartic spline collocation method on this problem is optimal.

In Table 6.2, the order of convergence for the approximation to $u(x)$ is sixth at midpoints and fifth at global points while the round-off dominates at grid points. We also obtain some superconvergence properties at collocation points. Namely, we obtain fourth order for the approximation to $u^{(2)}(x)$ at midpoints and grid points, and second order for the approximation to $u^{(4)}(x)$ at the midpoints. We also obtain third order for the approximation to $u^{(3)}(x)$ at the Gauss points.

Table 6.1: Errors and orders of convergence for Problem 1 where $u(x) = x^5$.

	N	τ_i		x_i		∞		λ_i	
u_Δ	8	1.4-15		1.5-15		7.5-07		7.3-07	
	16	2.7-15	-0.97	2.8-15	-0.91	2.3-08	5.00	2.3-08	5.00
	32	1.0-14	-1.90	1.0-14	-1.88	7.3-10	5.00	7.2-10	5.00
	64	1.4-14	-0.41	1.4-14	-0.39	2.3-11	5.00	2.2-11	5.00
	128	8.0-14	-2.56	8.0-14	-2.56	7.9-13	4.85	7.8-13	4.85
u'_Δ	8	3.6-05		4.1-05		4.1-05		6.8-06	
	16	2.2-06	4.00	2.5-06	4.00	2.5-06	4.00	4.2-07	4.00
	32	1.4-07	4.00	1.6-07	4.00	1.6-07	4.00	2.6-08	4.00
	64	8.7-09	4.00	9.9-09	4.00	9.9-09	4.00	1.7-09	4.00
	128	5.4-10	4.00	6.2-10	4.00	6.2-10	4.00	1.0-10	4.00
u''_Δ	8	8.9-16		2.2-16		1.9-03		1.9-03	
	16	2.8-14	-5.00	1.1-13	-9.00	2.3-04	3.00	2.3-04	3.00
	32	1.1-13	-2.00	3.4-13	-1.58	2.9-05	3.00	2.9-05	3.00
	64	5.7-13	-2.32	2.3-12	-2.74	3.7-06	3.00	3.7-06	3.00
	128	3.4-12	-2.58	2.7-12	-0.26	4.6-07	3.00	4.6-07	3.00
$u^{(3)}_\Delta$	8	7.8-02		1.6-01		1.6-01		7.1-14	
	16	2.0-02	2.00	3.9-02	2.00	3.9-02	2.00	4.4-12	-5.94
	32	4.9-03	2.00	9.8-03	2.00	9.8-03	2.00	3.4-11	-2.95
	64	1.2-03	2.00	2.4-03	2.00	2.4-03	2.00	1.8-10	-2.41
	128	3.1-04	2.00	6.1-04	2.00	6.1-04	2.00	1.1-09	-2.57
$u^{(4)}_\Delta$	8	1.8-12		7.5+00		7.5+00		4.3+00	
	16	1.2-10	-6.00	3.8+00	1.00	3.8+00	1.00	2.2+00	1.00
	32	1.6-09	-3.81	1.9+00	1.00	1.9+00	1.00	1.1+00	1.00
	64	1.3-08	-3.00	9.4-01	1.00	9.4-01	1.00	5.4-01	1.00
	128	3.0-07	-4.51	4.7-01	1.00	4.7-01	1.00	2.7-01	1.00

Table 6.2: Errors and orders of convergence for Problem 1 where $u(x) = x^6$.

	N	τ_i		x_i		∞		λ_i	
u_Δ	8	8.9-07		7.7-16		4.8-06		4.6-06	
	16	1.4-08	6.00	2.7-15	-1.79	1.4-07	5.05	1.4-07	5.04
	32	2.2-10	6.00	6.4-15	-1.26	4.4-09	5.03	4.4-09	5.02
	64	3.4-12	6.00	1.8-14	-1.45	1.4-10	5.01	1.4-10	5.01
	128	5.3-14	6.01	1.6-14	0.12	4.3-12	5.01	4.2-12	5.00
u'_Δ	8	2.0-04		2.4-04		2.4-04		5.7-05	
	16	1.3-05	3.95	1.5-05	4.00	1.5-05	4.00	3.1-06	4.23
	32	8.2-07	3.98	9.5-07	4.00	9.5-07	4.00	1.8-07	4.13
	64	5.2-08	3.99	6.0-08	4.00	6.0-08	4.00	1.0-08	4.07
	128	3.2-09	3.99	3.7-09	4.00	3.7-09	4.00	6.4-10	4.04
u''_Δ	8	6.4-04		7.3-04		1.1-02		1.1-02	
	16	4.0-05	4.00	4.6-05	4.00	1.4-03	2.97	1.4-03	2.97
	32	2.5-06	4.00	2.9-06	4.00	1.7-04	2.99	1.7-04	2.99
	64	1.6-07	4.00	1.8-07	4.00	2.2-05	3.00	2.2-05	2.99
	128	9.8-09	4.00	1.1-08	4.00	2.7-06	3.00	2.7-06	3.00
$u^{(3)}_\Delta$	8	4.4-01		9.4-01		9.4-01		2.3-02	
	16	1.1-01	1.95	2.3-01	2.00	2.3-01	2.00	2.8-03	3.00
	32	2.9-02	1.98	5.9-02	2.00	5.9-02	2.00	3.5-04	3.00
	64	7.3-03	1.99	1.5-02	2.00	1.5-02	2.00	4.4-05	3.00
	128	1.8-03	1.99	3.7-03	2.00	3.7-03	2.00	5.5-06	3.00
$u^{(4)}_\Delta$	8	4.7-01		4.4+01		4.4+01		2.5+01	
	16	1.2-01	2.00	2.2+01	0.98	2.2+01	0.98	1.3+01	0.98
	32	2.9-02	2.00	1.1+01	0.99	1.1+01	0.99	6.5+00	0.99
	64	7.3-03	2.00	5.6+00	1.00	5.6+00	1.00	3.2+00	1.00
	128	1.8-03	2.00	2.8+00	1.00	2.8+00	1.00	1.6+00	1.00

Problem 2

Consider the differential equation

$$Lu = u''(x) + u'(x) + u(x) = f(x), \quad x \in [0, 1]$$

with boundary conditions (6.1). The functions $f(x)$, g_1 and g_2 are chosen so that $u(x) = x^5$, $u(x) = x^{\frac{11}{2}}$, $u(x) = x^{\frac{13}{2}}$ and $u(x) = x^{\frac{15}{2}}$ respectively. Tables 6.3 - 6.6 show the errors and the orders of convergence for these different $u(x)$, respectively.

In these tables, the order of convergence is close to 6 at midpoints and grid points except in Table 6.4 in which the order of convergence at grid points is close to 5.5. This indicates that C^6 continuity for $u(x)$ is necessary to obtain the optimal orders of convergence. We emphasize that the condition $u \in C^{10}$, mentioned in the formulation of the quartic spline collocation methods, is only a sufficient and not a necessary condition. In all of these tables, the global order of convergence for u_Δ is close to 5. We obtain superconvergence for the approximation to $u^{(2)}(x)$ at midpoints and grid points, for the approximation to $u^{(4)}(x)$ at midpoints, and for the approximation to $u^{(3)}(x)$ at Gauss points.

Table 6.3: Errors and orders of convergence for Problem 2 where $u(x) = x^5$.

	N	τ_i		x_i		∞		λ_i	
u_Δ	8	2.0-09		2.2-09		7.5-07		7.4-07	
	16	2.5-11	6.34	2.5-11	6.50	2.3-08	5.00	2.3-08	5.00
	32	2.3-13	6.78	2.2-13	6.79	7.3-10	5.00	7.2-10	5.00
	64	1.6-14	3.80	1.6-14	3.77	2.3-11	5.00	2.2-11	5.00
	128	4.2-14	-1.34	4.2-14	-1.36	7.5-13	4.92	7.4-13	4.92
u'_Δ	8	3.6-05		4.1-05		4.1-05		6.8-06	
	16	2.2-06	4.00	2.5-06	4.00	2.5-06	4.00	4.2-07	4.00
	32	1.4-07	4.00	1.6-07	4.00	1.6-07	4.00	2.6-08	4.00
	64	8.7-09	4.00	9.9-09	4.00	9.9-09	4.00	1.7-09	4.00
	128	5.4-10	4.00	6.2-10	4.00	6.2-10	4.00	1.0-10	4.00
u''_Δ	8	1.7-07		2.0-06		1.9-03		1.9-03	
	16	5.3-09	5.03	6.0-08	5.04	2.3-04	3.00	2.3-04	3.00
	32	1.6-10	5.01	1.9-09	5.01	2.9-05	3.00	2.9-05	3.00
	64	5.1-12	5.01	5.8-11	5.00	3.7-06	3.00	3.7-06	3.00
	128	4.3-12	0.25	1.3-11	2.20	4.6-07	3.00	4.6-07	3.00
$u^{(3)}_\Delta$	8	7.8-02		1.6-01		1.6-01		3.1-05	
	16	2.0-02	2.00	3.9-02	2.00	3.9-02	2.00	1.9-06	4.04
	32	4.9-03	2.00	9.8-03	2.00	9.8-03	2.00	1.2-07	4.01
	64	1.2-03	2.00	2.4-03	2.00	2.4-03	2.00	7.2-09	4.00
	128	3.1-04	2.00	6.1-04	2.00	6.1-04	2.00	2.7-09	1.42
$u^{(4)}_\Delta$	8	3.8-04		7.5+00		7.5+00		4.3+00	
	16	4.6-05	3.04	3.8+00	1.00	3.7+00	1.00	2.2+00	1.00
	32	5.7-06	3.01	1.9+00	1.00	1.9+00	1.00	1.1+00	1.00
	64	7.1-07	3.00	9.4-01	1.00	9.4-01	1.00	5.4-01	1.00
	128	5.4-07	0.41	4.7-01	1.00	4.7-01	1.00	2.7-01	1.00

Table 6.4: Errors and orders of convergence for Problem 2 where $u(x) = x^{\frac{11}{2}}$.

	N	τ_i		x_i		∞		λ_i	
u_Δ	8	3.5-07		2.9-07		2.0-06		2.0-06	
	16	1.2-08	4.81	7.0-09	5.37	6.4-08	4.98	6.3-08	4.98
	32	3.3-10	5.21	2.4-10	4.90	2.0-09	5.02	2.0-09	5.01
	64	8.4-12	5.30	6.7-12	5.14	6.2-11	5.01	6.1-11	5.01
	128	1.2-13	6.17	1.6-13	5.38	1.9-12	5.00	1.9-12	5.00
u'_Δ	8	9.3-05		1.1-04		1.1-04		2.1-05	
	16	5.9-06	3.97	6.9-06	3.96	6.9-06	3.96	1.3-06	4.06
	32	3.7-07	3.99	4.3-07	4.00	4.3-07	4.00	7.6-08	4.07
	64	2.3-08	3.99	2.7-08	4.00	2.7-08	4.00	4.6-09	4.04
	128	1.5-09	4.00	1.7-09	4.00	1.7-09	4.00	2.8-10	4.02
u''_Δ	8	3.5-04		2.1-03		4.9-03		4.9-03	
	16	3.1-05	3.50	1.8-04	3.51	6.3-04	2.97	6.3-04	2.97
	32	2.7-06	3.50	1.6-05	3.50	7.9-05	2.99	7.9-05	2.99
	64	2.4-07	3.50	1.4-06	3.50	9.8-06	3.00	9.9-06	3.00
	128	2.1-08	3.50	1.2-07	3.50	1.2-06	3.00	1.2-06	3.00
$u^{(3)}_\Delta$	8	2.0-01		4.3-01		4.3-01		3.9-02	
	16	5.2-02	1.97	1.1-01	2.01	1.1-01	2.01	6.9-03	2.51
	32	1.3-02	1.99	2.6-02	2.00	2.6-02	2.00	1.2-03	2.50
	64	3.3-03	1.99	6.6-03	2.00	6.6-03	2.00	2.2-04	2.50
	128	8.2-04	2.00	1.7-03	2.00	1.7-03	2.00	3.8-05	2.50
$u^{(4)}_\Delta$	8	5.8-01		2.0+01		2.0+01		1.2+01	
	16	2.1-01	1.50	1.0+01	0.99	1.0+01	0.99	5.8+00	0.99
	32	7.3-02	1.50	5.1+00	1.00	5.1+00	1.00	2.9+00	1.00
	64	2.6-02	1.50	2.5+00	1.00	2.5+00	1.00	1.5+00	1.00
	128	9.1-03	1.50	1.3+00	1.00	1.3+00	1.00	7.3-01	1.00

Table 6.5: Errors and orders of convergence for Problem 2 where $u(x) = x^{\frac{13}{2}}$.

	N	τ_i		x_i		∞		λ_i	
	8	2.8-06		4.7-07		9.9-06		9.4-06	
	16	4.0-08	6.11	1.2-08	5.23	2.9-07	5.10	2.8-07	5.08
u_Δ	32	6.3-10	6.00	3.1-10	5.33	8.7-09	5.04	8.6-09	5.03
	64	9.9-12	5.99	6.0-12	5.67	2.7-10	5.02	2.7-10	5.01
	128	1.5-13	6.00	1.1-13	5.78	8.4-12	5.01	8.3-12	5.01
	8	3.8-04		4.9-04		4.9-04		1.3-04	
	16	2.5-05	3.93	3.0-05	4.03	3.0-05	4.03	6.5-06	4.33
u'_Δ	32	1.6-06	3.97	1.9-06	4.00	1.9-06	4.00	3.6-07	4.18
	64	1.0-07	3.98	1.2-07	4.00	1.2-07	4.00	2.1-08	4.10
	128	6.3-09	3.99	7.3-09	4.00	7.3-09	4.00	1.3-09	4.05
	8	1.9-03		2.0-03		2.1-02		2.1-02	
	16	1.2-04	4.01	1.3-04	3.96	2.7-03	2.97	2.7-03	2.97
u''_Δ	32	7.3-06	3.99	8.4-06	3.96	3.4-04	2.98	3.4-04	2.98
	64	4.6-07	4.00	5.3-07	4.00	4.2-05	3.00	4.3-05	2.99
	128	2.9-08	4.00	3.3-08	4.00	5.3-06	3.00	5.4-06	2.99
	8	8.3-01		1.8+00		1.8+00		6.4-02	
	16	2.2-01	1.93	4.6-01	1.99	4.6-01	1.99	8.1-03	2.98
$u^{(3)}_\Delta$	32	5.6-02	1.97	1.1-01	2.00	1.1-01	2.00	1.0-03	2.98
	64	1.4-02	1.98	2.9-02	2.00	2.9-02	2.00	1.3-04	3.00
	128	3.6-03	1.99	7.2-03	2.00	7.2-03	2.00	1.6-05	3.00
	8	1.3+00		8.5+01		8.5+01		4.9+01	
	16	3.4-01	1.91	4.3+01	0.98	4.3+01	0.98	2.5+01	0.97
$u^{(4)}_\Delta$	32	8.5-02	1.98	2.2+01	0.99	2.2+01	0.99	1.3+01	0.99
	64	2.1-02	1.99	1.1+01	0.99	1.1+01	0.99	6.3+00	0.99
	128	5.4-03	2.00	5.5+00	1.00	5.5+00	1.00	3.2+00	1.00

Table 6.6: Errors and orders of convergence for Problem 2 where $u(x) = x^{\frac{15}{2}}$.

	N	τ_i		x_i		∞		λ_i	
u_Δ	8	1.1-05		1.0-06		2.9-05		2.8-05	
	16	2.0-07	5.82	6.3-08	4.02	8.9-07	5.05	8.5-07	5.02
	32	3.1-09	5.96	1.2-09	5.67	2.7-08	5.05	2.6-08	5.04
	64	5.0-11	5.99	2.1-11	5.90	8.2-10	5.03	8.0-10	5.02
	128	7.8-13	5.99	3.5-13	5.91	2.5-11	5.02	2.5-11	5.01
u'_Δ	8	1.0-03		1.3-03		1.4-03		4.4-04	
	16	7.2-05	3.86	8.9-05	3.92	8.9-05	3.94	2.2-05	4.32
	32	4.7-06	3.94	5.6-06	3.99	5.6-06	3.99	1.2-06	4.25
	64	3.0-07	3.97	3.5-07	4.00	3.5-07	4.00	6.6-08	4.15
	128	1.9-08	3.99	2.2-08	4.00	2.2-08	4.00	3.9-09	4.08
u''_Δ	8	8.3-03		1.3-02		5.9-02		5.8-02	
	16	5.6-04	3.90	7.0-04	4.20	7.9-03	2.90	7.9-03	2.89
	32	3.6-05	3.96	4.3-05	4.05	1.0-03	2.96	1.0-03	2.96
	64	2.3-06	3.98	2.6-06	4.01	1.3-04	2.99	1.3-04	2.98
	128	1.4-07	3.99	1.6-07	4.01	1.6-05	3.00	1.6-05	2.99
$u^{(3)}_\Delta$	8	2.3+00		5.5+00		5.5+00		3.5-01	
	16	6.3-01	1.86	1.4+00	2.01	1.4+00	2.01	4.1-02	3.07
	32	1.7-01	1.94	3.4-01	2.00	3.4-01	2.00	5.1-03	3.01
	64	4.2-02	1.97	8.6-02	2.00	8.6-02	2.00	6.4-04	3.00
	128	1.1-02	1.99	2.1-02	2.00	2.1-02	2.00	8.1-05	3.00
$u^{(4)}_\Delta$	8	6.6+00		2.5+02		2.5+02		1.4+02	
	16	1.7+00	2.00	1.3+02	0.96	1.3+02	0.96	7.4+01	0.95
	32	4.2-01	1.98	6.5+01	0.98	6.5+01	0.98	3.7+01	0.98
	64	1.1-01	1.99	3.3+01	0.99	3.3+01	0.99	1.9+01	0.99
	128	2.7-02	1.99	1.6+01	1.00	1.6+01	1.00	9.5+00	0.99

Problem 3

Consider the differential equation

$$Lu = u''(x) + u'(x) + u(x) = f(x), \quad x \in [0, 1]$$

with boundary conditions (6.2). The functions $f(x)$, g_1 and g_2 are chosen so that $u(x) = e^x \sin(\pi x)$, $u(x) = x^{\frac{11}{2}}$, $u(x) = x^{\frac{13}{2}}$ and $u(x) = x^{\frac{15}{2}}$, respectively.

Table 6.7 presents the errors and convergence of the 5-th and 6-th derivatives of the approximation v in the first step of the collocation method for $u(x) = e^x \sin(\pi x)$. We denote by $\|\cdot\|_b$ the errors at the boundary points x_0, x_N, and near-boundary points x_1, x_{N-1}. Suppose S be the quartic spline interpolant defined in (2.11)-(2.12), and $v \in S^4_\Delta$ obtained in the first step of the optimal collocation method. By (4.18) and (2.18)-(2.19), we have

$$u_i^{(k+4)} = \frac{\delta^2 S_i^{(k)}}{h^4} + O(h^2) = \frac{\delta^2 v_i^{(k)}}{h^4} + O(h^2),$$

for $k = 1, 2$. (Note that the second equality in the above relation is an assumption.) Therefore, we expect the errors $\| \; u^{(5)} - \frac{\delta^2 v_i^{(1)}}{h^4} \; \|$ and $\| \; u^{(6)} - \frac{\delta^2 v_i^{(2)}}{h^4} \; \|$ to be second order. From Table 6.7, it is clear that the errors are second order as we expect. In Table 6.7, we denote $\frac{\delta^2 v^{(1)}}{h^4}$ by $v^{(5)}$ and $\frac{\delta^2 v^{(2)}}{h^4}$ by $v^{(6)}$, for simplicity.

Tables 6.8-6.11 show the errors and the orders of convergence for the different $u(x)$ respectively. In Table 6.8 the collocation point error and global error orders of convergence are about 6 while in Table 6.9 the collocation point error and global error orders of convergence are about 5. In Table 6.10 we obtain sixth order of collocation point error and fifth order of global error. In Table 6.11 the collocation point error and global error orders of convergence are about 6 as well. These results indicate that the boundary conditions (6.2) which involve the derivative of $u(x)$ do not affect the optimal orders of convergence.

Table 6.7: Errors and orders of convergence for the 5-th and 6-th derivatives of $u(x) = e^x \sin(\pi x)$.

N	$\| u^{(5)} - v^{(5)} \|_{\tau_i}$		$\| u^{(6)} - v^{(6)} \|_{\tau_i}$		$\| u^{(5)} - v^{(5)} \|_b$		$\| u^{(6)} - v^{(6)} \|_b$	
8	3.6+02		7.9+02		5.3+02		1.1+03	
16	8.8+01	2.02	1.6+02	2.29	1.3+02	2.03	2.3+02	2.29
32	1.9+01	2.25	4.0+01	2.01	2.8+01	2.22	6.3+01	1.87
64	3.5+00	2.41	1.2+01	1.73	5.6+00	2.33	1.9+01	1.75
128	6.5-01	2.43	3.0+00	2.01	8.8-01	2.66	4.8+00	1.98

Table 6.8: Errors and orders of convergence for Problem 3 where $u(x) = e^x \sin(\pi x)$.

	N	τ_i		x_i		∞		λ_i	
u_Δ	8	1.3-04		1.3-04		1.3-04		1.3-04	
	16	2.6-06	5.63	2.6-06	5.63	2.7-06	5.57	2.7-06	5.57
	32	3.6-08	6.15	3.7-08	6.13	4.0-08	6.07	4.0-08	6.07
	64	4.4-10	6.37	4.5-10	6.35	5.5-10	6.18	5.5-10	6.19
	128	5.6-12	6.31	5.8-12	6.28	8.6-12	6.00	8.6-12	6.01
u'_Δ	8	1.2-04		3.1-04		3.1-04		1.6-04	
	16	1.2-05	3.31	1.4-05	4.45	1.4-05	4.45	3.7-06	5.42
	32	7.8-07	3.99	8.8-07	4.03	8.8-07	4.04	1.6-07	4.58
	64	4.9-08	4.01	5.5-08	4.00	5.5-08	3.99	9.3-09	4.08
	128	3.0-09	4.00	3.5-09	4.00	3.4-09	4.00	5.8-10	4.01
u''_Δ	8	2.5-03		5.2-03		1.1-02		1.1-02	
	16	1.8-04	3.81	1.9-04	4.76	1.3-03	3.02	1.3-03	3.02
	32	1.1-05	3.97	1.3-05	3.91	1.6-04	3.01	1.6-04	3.01
	64	7.2-07	3.99	8.1-07	3.96	2.0-05	3.00	2.0-05	3.00
	128	4.5-08	3.99	5.2-08	3.97	2.6-06	3.00	2.6-06	3.00
$u^{(3)}_\Delta$	8	4.4-01		8.7-01		8.7-01		1.3-01	
	16	1.1-01	2.01	2.2-01	2.00	2.1-01	2.02	1.2-02	3.38
	32	2.7-02	2.00	5.4-02	2.00	5.3-02	2.02	1.6-03	2.94
	64	6.8-03	2.00	1.4-02	2.00	1.3-02	2.04	2.0-04	2.97
	128	1.7-03	2.00	3.4-03	2.00	3.3-03	1.95	2.5-05	2.98
$u^{(4)}_\Delta$	8	2.2+00		4.1+01		4.1+01		2.4+01	
	16	4.8-01	2.22	2.1+01	0.98	2.1+01	1.00	1.2+01	1.01
	32	1.3-01	1.91	1.0+01	1.00	1.0+01	1.01	6.0+00	1.00
	64	3.3-02	1.95	5.2+00	1.00	5.1+00	1.00	3.0+00	1.00
	128	8.5-03	1.98	2.6+00	1.00	2.6+00	0.99	1.5+00	1.00

Table 6.9: Errors and orders of convergence for Problem 3 where $u(x) = x^{\frac{11}{2}}$.

	N	τ_i		x_i		∞		λ_i	
u_Δ	8	9.8-06		9.9-06		1.2-05		1.2-05	
	16	2.5-07	5.32	2.5-07	5.32	3.1-07	5.26	3.1-07	5.26
	32	9.9-09	4.63	1.0-08	4.57	1.2-08	4.72	1.2-08	4.72
	64	4.5-10	4.44	4.7-10	4.48	4.8-10	4.60	4.8-10	4.60
	128	2.0-11	4.48	2.1-11	4.50	2.1-11	4.54	2.1-11	4.54
u'_Δ	8	9.7-05		1.0-04		1.0-04		2.1-05	
	16	6.1-06	3.98	6.7-06	3.95	6.7-06	3.95	1.1-06	4.30
	32	3.8-07	4.00	4.2-07	3.99	4.2-07	3.99	6.6-08	4.00
	64	2.4-08	4.00	2.6-08	3.99	2.6-08	3.99	4.2-09	3.98
	128	1.5-09	4.00	1.7-09	3.99	1.7-09	3.99	2.6-10	3.98
u''_Δ	8	3.7-04		2.0-03		4.9-03		4.9-03	
	16	3.2-05	3.55	1.8-04	3.50	6.3-04	2.97	6.3-04	2.97
	32	2.8-06	3.51	1.6-05	3.50	7.9-05	3.00	7.9-05	2.99
	64	2.4-07	3.50	1.4-06	3.50	9.8-06	3.00	9.9-06	3.00
	128	2.1-08	3.50	1.2-07	3.50	1.2-06	3.00	1.2-06	3.00
$u^{(3)}_\Delta$	8	2.0-01		4.3-01		4.3-01		3.9-02	
	16	5.2-02	1.97	1.1-01	2.01	1.1-01	2.01	6.9-03	2.51
	32	1.3-02	1.99	2.6-02	2.00	2.6-02	2.00	1.2-03	2.50
	64	3.3-03	1.99	6.6-03	2.00	6.6-03	2.00	2.2-04	2.50
	128	8.2-04	2.00	1.7-03	2.00	1.7-03	2.00	3.8-05	2.50
$u^{(4)}_\Delta$	8	5.8-01		2.0+01		2.0+01		1.2+01	
	16	2.1-01	1.50	1.0+01	0.99	1.0+01	0.99	5.8+00	0.99
	32	7.3-02	1.50	5.1+00	1.00	5.1+00	1.00	2.9+00	1.00
	64	2.6-02	1.50	2.5+00	1.00	2.5+00	1.00	1.5+00	1.00
	128	9.1-03	1.50	1.3+00	1.00	1.3+00	1.00	7.3-01	1.00

Table 6.10: Errors and orders of convergence for Problem 3 where $u(x) = x^{\frac{13}{2}}$.

	N	τ_i		x_i		∞		λ_i	
	8	2.6-05		2.8-05		3.3-05		3.3-05	
	16	2.4-07	6.75	2.6-07	6.73	4.5-07	6.20	4.4-07	6.21
u_Δ	32	2.7-09	6.45	2.9-09	6.50	8.9-09	5.64	8.9-09	5.65
	64	3.8-11	6.16	3.1-11	6.56	3.0-10	4.91	2.9-10	4.91
	128	1.9-12	4.35	1.7-12	4.16	1.0-11	4.88	1.0-11	4.88
	8	3.9-04		4.8-04		4.8-04		1.2-04	
	16	2.5-05	3.95	3.0-05	4.01	3.0-05	4.01	6.3-06	4.26
u'_Δ	32	1.6-06	3.97	1.9-06	4.00	1.9-06	4.00	3.6-07	4.16
	64	1.0-07	3.98	1.2-07	4.00	1.2-07	4.00	2.1-08	4.09
	128	6.3-09	3.99	7.3-09	4.00	7.3-09	4.00	1.3-09	4.05
	8	1.8-03		2.1-03		2.1-02		2.1-02	
	16	1.2-04	4.00	1.3-04	3.97	2.7-03	2.97	2.7-03	2.97
u''_Δ	32	7.3-06	3.99	8.4-06	3.97	3.4-04	2.98	3.4-04	2.98
	64	4.6-07	4.00	5.3-07	4.00	4.2-05	3.00	4.3-05	2.99
	128	2.9-08	4.00	3.3-08	4.00	5.3-06	3.00	5.4-06	2.99
	8	8.3-01		1.8+00		1.8+00		6.4-02	
	16	2.2-01	1.93	4.6-01	1.99	4.6-01	1.99	8.1-03	2.98
$u^{(3)}_\Delta$	32	5.6-02	1.97	1.1-01	2.00	1.1-01	2.00	1.0-03	2.98
	64	1.4-02	1.98	2.9-02	2.00	2.9-02	2.00	1.3-04	3.00
	128	3.6-03	1.99	7.2-03	2.00	7.2-03	2.00	1.6-05	3.00
	8	1.3+00		8.5+01		8.5+01		4.9+01	
	16	3.4-01	1.91	4.3+01	0.98	4.3+01	0.98	2.5+01	0.97
$u^{(4)}_\Delta$	32	8.5-02	1.98	2.2+01	0.99	2.2+01	0.99	1.3+01	0.99
	64	2.1-02	1.99	1.1+01	0.99	1.1+01	0.99	6.3+00	0.99
	128	5.4-03	2.00	5.5+00	1.00	5.5+00	1.00	3.2+00	1.00

Table 6.11: Errors and orders of convergence for Problem 3 where $u(x) = x^{\frac{15}{2}}$.

	N	τ_i		x_i		∞		λ_i	
u_Δ	8	2.0-04		2.0-04		2.0-04		2.0-04	
	16	3.3-06	5.90	3.3-06	5.93	3.3-06	5.92	3.3-06	5.92
	32	4.7-08	6.11	4.8-08	6.10	6.2-08	5.72	6.2-08	5.72
	64	6.1-10	6.29	6.2-10	6.27	1.3-09	5.59	1.3-09	5.59
	128	5.0-12	6.93	5.0-12	6.94	2.9-11	5.49	2.8-11	5.50
u'_Δ	8	1.2-03		1.2-03		1.2-03		3.3-04	
	16	7.4-05	3.98	8.7-05	3.82	8.7-05	3.84	2.0-05	4.00
	32	4.7-06	3.96	5.6-06	3.97	5.6-06	3.97	1.1-06	4.16
	64	3.0-07	3.98	3.5-07	3.99	3.5-07	3.99	6.5-08	4.13
	128	1.9-08	3.99	2.2-08	4.00	2.2-08	4.00	3.9-09	4.08
u''_Δ	8	8.3-03		1.3-02		5.9-02		5.8-02	
	16	5.6-04	3.90	7.1-04	4.19	7.9-03	2.90	7.9-03	2.90
	32	3.6-05	3.96	4.3-05	4.05	1.0-03	2.96	1.0-03	2.96
	64	2.3-06	3.98	2.6-06	4.01	1.3-04	2.99	1.3-04	2.98
	128	1.4-07	3.99	1.6-07	4.01	1.6-05	3.00	1.6-05	2.99
$u^{(3)}_\Delta$	8	2.3+00		5.5+00		5.5+00		3.5-01	
	16	6.3-01	1.86	1.4+00	2.01	1.4+00	2.01	4.1-02	3.07
	32	1.7-01	1.94	3.4-01	2.00	3.4-01	2.00	5.1-03	3.01
	64	4.2-02	1.97	8.6-02	2.00	8.6-02	2.00	6.4-04	3.00
	128	1.1-02	1.99	2.1-02	2.00	2.1-02	2.00	8.1-05	3.00
$u^{(4)}_\Delta$	8	6.6+00		2.5+02		2.5+02		1.4+02	
	16	1.7+00	2.00	1.3+02	0.96	1.3+02	0.96	7.4+01	0.95
	32	4.2-01	1.98	6.5+01	0.98	6.5+01	0.98	3.7+01	0.98
	64	1.1-01	1.99	3.3+01	0.99	3.3+01	0.99	1.9+01	0.99
	128	2.7-02	1.99	1.6+01	1.00	1.6+01	1.00	9.5+00	0.99

Problem 4

Consider the differential equation

$$Lu = e^x u''(x) + e^x u'(x) - \frac{1}{1+x} u(x) = f(x), \quad x \in [0, 1]$$

with boundary conditions (6.1). The functions $f(x)$, g_1 and g_2 are chosen so that $u(x) = x^{\frac{11}{2}}$ and $u(x) = x^{\frac{13}{2}}$ respectively. Tables 6.12 and 6.13 show the errors and the orders of convergence for these different $u(x)$ respectively.

In Table 6.12 the collocation point error and global error orders of convergence are all about 5, while in Table 6.13 the collocation point error order of convergence is about 6, and global error order of convergence is about 5. In Table 6.13 we also achieve superconvergence for the approximation to $u^{(2)}(x)$ at midpoints and grid points, for the approximation to $u^{(4)}(x)$ at midpoints, and for the approximation to $u^{(3)}(x)$ at the Gauss points. These results indicate that we obtain optimal orders of convergence even with a general operator with variable coefficients.

Table 6.12: Errors and orders of convergence for Problem 4 where $u(x) = x^{\frac{11}{2}}$.

	N	τ_i		x_i		∞		λ_i	
u_Δ	8	3.0-07		3.1-07		2.0-06		2.0-06	
	16	1.1-08	4.82	6.0-09	5.70	6.4-08	4.98	6.3-08	4.98
	32	2.9-10	5.18	2.0-10	4.92	2.0-09	5.02	2.0-09	5.01
	64	7.7-12	5.26	5.9-12	5.06	6.2-11	5.01	6.1-11	5.01
	128	1.9-13	5.37	1.5-13	5.25	1.9-12	5.00	1.9-12	5.00
u'_Δ	8	9.3-05		1.1-04		1.1-04		2.1-05	
	16	5.9-06	3.97	6.9-06	3.95	6.9-06	3.96	1.3-06	4.06
	32	3.7-07	3.99	4.3-07	4.00	4.3-07	4.00	7.6-08	4.07
	64	2.3-08	3.99	2.7-08	4.00	2.7-08	4.00	4.6-09	4.04
	128	1.5-09	4.00	1.7-09	4.00	1.7-09	4.00	2.8-10	4.02
u''_Δ	8	3.5-04		2.1-03		4.9-03		4.9-03	
	16	3.1-05	3.50	1.8-04	3.51	6.3-04	2.97	6.3-04	2.97
	32	2.7-06	3.50	1.6-05	3.50	7.9-05	2.99	7.9-05	2.99
	64	2.4-07	3.50	1.4-06	3.50	9.8-06	3.00	9.9-06	3.00
	128	2.1-08	3.50	1.2-07	3.50	1.2-06	3.00	1.2-06	3.00
$u^{(3)}_\Delta$	8	2.0-01		4.3-01		4.3-01		3.9-02	
	16	5.2-02	1.97	1.1-01	2.01	1.1-01	2.01	6.9-03	2.51
	32	1.3-02	1.99	2.6-02	2.00	2.6-02	2.00	1.2-03	2.50
	64	3.3-03	1.99	6.6-03	2.00	6.6-03	2.00	2.2-04	2.50
	128	8.2-04	2.00	1.7-03	2.00	1.7-03	2.00	3.8-05	2.50
$u^{(4)}_\Delta$	8	5.8-01		2.0+01		2.0+01		1.2+01	
	16	2.1-01	1.51	1.0+01	0.99	1.0+01	0.99	5.8+00	0.99
	32	7.3-02	1.50	5.1+00	1.00	5.1+00	1.00	2.9+00	1.00
	64	2.6-02	1.50	2.5+00	1.00	2.5+00	1.00	1.5+00	1.00
	128	9.1-03	1.50	1.3+00	1.00	1.3+00	1.00	7.3-01	1.00

Table 6.13: Errors and orders of convergence for Problem 4 where $u(x) = x^{\frac{13}{2}}$.

	N	τ_i		x_i		∞		λ_i	
u_Δ	8	2.7-06		3.2-07		9.8-06		9.4-06	
	16	4.0-08	6.09	1.5-08	4.36	2.9-07	5.09	2.8-07	5.07
	32	6.2-10	5.99	3.4-10	5.52	8.7-09	5.04	8.6-09	5.03
	64	9.8-12	5.98	6.2-12	5.75	2.7-10	5.02	2.7-10	5.01
	128	1.5-13	5.99	9.5-14	6.04	8.4-12	5.01	8.3-12	5.01
u'_Δ	8	3.8-04		4.9-04		4.9-04		1.3-04	
	16	2.5-05	3.93	3.0-05	4.03	3.0-05	4.03	6.4-06	4.32
	32	1.6-06	3.97	1.9-06	4.00	1.9-06	4.00	3.6-07	4.17
	64	1.0-07	3.98	1.2-07	4.00	1.2-07	4.00	2.1-08	4.10
	128	6.3-09	3.99	7.3-09	4.00	7.3-09	4.00	1.3-09	4.05
u''_Δ	8	1.9-03		2.0-03		2.1-02		2.1-02	
	16	1.2-04	4.01	1.3-04	3.96	2.7-03	2.97	2.7-03	2.97
	32	7.3-06	3.99	8.4-06	3.97	3.4-04	2.98	3.4-04	2.98
	64	4.6-07	4.00	5.3-07	4.00	4.2-05	3.00	4.3-05	2.99
	128	2.9-08	4.00	3.3-08	4.00	5.3-06	3.00	5.4-06	2.99
$u^{(3)}_\Delta$	8	8.3-01		1.8+00		1.8+00		6.4-02	
	16	2.2-01	1.93	4.6-01	1.99	4.6-01	1.99	8.1-03	2.98
	32	5.6-02	1.97	1.1-01	2.00	1.1-01	2.00	1.0-03	2.98
	64	1.4-02	1.98	2.9-02	2.00	2.9-02	2.00	1.3-04	3.00
	128	3.6-03	1.99	7.2-03	2.00	7.2-03	2.00	1.6-05	3.00
$u^{(4)}_\Delta$	8	1.3+00		8.5+01		8.5+01		4.9+01	
	16	3.4-01	1.91	4.3+01	0.98	4.3+01	0.98	2.5+01	0.97
	32	8.5-02	1.98	2.2+01	0.99	2.2+01	0.99	1.3+01	0.99
	64	2.1-02	1.99	1.1+01	0.99	1.1+01	0.99	6.3+00	0.99
	128	5.4-03	2.00	5.5+00	1.00	5.5+00	1.00	3.2+00	1.00

6.2 Numerical results with extra boundary conditions $Lu = f$ at x_0 and x_N

We apply the two-step quartic spline collocation method to Problem 1 with the extra boundary conditions (3.52) instead of (3.3).

The functions $f(x)$, g_1 and g_2 are chosen so that $u(x) = x^6$. The errors and the orders of convergence are presented in Table 6.14.

From Table 6.14, it is clear that we obtain the same order of convergence as that in Table 6.2, that is, sixth order of convergence at mid points, fifth global order of convergence. The round-off effects occur at grid points like Table 6.2. Also like Table 6.2, we achieve superconvergence for the approximation to $u^{(2)}(x)$ at midpoints and grid points, for the approximation to $u^{(4)}(x)$ at midpoints, and for the approximation to $u^{(3)}(x)$ at the Gauss points. It implies that the quartic spline collocation methods with extra boundary conditions (3.3) and (3.52) achieve the same order of convergence for this problem set.

Table 6.14: Errors and orders of convergence for Problem 1 where $u(x) = x^6$.

	N	τ_i		x_i		∞		λ_i	
u_Δ	8	8.9-07		1.2-15		4.8-06		4.6-06	
	16	1.4-08	6.00	8.2-15	-2.75	1.4-07	5.05	1.4-07	5.04
	32	2.2-10	6.00	2.4-15	1.77	4.4-09	5.03	4.4-09	5.02
	64	3.4-12	6.00	1.8-14	-2.88	1.4-10	5.01	1.4-10	5.01
	128	5.2-14	6.03	4.5-14	-1.33	4.3-12	5.01	4.2-12	5.01
u'_Δ	8	2.0-04		2.4-04		2.4-04		5.7-05	
	16	1.3-05	3.95	1.5-05	4.00	1.5-05	4.00	3.1-06	4.23
	32	8.2-07	3.98	9.5-07	4.00	9.5-07	4.00	1.8-07	4.13
	64	5.2-08	3.99	6.0-08	4.00	6.0-08	4.00	1.0-08	4.07
	128	3.2-09	3.99	3.7-09	4.00	3.7-09	4.00	6.4-10	4.04
u''_Δ	8	6.4-04		7.3-04		1.1-02		1.1-02	
	16	4.0-05	4.00	4.6-05	4.00	1.4-03	2.97	1.4-03	2.97
	32	2.5-06	4.00	2.9-06	4.00	1.7-04	2.99	1.7-04	2.99
	64	1.6-07	4.00	1.8-07	4.00	2.2-05	3.00	2.2-05	2.99
	128	9.8-09	4.00	1.1-08	4.00	2.7-06	3.00	2.7-06	3.00
$u^{(3)}_\Delta$	8	4.4-01		9.4-01		9.4-01		2.3-02	
	16	1.1-01	1.95	2.3-01	2.00	2.3-01	2.00	2.8-03	3.00
	32	2.9-02	1.98	5.9-02	2.00	5.9-02	2.00	3.5-04	3.00
	64	7.3-03	1.99	1.5-02	2.00	1.5-02	2.00	4.4-05	3.00
	128	1.8-03	1.99	3.7-03	2.00	3.7-03	2.00	5.5-06	3.00
$u^{(4)}_\Delta$	8	4.7-01		4.4+01		4.4+01		2.5+01	
	16	1.2-01	2.00	2.2+01	0.98	2.2+01	0.98	1.3+01	0.98
	32	2.9-02	2.00	1.1+01	0.99	1.1+01	0.99	6.5+00	0.99
	64	7.3-03	2.00	5.6+00	1.00	5.6+00	1.00	3.2+00	1.00
	128	1.8-03	2.00	2.8+00	1.00	2.8+00	1.00	1.6+00	1.00

We also apply the quartic spline collocation method to Problem 3, and we use the extra boundary conditions (3.52) instead of (3.3). The functions $f(x)$, g_1 and g_2 are chosen so that $u(x) = e^x \sin(\pi x)$ and $u(x) = x^{\frac{13}{2}}$ respectively. The errors and the orders of convergence is presented in Tables 6.15 and 6.16.

Comparing Tables 6.15 and 6.16 with Tables 6.8 and 6.10 respectively, we see that the quartic spline collocation methods with extra boundary conditions (3.52) have the same order of convergence for these problem sets as extra boundary conditions (3.3). In Table 6.15 the collocation point error and global error order of convergence are all about 6. In Table 6.16 we obtain sixth order of collocation point error and fifth order of global error.

In general, the two optimal quartic spline collocation methods with extra boundary conditions (3.3) and (3.52) seem to be equivalent in orders of convergence and accuracy, with the method using (3.3) being only slightly more accurate in the approximation of $u(x)$.

Table 6.15: Errors and orders of convergence for Problem 3 where $u(x) = e^x \sin(\pi x)$.

	N	τ_i		x_i		∞		λ_i	
u_Δ	8	1.3-04		1.3-04		1.3-04		1.3-04	
	16	2.4-06	5.75	2.5-06	5.73	2.6-06	5.69	2.6-06	5.68
	32	3.7-08	6.06	3.7-08	6.05	4.0-08	5.99	4.0-08	5.99
	64	5.3-10	6.11	5.4-10	6.10	6.5-10	5.97	6.5-10	5.97
	128	9.1-12	5.87	9.3-12	5.86	1.2-11	5.70	1.2-11	5.71
u'_Δ	8	1.7-04		2.8-04		2.8-04		1.5-04	
	16	1.3-05	3.70	1.4-05	4.30	1.4-05	4.30	3.5-06	5.44
	32	7.8-07	4.02	8.8-07	4.00	8.8-07	4.01	1.6-07	4.51
	64	4.9-08	4.01	5.5-08	3.99	5.5-08	3.99	9.3-09	4.07
	128	3.0-09	4.00	3.5-09	4.00	3.4-09	4.00	5.7-10	4.01
u''_Δ	8	2.5-03		3.4-03		1.1-02		1.1-02	
	16	1.8-04	3.79	2.3-04	3.91	1.3-03	3.03	1.3-03	3.03
	32	1.1-05	3.97	1.4-05	4.06	1.6-04	3.01	1.6-04	3.01
	64	7.2-07	3.99	8.4-07	4.03	2.0-05	3.00	2.0-05	3.00
	128	4.5-08	3.99	5.2-08	4.01	2.6-06	3.00	2.6-06	3.00
$u^{(3)}_\Delta$	8	4.4-01		8.7-01		8.7-01		9.6-02	
	16	1.1-01	2.01	2.2-01	2.00	2.1-01	2.02	1.3-02	2.85
	32	2.7-02	2.00	5.4-02	2.00	5.3-02	2.02	1.7-03	3.02
	64	6.8-03	2.00	1.4-02	2.00	1.3-02	2.04	2.1-04	3.01
	128	1.7-03	2.00	3.4-03	2.00	3.3-03	1.95	2.6-05	3.00
$u^{(4)}_\Delta$	8	1.9+00		4.1+01		4.1+01		2.4+01	
	16	5.3-01	1.82	2.1+01	0.98	2.1+01	1.00	1.2+01	1.01
	32	1.3-01	1.98	1.0+01	1.00	1.0+01	1.01	6.0+00	1.00
	64	3.4-02	1.99	5.2+00	1.00	5.1+00	1.00	3.0+00	1.00
	128	8.5-03	1.99	2.6+00	1.00	2.6+00	0.99	1.5+00	1.00

Table 6.16: Errors and orders of convergence for Problem 3 where $u(x) = x^{\frac{13}{2}}$.

	N	τ_i		x_i		∞		λ_i	
	8	2.5-05		2.7-05		2.7-05		2.6-05	
	16	4.0-07	5.97	4.2-07	5.98	4.2-07	5.98	4.1-07	5.98
u_Δ	32	7.3-09	5.76	7.6-09	5.79	8.7-09	5.60	8.6-09	5.58
	64	1.5-10	5.65	1.5-10	5.68	2.9-10	4.92	2.8-10	4.93
	128	3.0-12	5.59	3.1-12	5.61	9.3-12	4.94	9.2-12	4.94
	8	3.9-04		4.7-04		4.7-04		1.2-04	
	16	2.5-05	3.94	3.0-05	3.98	3.0-05	3.99	6.2-06	4.26
u'_Δ	32	1.6-06	3.97	1.9-06	3.99	1.9-06	3.99	3.5-07	4.15
	64	1.0-07	3.99	1.2-07	4.00	1.2-07	4.00	2.1-08	4.08
	128	6.3-09	3.99	7.3-09	4.00	7.3-09	4.00	1.3-09	4.05
	8	1.9-03		2.2-03		2.1-02		2.1-02	
	16	1.2-04	4.00	1.4-04	4.05	2.7-03	2.96	2.7-03	2.96
u''_Δ	32	7.3-06	4.00	8.4-06	4.01	3.4-04	2.98	3.4-04	2.98
	64	4.6-07	4.00	5.2-07	4.00	4.2-05	3.00	4.3-05	2.99
	128	2.9-08	4.00	3.3-08	4.00	5.3-06	3.00	5.4-06	2.99
	8	8.3-01		1.8+00		1.8+00		6.7-02	
	16	2.2-01	1.93	4.6-01	2.00	4.6-01	2.00	8.3-03	3.03
$u^{(3)}_\Delta$	32	5.6-02	1.96	1.1-01	2.00	1.1-01	2.00	1.0-03	3.00
	64	1.4-02	1.98	2.9-02	2.00	2.9-02	2.00	1.3-04	3.00
	128	3.6-03	1.99	7.2-03	2.00	7.2-03	2.00	1.6-05	3.00
	8	1.4+00		8.5+01		8.5+01		4.9+01	
	16	3.4-01	2.00	4.3+01	0.98	4.3+01	0.98	2.5+01	0.97
$u^{(4)}_\Delta$	32	8.5-02	1.99	2.2+01	0.99	2.2+01	0.99	1.3+01	0.99
	64	2.1-02	2.00	1.1+01	0.99	1.1+01	0.99	6.3+00	0.99
	128	5.4-03	2.00	5.5+00	1.00	5.5+00	1.00	3.2+00	1.00

6.3 Numerical results with other extra boundary conditions

Consider the BVP problem

$$Lu = u''(x) + u'(x) + u(x) = f(x), \quad x \in [0, 1],$$

with the boundary conditions (6.1). We choose $f(x)$, g_1 and g_2 so that the exact solution is $u(x) = x^6$. We apply the non-optimal quartic spline collocation methods, specifically, method 1, method 2 and method 3, outlined in Section 3.5.

Table 6.17 presents the errors and convergence of the last step of method 1. In Tables 6.18 and 6.19, we present the errors and convergence of each of steps of method 2 and method 3, respectively.

From Table 6.17, it is clear that the order of the convergence of method 1 is close to 2. Table 6.18 shows that method 2, like method 1, has order of convergence close to 2. From Table 6.19, we can see that method 3 is third order.

Table 6.17: Errors and corresponding orders of convergence for method 1.

	N	τ_i		x_i		∞		λ_i	
	8	4.7e-003		5.0e-003		5.1e-003		5.1e-003	
	16	1.3e-003	1.81	1.4e-003	1.82	1.5e-003	1.82	1.5e-003	1.82
u_Δ	32	3.5e-004	1.93	3.7e-004	1.93	3.8e-004	1.93	3.8e-004	1.93
	64	9.0e-005	1.97	9.5e-005	1.96	9.8e-005	1.97	9.8e-005	1.97
	128	2.3e-005	1.98	2.4e-005	1.98	2.5e-005	1.98	2.5e-005	1.98

Table 6.18: Errors and corresponding orders of convergence for method 2.

	N	τ_i		x_i		∞		λ_i	
w	8	1.9e-002		2.0e-002		2.1e-002		2.0e-002	
	16	5.0e-003	1.88	5.4e-003	1.90	5.5e-003	1.90	5.5e-003	1.90
	32	1.3e-003	1.95	1.4e-003	1.95	1.4e-003	1.95	1.4e-003	1.95
	64	3.3e-004	1.98	3.5e-004	1.98	3.6e-004	1.98	3.6e-004	1.98
	128	8.4e-005	1.99	8.9e-005	1.99	9.1e-005	1.99	9.1e-005	1.99
v	8	4.5e-003		4.8e-003		5.0e-003		4.9e-003	
	16	1.3e-003	1.79	1.4e-003	1.80	1.4e-003	1.80	1.4e-003	1.80
	32	3.5e-004	1.91	3.7e-004	1.91	3.8e-004	1.91	3.8e-004	1.91
	64	8.9e-005	1.96	9.5e-005	1.96	9.7e-005	1.96	9.7e-005	1.96
	128	2.3e-005	1.98	2.4e-005	1.98	2.5e-005	1.98	2.5e-005	1.98
u_Δ	8	1.2e-003		1.3e-003		1.4e-003		1.4e-003	
	16	3.5e-004	1.84	3.7e-004	1.85	3.8e-004	1.85	3.8e-004	1.85
	32	9.3e-005	1.89	9.9e-005	1.89	1.0e-004	1.89	1.0e-004	1.89
	64	2.4e-005	1.95	2.6e-005	1.95	2.6e-005	1.95	2.6e-005	1.95
	128	6.2e-006	1.97	6.5e-006	1.97	6.7e-006	1.97	6.7e-006	1.97

Table 6.19: Errors and corresponding orders of convergence for method 3.

	N	τ_i		x_i		∞		λ_i	
w	8	3.3e-003		3.5e-003		3.6e-003		3.6e-003	
	16	4.5e-004	2.87	4.8e-004	2.89	4.9e-004	2.89	4.9e-004	2.89
	32	5.8e-005	2.95	6.1e-005	2.95	6.3e-005	2.95	6.3e-005	2.95
	64	7.4e-006	2.98	7.8e-006	2.98	8.0e-006	2.98	8.0e-006	2.98
	128	9.3e-007	2.99	9.8e-007	2.99	1.0e-006	2.99	1.0e-006	2.99
v	8	3.4e-004		3.6e-004		3.7e-004		3.7e-004	
	16	5.3e-005	2.70	5.6e-005	2.69	5.7e-005	2.68	5.7e-005	2.68
	32	7.0e-006	2.93	7.4e-006	2.92	7.6e-006	2.92	7.5e-006	2.92
	64	8.9e-007	2.97	9.4e-007	2.97	9.7e-007	2.97	9.6e-007	2.97
	128	1.1e-007	2.98	1.2e-007	2.98	1.2e-007	2.98	1.2e-007	2.98
u_Δ	8	9.7e-005		9.8e-005		1.1e-004		1.1e-004	
	16	6.8e-006	3.84	7.1e-006	3.79	7.5e-006	3.84	7.4e-006	3.84
	32	8.4e-007	3.02	8.9e-007	3.00	9.2e-007	3.03	9.1e-007	3.03
	64	1.1e-007	2.96	1.1e-007	2.95	1.2e-007	2.96	1.2e-007	2.96
	128	1.4e-008	2.98	1.5e-008	2.98	1.5e-008	2.98	1.5e-008	2.98

6.4 Numerical results for layer problems

We apply the quartic spline collocation method described in Chapter 5 to the layer problems with extra boundary conditions (3.3) and (3.52) respectively. Problems 5, 6, 7 and 8 are from [13]. To compute the global errors for Problem 5 to 8, 5237 global points are used. Besides the errors and corresponding orders of convergence, the tables in this section also have more specific information about the layer area. The data in column “Layer” are the indices of grid points where layer starts and ends as determined by (5.9). If the layer identified has less than 10 subintervals, for example, the indices are $[1, 9]$, we will not refine this layer by the quartic spline collocation method as we consider that the results may be unreliable, but we proceed to double N, and resolve the problem. The number of subintervals in the layer area, N', in column $N + N'$, is determined by (5.11), (5.13) and (5.14). If we use quartic spline collocation methods described in chapter 3 instead of the grid refinement technique, the number of collocation points needed to achieve comparable results with the grid refinement technique would be much larger than $N + N'$. In the tables, the absolute errors for high derivatives are very large. However, the relative errors for high derivatives are not large. We can see it from the quantities $\| u^{(3)} \|_\infty$ and $\| u^{(4)} \|_\infty$ listed for each problem.

Problem 5

Consider the differential equation

$$Lu = \epsilon u''(x) + 2xu'(x) = f(x), \quad x \in [-1, 1]$$

with the boundary conditions

$$u(-1) = -1, \quad u(1) = 1.$$

The function $f(x)$ is chosen so that equation (5.1) is the solution $u(x)$. Figures 5.2 and 5.3 show the plot of $u(x)$ and $u^{(5)}(x)$ respectively. We choose $\epsilon = 10^{-4}$ when applying the quartic spline collocation method. Tables 6.20 and 6.21 show the errors and orders of convergence with extra boundary conditions (3.3) and (3.52) at midpoints, grid points and global points respectively. Note that $\| u^{(3)} \|_\infty = 2.3e + 06$ and $\| u^{(4)} \|_\infty = 4.4e + 08$, so the relative errors of the high derivatives are not large. From Figure 6.1 we can see the decrease of error at grid points with respect to $N + N'$.

Table 6.20: Errors and orders of convergence for Problem 5 with extra boundary conditions (3.3)

	N	Layer	$N+N'$	τ_i		x_i		∞	
u_Δ	256	[119, 139]	298	2.6-04		9.6-05		2.0-04	
	512	[244, 270]	884	2.4-10	12.75	2.4-10	11.85	2.5-10	12.50
	1024	[489, 537]	1606	7.4-12	5.84	7.4-12	5.84	8.0-12	5.74
	2048	[978,1072]	3022	2.3-12	1.87	2.3-12	1.87	2.3-12	1.98
	4096	[1958,2140]	5180	7.2-13	2.14	7.2-13	2.14	7.2-13	2.14
u'_Δ	256	[119, 139]	298	4.7-02		9.0-02		9.1-02	
	512	[244, 270]	884	8.9-07	10.00	1.1-06	10.43	1.1-06	10.44
	1024	[489, 537]	1606	1.1-07	3.50	1.3-07	3.57	1.3-07	3.57
	2048	[978,1072]	3022	1.3-08	3.40	1.5-08	3.38	1.5-08	3.38
	4096	[1958,2140]	5180	7.4-09	1.01	8.5-09	1.05	8.5-09	1.05
u''_Δ	256	[119, 139]	298	3.8+01		4.3+01		6.1+01	
	512	[244, 270]	884	7.5-04	9.97	8.5-04	9.95	2.2-02	7.28
	1024	[489, 537]	1606	9.1-05	3.54	1.0-04	3.53	4.5-03	2.66
	2048	[978,1072]	3022	1.1-05	3.39	1.2-05	3.39	9.1-04	2.55
	4096	[1958,2140]	5180	6.1-06	1.03	7.0-06	1.03	6.0-04	0.77
$u^{(3)}_\Delta$	256	[119, 139]	298	7.6+04		1.6+05		1.6+05	
	512	[244, 270]	884	4.2+02	4.77	8.4+02	4.83	8.4+02	4.83
	1024	[489, 537]	1606	1.5+02	1.77	2.9+02	1.77	2.9+02	1.77
	2048	[978,1072]	3022	5.0+01	1.70	1.0+02	1.70	1.0+02	1.70
	4096	[1958,2140]	5180	3.8+01	0.52	7.6+01	0.52	7.6+01	0.52
$u^{(4)}_\Delta$	256	[119, 139]	298	1.9+07		2.5+08		2.5+08	
	512	[244, 270]	884	1.1+05	4.72	1.8+07	2.39	1.8+07	2.39
	1024	[489, 537]	1606	4.0+04	1.77	1.1+07	0.88	1.1+07	0.88
	2048	[978,1072]	3022	1.4+04	1.69	6.4+06	0.85	6.4+06	0.85
	4096	[1958,2140]	5180	1.0+04	0.51	5.6+06	0.26	5.6+06	0.26

Table 6.21: Errors and orders of convergence for Problem 5 with extra boundary conditions (3.52)

	N	Layer	$N+N'$	τ_i		x_i		∞	
	256	[119, 139]	298	2.3-04		1.2-04		2.0-04	
	512	[244, 270]	884	2.4-10	12.67	2.4-10	12.03	2.5-10	12.52
u_Δ	1024	[489, 537]	1606	8.5-12	5.60	8.5-12	5.60	9.0-12	5.53
	2048	[978,1072]	3022	6.1-12	0.52	6.1-12	0.52	6.2-12	0.61
	4096	[1958,2140]	5180	7.9-12	-0.48	7.9-12	-0.48	7.9-12	-0.47
	256	[119, 139]	298	4.5-02		9.0-02		9.1-02	
	512	[244, 270]	884	8.9-07	9.95	1.1-06	10.43	1.1-06	10.44
u'_Δ	1024	[489, 537]	1606	1.1-07	3.50	1.3-07	3.56	1.3-07	3.56
	2048	[978,1072]	3022	1.3-08	3.35	1.5-08	3.43	1.5-08	3.43
	4096	[1958,2140]	5180	7.0-09	1.19	8.9-09	0.91	8.9-09	0.91
	256	[119, 139]	298	3.8+01		4.2+01		6.1+01	
	512	[244, 270]	884	7.5-04	9.97	8.5-04	9.95	2.2-02	7.28
u''_Δ	1024	[489, 537]	1606	9.1-05	3.54	1.0-04	3.53	4.5-03	2.66
	2048	[978,1072]	3022	1.1-05	3.39	1.2-05	3.39	9.1-04	2.55
	4096	[1958,2140]	5180	6.1-06	1.02	6.9-06	1.05	6.0-04	0.77
	256	[119, 139]	298	7.6+04		1.6+05		1.6+05	
	512	[244, 270]	884	4.2+02	4.77	8.4+02	4.83	8.4+02	4.83
$u^{(3)}_\Delta$	1024	[489, 537]	1606	1.5+02	1.77	2.9+02	1.77	2.9+02	1.77
	2048	[978,1072]	3022	5.0+01	1.70	1.0+02	1.70	1.0+02	1.70
	4096	[1958,2140]	5180	3.8+01	0.52	7.6+01	0.52	7.6+01	0.52
	256	[119, 139]	298	1.9+07		2.5+08		2.5+08	
	512	[244, 270]	884	1.1+05	4.72	1.8+07	2.39	1.8+07	2.39
$u^{(4)}_\Delta$	1024	[489, 537]	1606	4.0+04	1.77	1.1+07	0.88	1.1+07	0.88
	2048	[978,1072]	3022	1.4+04	1.70	6.4+06	0.85	6.4+06	0.85
	4096	[1958,2140]	5180	1.0+04	0.52	5.6+06	0.26	5.6+06	0.26

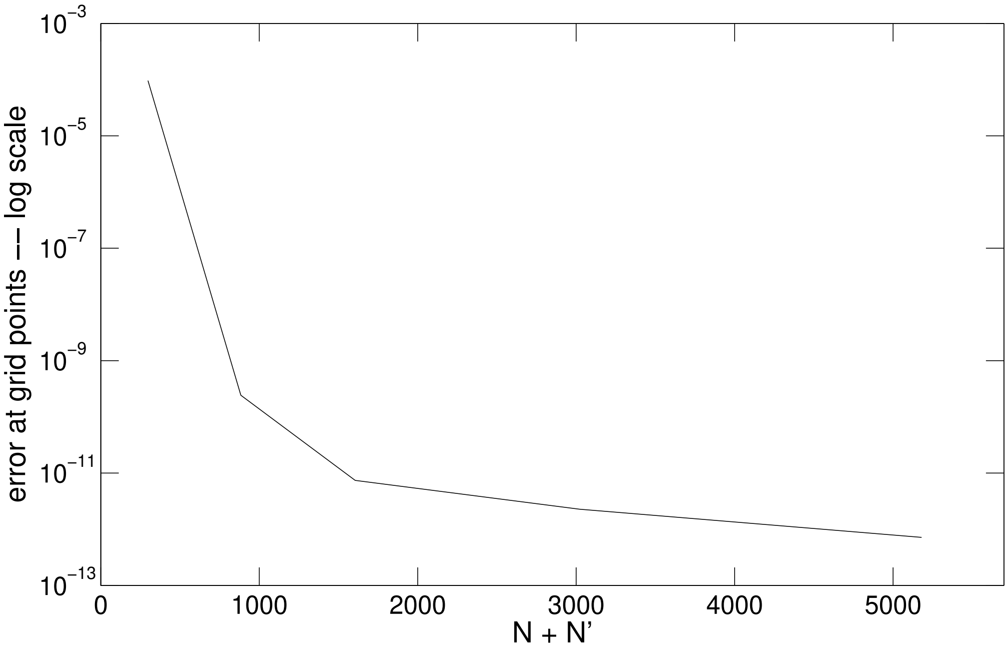

Figure 6.1: Error at grid points with respect to $N + N'$ for Problem 5

Problem 6

Consider the differential equation

$$Lu = \epsilon u''(x) + xu'(x) = f(x), \quad x \in [-1, 1]$$

with the boundary conditions

$$u(-1) = -2, \;\; u(1) = 0.$$

The function $f(x)$ is chosen so that

$$u(x) = \cos(\pi x) + \frac{\text{erf}(x/\sqrt{2\epsilon})}{\text{erf}(1/\sqrt{2\epsilon})}. \tag{6.12}$$

Figures 6.2 and 6.3 show the plot of $u(x)$ and $u^{(5)}(x)$ respectively. From Figure 6.3 it is clear that $u(x)$ of (6.12) has a layer in the middle of the domain. We choose $\epsilon = 10^{-4}$ when applying the quartic spline collocation method. Tables 6.22 and 6.23 show the errors and orders of convergence with extra boundary conditions (3.3) and (3.52) at midpoints, grid points and global points respectively. Note that $\| \ u^{(3)} \ \|_\infty = 8.0e + 05$ and $\| \ u^{(4)} \ \|_\infty = 1.1e + 08$, so the relative errors of the high derivatives are not large. Figure 6.4 shows the decrease of error at grid points with respect to $N + N'$.

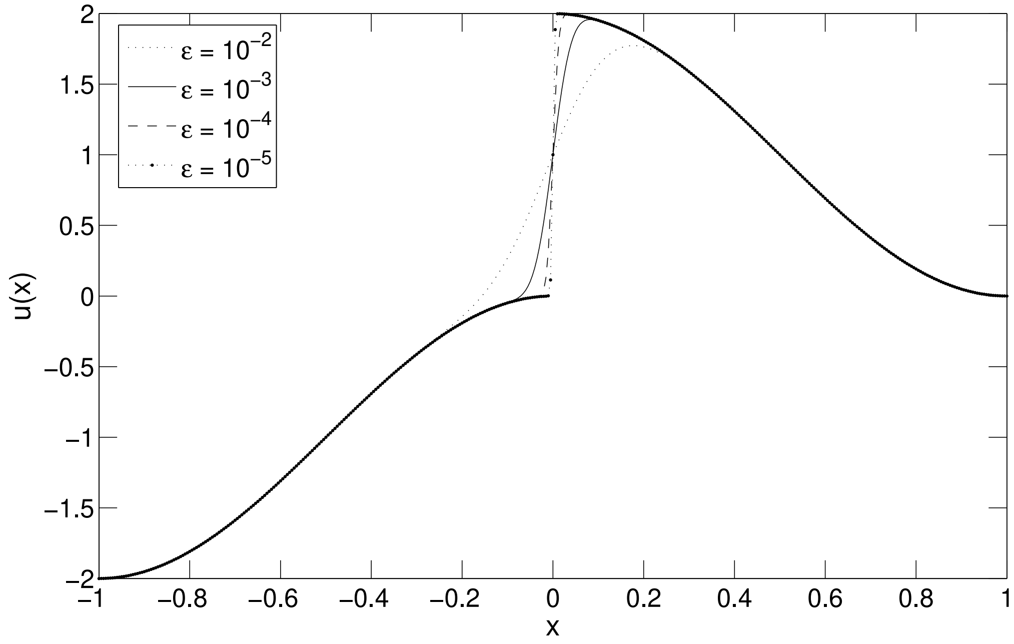

Figure 6.2: Plot of $u(x)$ of (6.12) for some values of ϵ

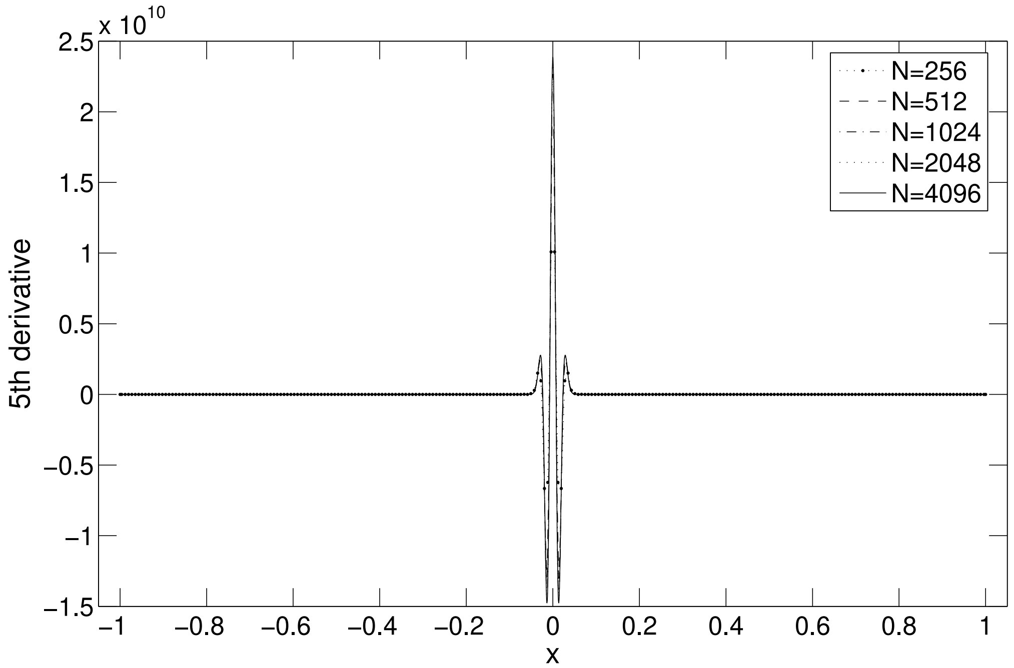

Figure 6.3: Plot of $u^{(5)}(x)$ corresponding to (6.12) where $\epsilon = 10^{-4}$

Table 6.22: Errors and orders of convergence for Problem 6 with extra boundary conditions (3.3)

	N	Layer	$N+N'$	τ_i		x_i		∞	
u_Δ	256	[119, 139]	321	1.4-06		1.2-06		1.3-06	
	512	[242, 272]	665	2.6-09	8.61	2.7-09	8.35	3.4-09	8.13
	1024	[483, 543]	1339	3.7-11	6.08	3.9-11	6.08	5.7-11	5.82
	2048	[966,1084]	2594	1.6-12	4.76	1.6-12	4.78	2.8-12	4.58
	4096	[1931,2167]	5090	2.8-12	-0.85	2.8-12	-0.81	2.8-12	-0.03
u'_Δ	256	[119, 139]	321	1.3-03		9.0-04		1.3-03	
	512	[242, 272]	665	1.0-05	6.68	1.2-05	5.99	1.1-05	6.49
	1024	[483, 543]	1339	5.6-07	4.13	6.4-07	4.13	6.3-07	4.14
	2048	[966,1084]	2594	5.8-08	3.43	6.6-08	3.43	6.6-08	3.42
	4096	[1931,2167]	5090	5.1-09	3.59	6.1-09	3.53	6.1-09	3.53
u''_Δ	256	[119, 139]	321	5.2-01		7.1-01		2.8+00	
	512	[242, 272]	665	5.8-03	6.18	6.6-03	6.41	8.7-02	4.76
	1024	[483, 543]	1339	3.2-04	4.12	3.7-04	4.13	9.9-03	3.10
	2048	[966,1084]	2594	3.3-05	3.43	3.8-05	3.43	1.8-03	2.57
	4096	[1931,2167]	5090	3.1-06	3.55	3.5-06	3.55	3.0-04	2.67
$u^{(3)}_\Delta$	256	[119, 139]	321	5.9+03		1.1+04		1.1+04	
	512	[242, 272]	665	5.9+02	3.18	1.2+03	3.11	1.1+03	3.11
	1024	[483, 543]	1339	1.4+02	2.07	2.8+02	2.06	2.7+02	2.07
	2048	[966,1084]	2594	4.4+01	1.72	8.9+01	1.71	8.9+01	1.67
	4096	[1931,2167]	5090	1.3+01	1.78	2.7+01	1.78	2.7+01	1.78
$u^{(4)}_\Delta$	256	[119, 139]	321	1.1+06		2.8+07		2.8+07	
	512	[242, 272]	665	1.1+05	3.10	9.2+06	1.56	9.1+06	1.56
	1024	[483, 543]	1339	2.7+04	2.07	4.5+06	1.03	4.4+06	1.04
	2048	[966,1084]	2594	8.6+03	1.71	2.5+06	0.86	2.5+06	0.83
	4096	[1931,2167]	5090	2.6+03	1.77	1.4+06	0.89	1.4+06	0.89

Table 6.23: Errors and orders of convergence for Problem 6 with extra boundary conditions (3.52)

	N	Layer	$N+N'$	τ_i		x_i		∞	
u_Δ	256	[119, 139]	321	1.4-06		1.2-06		1.3-06	
	512	[242, 272]	665	2.6-09	8.61	2.7-09	8.36	3.4-09	8.13
	1024	[483, 543]	1339	3.7-11	6.09	3.8-11	6.10	5.7-11	5.83
	2048	[966,1084]	2594	8.3-13	5.73	8.3-13	5.79	2.3-12	4.86
	4096	[1931,2167]	5090	2.4-12	-1.58	2.4-12	-1.58	2.4-12	-0.09
u'_Δ	256	[119, 139]	321	1.3-03		9.0-04		1.3-03	
	512	[242, 272]	665	1.0-05	6.68	1.2-05	5.99	1.1-05	6.49
	1024	[483, 543]	1339	5.6-07	4.13	6.4-07	4.13	6.3-07	4.14
	2048	[966,1084]	2594	5.8-08	3.43	6.6-08	3.43	6.6-08	3.42
	4096	[1931,2167]	5090	5.4-09	3.53	5.9-09	3.58	5.9-09	3.58
u''_Δ	256	[119, 139]	321	5.2-01		7.1-01		2.8+00	
	512	[242, 272]	665	5.8-03	6.18	6.6-03	6.41	8.7-02	4.76
	1024	[483, 543]	1339	3.2-04	4.12	3.7-04	4.13	9.9-03	3.10
	2048	[966,1084]	2594	3.3-05	3.43	3.8-05	3.43	1.8-03	2.57
	4096	[1931,2167]	5090	3.1-06	3.55	3.5-06	3.55	3.0-04	2.67
$u^{(3)}_\Delta$	256	[119, 139]	321	5.9+03		1.1+04		1.1+04	
	512	[242, 272]	665	5.9+02	3.18	1.2+03	3.11	1.1+03	3.11
	1024	[483, 543]	1339	1.4+02	2.07	2.8+02	2.06	2.7+02	2.07
	2048	[966,1084]	2594	4.4+01	1.72	8.9+01	1.71	8.9+01	1.67
	4096	[1931,2167]	5090	1.3+01	1.78	2.7+01	1.78	2.7+01	1.78
$u^{(4)}_\Delta$	256	[119, 139]	321	1.1+06		2.8+07		2.8+07	
	512	[242, 272]	665	1.1+05	3.10	9.2+06	1.56	9.1+06	1.56
	1024	[483, 543]	1339	2.7+04	2.07	4.5+06	1.03	4.4+06	1.04
	2048	[966,1084]	2594	8.6+03	1.71	2.5+06	0.86	2.5+06	0.83
	4096	[1931,2167]	5090	2.6+03	1.77	1.4+06	0.89	1.4+06	0.89

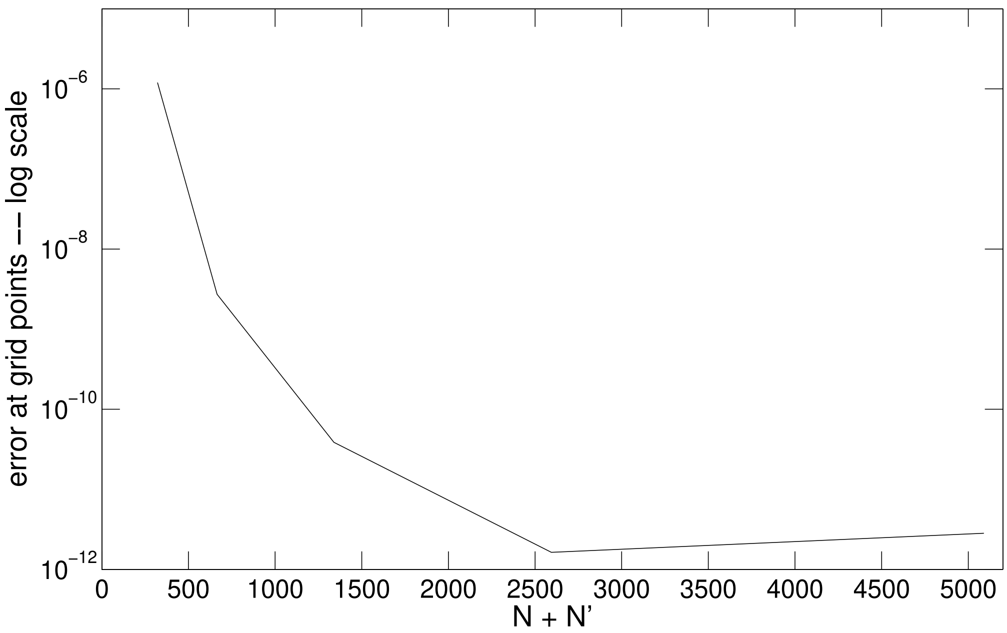

Figure 6.4: Error at grid points with respect to $N + N'$ for Problem 6

Problem 7

Consider the differential equation

$$Lu = \epsilon u''(x) - xu'(x) - \frac{u(x)}{2} = f(x), \quad x \in [-1, 1]$$

with boundary conditions (6.1). The functions $f(x)$, g_1 and g_2 are chosen so that (5.2) is the solution. Figures 5.4 and 5.5 show the plot of $u(x)$ and $u^{(5)}(x)$ respectively. We choose $\epsilon = 10^{-4}$ when applying the quartic spline collocation method. Figure 6.5 shows the decrease of error at grid points with respect to $N + N'$. The errors and orders of convergence with extra boundary conditions (3.3) and (3.52) at midpoints, grid points and global points respectively are listed in Tables 6.24 and 6.25. Note that $\parallel u^{(3)} \parallel_\infty = 2.0e + 12$ and $\parallel u^{(4)} \parallel_\infty = 2.0e + 16$, so the relative errors of the high derivatives are not large.

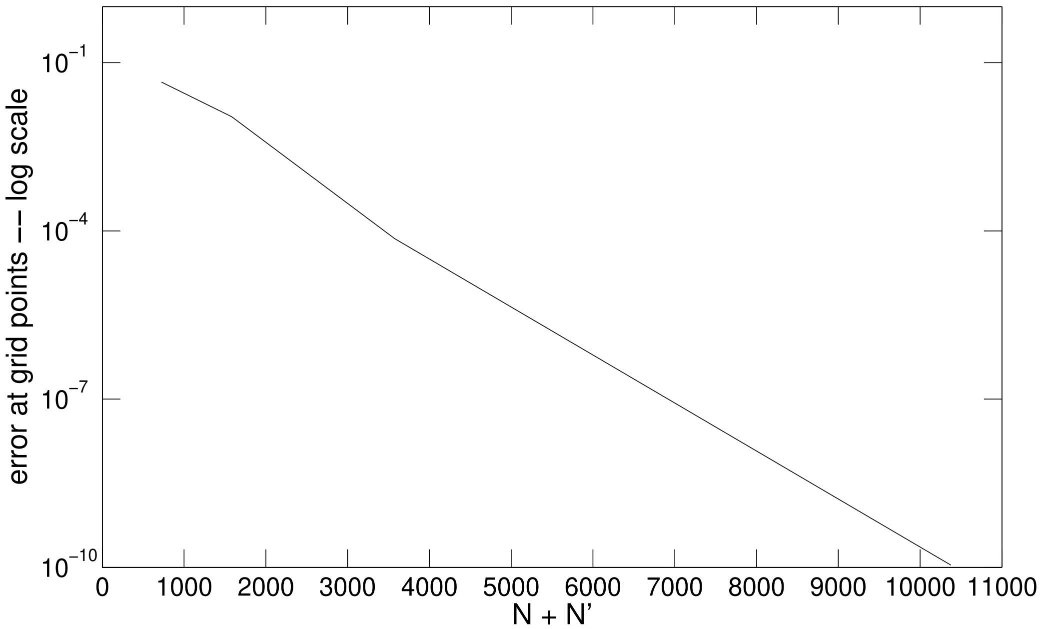

Figure 6.5: Error at grid points with respect to $N + N'$ for Problem 7

Table 6.24: Errors and orders of convergence for Problem 7 with extra boundary conditions (3.3)

	N	Layers	$N+N'$	τ_i		x_i		∞	
u_Δ	256	Too small							
	512	[1,49][456, 512]	724	1.5-01		4.5-02		8.3-01	
	1024	[1,71][950,1024]	1582	2.1-02	2.52	1.1-02	1.82	1.4-01	2.27
	2048	[1,64][1983,2048]	3577	8.1-05	6.80	7.3-05	6.14	8.3-05	9.10
	4096	[1,47][4050,4096]	10376	1.4-10	12.49	1.1-10	12.58	6.0-10	11.13
u'_Δ	256	Too small							
	512	[1,49][456, 512]	724	1.9+02		1.4+04		1.4+04	
	1024	[1,71][950,1024]	1582	5.7+02	-1.41	4.8+03	1.42	4.8+03	1.42
	2048	[1,64][1983,2048]	3577	3.1+00	6.37	4.8+00	8.46	4.4+00	8.56
	4096	[1,47][4050,4096]	10376	5.1-04	8.19	6.1-04	8.43	6.1-04	8.35
u''_Δ	256	Too small							
	512	[1,49][456, 512]	724	1.8+06		1.9+08		1.9+08	
	1024	[1,71][950,1024]	1582	6.1+06	-1.53	1.1+08	0.70	1.1+08	0.70
	2048	[1,64][1983,2048]	3577	9.1+04	5.15	1.0+06	5.75	1.0+06	5.75
	4096	[1,47][4050,4096]	10376	1.5+01	8.16	2.0+01	10.19	4.3+02	7.28
$u^{(3)}_\Delta$	256	Too small							
	512	[1,49][456, 512]	724	5.3+09		2.0+12		2.0+12	
	1024	[1,71][950,1024]	1582	4.7+09	0.14	1.6+12	0.25	1.6+12	0.25
	2048	[1,64][1983,2048]	3577	2.2+10	-1.91	1.2+11	3.19	1.2+11	3.19
	4096	[1,47][4050,4096]	10376	3.8+08	3.84	7.8+08	4.73	7.8+08	4.73
$u^{(4)}_\Delta$	256	Too small							
	512	[1,49][456, 512]	724	5.2+12		2.0+16		2.0+16	
	1024	[1,71][950,1024]	1582	1.1+15	-6.81	1.9+16	0.05	1.9+16	0.05
	2048	[1,64][1983,2048]	3577	6.7+14	0.59	7.1+15	1.22	7.1+15	1.22
	4096	[1,47][4050,4096]	10376	3.8+12	4.84	6.8+14	2.21	6.8+14	2.21

Table 6.25: Errors and orders of convergence for Problem 7 with extra boundary conditions (3.52)

	N	Layers	$N+N'$	τ_i		x_i		∞	
u_Δ	256	Too small							
	512	[1,48][456, 512]	722	2.9+00		8.2-01		2.9+00	
	1024	[1,71][950,1024]	1657	7.4-01	1.65	5.4-01	0.49	8.0-01	1.55
	2048	[1,64][1984,2048]	3980	3.4-05	11.39	7.1-05	10.20	8.1-05	10.49
	4096	[1,48][4049,4096]	9074	6.9-10	13.11	5.9-10	14.19	1.7-09	13.05
u'_Δ	256	Too small							
	512	[1,48][456, 512]	722	7.2+02		1.9+04		1.9+04	
	1024	[1,71][950,1024]	1657	2.1+03	-1.28	6.0+03	1.42	6.0+03	1.42
	2048	[1,64][1984,2048]	3980	3.7+00	7.24	2.2+00	9.02	3.4+00	8.53
	4096	[1,48][4049,4096]	9074	1.4-03	9.55	1.7-03	8.73	1.7-03	9.24
u''_Δ	256	Too small							
	512	[1,48][456, 512]	722	5.5+06		1.9+08		1.9+08	
	1024	[1,71][950,1024]	1657	2.1+07	-1.61	5.5+07	1.49	5.5+07	1.49
	2048	[1,64][1984,2048]	3980	4.1+04	7.12	1.0+05	7.16	2.9+05	6.01
	4096	[1,48][4049,4096]	9074	4.1+01	8.39	4.8+01	9.32	6.5+02	7.38
$u^{(3)}_\Delta$	256	Too small							
	512	[1,48][456, 512]	722	4.9+09		2.0+12		2.0+12	
	1024	[1,71][950,1024]	1657	1.3+11	-3.90	1.0+12	0.79	1.0+12	0.79
	2048	[1,64][1984,2048]	3980	2.4+10	1.90	5.6+10	3.33	5.6+10	3.33
	4096	[1,48][4049,4096]	9074	6.2+08	4.43	1.3+09	4.57	1.3+09	4.57
$u^{(4)}_\Delta$	256	Too small							
	512	[1,48][456, 512]	722	1.7+13		2.0+16		2.0+16	
	1024	[1,71][950,1024]	1657	9.4+14	-4.85	1.7+16	0.19	1.7+16	0.19
	2048	[1,64][1984,2048]	3980	1.6+14	2.01	5.4+15	1.33	5.4+15	1.33
	4096	[1,48][4049,4096]	9074	6.0+12	3.98	8.7+14	2.21	8.7+14	2.21

Problem 8

Consider the differential equation

$$Lu = \epsilon u''(x) + (2 + \cos(\pi x))u'(x) - u(x) = f(x), \quad x \in [0, 1]$$

with boundary conditions (6.1). The functions $f(x)$, g_1 and g_2 are chosen so that equation (5.3) is the solution. Figures 5.6 and 5.7 show the plot of $u(x)$ and $u^{(5)}$ respectively. We choose $\epsilon = 10^{-4}$ when applying the quartic spline collocation method. We can see the decrease of error at grid points with respect to $N + N'$ from Figure 6.6. Tables 6.26 and 6.27 present the errors and orders of convergence with extra boundary conditions (3.3) and (3.52) at midpoints, grid points and global points respectively. Note that $\| u^{(3)} \|_\infty = 2.7e + 13$ and $\| u^{(4)} \|_\infty = 8.1e + 17$, so the relative errors of the high derivatives are not large.

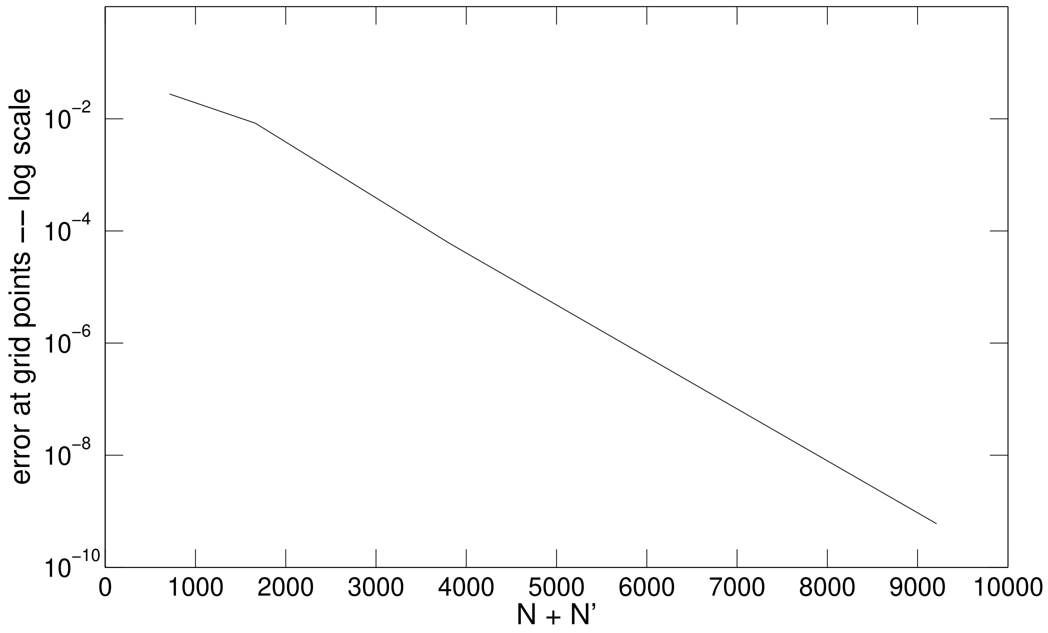

Figure 6.6: Error at grid points with respect to $N + N'$ for Problem 8

Table 6.26: Errors and orders of convergence for Problem 8 with extra boundary conditions (3.3)

	N	Layer	$N+N'$	τ_i		x_i		∞	
	256	Too small							
	512	[1,101]	714	9.6-02		2.8-02		5.1-01	
u_Δ	1024	[1,128]	1663	7.8-03	2.97	8.3-03	1.42	1.0-01	1.90
	2048	[1,106]	3825	6.5-05	5.74	5.9-05	5.95	6.5-05	8.83
	4096	[1, 72]	9210	7.5-10	12.94	6.0-10	13.08	4.7-10	13.48
	256	Too small							
	512	[1,101]	714	1.3+02		2.4+04		2.4+04	
u'_Δ	1024	[1,128]	1663	9.7+02	-2.36	9.4+03	1.12	9.4+03	1.12
	2048	[1,106]	3825	5.8+00	6.16	1.2+01	8.01	1.2+01	8.01
	4096	[1, 72]	9210	3.7-03	8.38	4.4-03	8.98	4.4-03	8.98
	256	Too small							
	512	[1,101]	714	3.8+06		8.8+08		8.8+08	
u''_Δ	1024	[1,128]	1663	3.0+07	-2.44	5.8+08	0.49	5.8+08	0.49
	2048	[1,106]	3825	5.5+05	4.81	7.0+06	5.30	7.0+06	5.30
	4096	[1, 72]	9210	3.3+02	8.46	4.7+02	10.94	4.0+03	8.51
	256	Too small							
	512	[1,101]	714	2.4+10		2.7+13		2.7+13	
$u^{(3)}_\Delta$	1024	[1,128]	1663	1.7+11	-2.34	2.4+13	0.15	2.4+13	0.15
	2048	[1,106]	3825	3.2+11	-0.75	2.1+12	2.93	2.1+12	2.93
	4096	[1, 72]	9210	1.1+10	3.84	2.3+10	5.10	2.3+10	5.10
	256	Too small							
	512	[1,101]	714	5.8+13		8.1+17		8.1+17	
$u^{(4)}_\Delta$	1024	[1,128]	1663	2.7+16	-7.27	7.9+17	0.03	7.9+17	0.03
	2048	[1,106]	3825	3.3+16	-0.23	3.2+17	1.10	3.2+17	1.10
	4096	[1, 72]	9210	3.5+14	5.18	4.1+16	2.34	4.1+16	2.34

Table 6.27: Errors and orders of convergence for Problem 8 with extra boundary conditions (3.52)

	N	Layer	$N+N'$	τ_i		x_i		∞	
	256	Too small							
	512	[1,101]	714	1.6+00		3.0-01		1.6+00	
u_Δ	1024	[1,129]	1741	6.1-01	1.06	3.9-01	-0.28	6.3-01	1.00
	2048	[1,107]	4301	3.4-05	10.83	7.0-05	9.53	8.1-05	9.92
	4096	[1, 75]	9739	1.0-09	12.72	8.2-10	13.90	2.1-09	12.92
	256	Too small							
	512	[1,101]	714	5.8+02		3.0+04		3.0+04	
u'_Δ	1024	[1,129]	1741	3.7+03	-2.08	1.4+04	0.83	1.4+04	0.83
	2048	[1,107]	4301	9.3+00	6.62	5.3+00	8.73	8.6+00	8.18
	4096	[1, 75]	9739	3.0-03	9.85	3.5-03	8.94	3.5-03	9.55
	256	Too small							
	512	[1,101]	714	1.4+07		8.8+08		8.8+08	
u''_Δ	1024	[1,129]	1741	1.1+08	-2.29	3.9+08	0.90	3.9+08	0.90
	2048	[1,107]	4301	2.8+05	6.56	7.5+05	6.92	1.8+06	5.95
	4096	[1, 75]	9739	2.6+02	8.56	3.1+02	9.56	4.7+03	7.27
	256	Too small							
	512	[1,101]	714	2.3+10		2.7+13		2.7+13	
$u^{(3)}_\Delta$	1024	[1,129]	1741	1.2+12	-4.49	1.8+13	0.47	1.8+13	0.47
	2048	[1,107]	4301	4.0+11	1.25	9.3+11	3.26	9.3+11	3.26
	4096	[1, 75]	9739	9.9+09	4.52	2.1+10	4.65	2.1+10	4.65
	256	Too small							
	512	[1,101]	714	1.7+14		8.1+17		8.1+17	
$u^{(4)}_\Delta$	1024	[1,129]	1741	2.8+16	-5.71	7.4+17	0.10	7.4+17	0.10
	2048	[1,107]	4301	7.9+15	1.40	2.4+17	1.24	2.4+17	1.24
	4096	[1, 75]	9739	2.9+14	4.05	3.8+16	2.25	3.8+16	2.25

6.5 Numerical results for one-step methods

We apply the one-step quartic spline collocation method to the same problems as those in Section 6.1 and compare the results with those of the two-step collocation method. In general, we found that the one-step quartic spline collocation method gives almost equivalent results as the two-step method. There are a few problems where the one-step method gives slightly better results than the two-step method, but more problems where the opposite happens. We present selected results to give the reader an understanding of the differences between the two methods.

First we apply the one-step collocation method to Problem 2 with extra boundary conditions (3.3). The functions $f(x)$, g_1 and g_2 are chosen so that $u(x) = x^5$. Table 6.28 shows the errors and the orders of convergence.

Comparing with Table 6.3 obtained by the two-step collocation method, we can see that the round-off dominates the approximation to $u(x)$, $u^{(2)}(x)$ and $u^{(4)}(x)$ at the midpoints, the approximation to $u(x)$ and $u^{(2)}(x)$ at the grid points, and the approximation to $u^{(3)}(x)$ at Gauss points. Except those round-off effect data, the errors and orders of convergence in Table 6.28 are almost equal to those in Table 6.3.

Table 6.28: Errors and orders of convergence for Problem 2 where $u(x) = x^5$ for the one-step method.

	N	τ_i		x_i		∞		λ_i	
u_Δ	8	2.0-15		2.2-15		7.5-07		7.3-07	
	16	2.2-15	-0.16	2.3-15	-0.08	2.3-08	5.00	2.3-08	5.00
	32	2.3-14	-3.37	2.3-14	-3.35	7.3-10	5.00	7.2-10	5.00
	64	6.5-14	-1.52	6.6-14	-1.50	2.3-11	5.00	2.2-11	5.00
	128	1.1-14	2.51	1.1-14	2.53	7.2-13	4.98	7.1-13	4.98
u'_Δ	8	3.6-05		4.1-05		4.1-05		6.8-06	
	16	2.2-06	4.00	2.5-06	4.00	2.5-06	4.00	4.2-07	4.00
	32	1.4-07	4.00	1.6-07	4.00	1.6-07	4.00	2.6-08	4.00
	64	8.7-09	4.00	9.9-09	4.00	9.9-09	4.00	1.7-09	4.00
	128	5.4-10	4.00	6.2-10	4.00	6.2-10	4.00	1.0-10	4.00
u''_Δ	8	8.9-15		4.3-14		1.9-03		1.9-03	
	16	5.7-14	-2.68	4.5-13	-3.42	2.3-04	3.00	2.3-04	3.00
	32	2.6-13	-2.17	1.0-12	-1.17	2.9-05	3.00	2.9-05	3.00
	64	1.0-12	-2.00	3.2-12	-1.64	3.7-06	3.00	3.7-06	3.00
	128	4.5-12	-2.15	9.1-12	-1.51	4.6-07	3.00	4.6-07	3.00
$u^{(3)}_\Delta$	8	7.8-02		1.6-01		1.6-01		6.7-13	
	16	2.0-02	2.00	3.9-02	2.00	3.9-02	2.00	1.4-11	-4.37
	32	4.9-03	2.00	9.8-03	2.00	9.8-03	2.00	5.3-11	-1.95
	64	1.2-03	2.00	2.4-03	2.00	2.4-03	2.00	4.6-10	-3.10
	128	3.1-04	2.00	6.1-04	2.00	6.1-04	2.00	3.6-09	-2.96
$u^{(4)}_\Delta$	8	1.1-11		7.5+00		7.5+00		4.3+00	
	16	3.2-10	-4.87	3.8+00	1.00	3.7+00	1.00	2.2+00	1.00
	32	3.0-09	-3.24	1.9+00	1.00	1.9+00	1.00	1.1+00	1.00
	64	4.5-08	-3.88	9.4-01	1.00	9.4-01	1.00	5.4-01	1.00
	128	8.9-07	-4.32	4.7-01	1.00	4.7-01	1.00	2.7-01	1.00

Then we apply the one-step collocation method to Problem 3 with extra boundary conditions (3.3). The functions $f(x)$, g_1 and g_2 are chosen so that $u(x) = x^{\frac{15}{2}}$. Table 6.29 shows the errors and the orders of convergence.

We compare the results in Table 6.29 with those in Table 6.11 obtained by the two-step collocation method. It is clear that the results by the one-step collocation method are slightly worse than those by the two-step collocation method, especially, for the approximation to $u(x)$ at the midpoints, the grid points, the global points and Gauss points.

Table 6.29: Errors and orders of convergence for Problem 3 where $u(x) = x^{\frac{15}{2}}$ for the one-step method.

	N	τ_i		x_i		∞		λ_i	
	8	2.1-04		2.1-04		2.1-04		2.1-04	
	16	3.6-06	5.85	3.6-06	5.87	3.6-06	5.87	3.6-06	5.87
u_Δ	32	5.9-08	5.95	5.9-08	5.94	7.4-08	5.62	7.4-08	5.63
	64	9.4-10	5.96	9.6-10	5.95	1.6-09	5.51	1.6-09	5.51
	128	1.5-11	5.97	1.5-11	5.96	3.9-11	5.39	3.8-11	5.40
	8	1.2-03		1.2-03		1.2-03		3.3-04	
	16	7.4-05	3.98	8.7-05	3.83	8.7-05	3.84	2.0-05	4.00
u'_Δ	32	4.7-06	3.97	5.6-06	3.97	5.6-06	3.97	1.1-06	4.16
	64	3.0-07	3.98	3.5-07	3.99	3.5-07	3.99	6.5-08	4.13
	128	1.9-08	3.99	2.2-08	4.00	2.2-08	4.00	3.9-09	4.08
	8	8.3-03		1.3-02		5.9-02		5.9-02	
	16	5.6-04	3.90	6.9-04	4.18	7.9-03	2.91	7.9-03	2.90
u''_Δ	32	3.6-05	3.96	4.2-05	4.04	1.0-03	2.96	1.0-03	2.96
	64	2.3-06	3.98	2.6-06	4.01	1.3-04	2.99	1.3-04	2.98
	128	1.4-07	3.99	1.6-07	4.00	1.6-05	3.00	1.6-05	2.99
	8	2.3+00		5.5+00		5.5+00		3.4-01	
	16	6.3-01	1.87	1.4+00	2.01	1.4+00	2.01	4.1-02	3.06
$u^{(3)}_\Delta$	32	1.7-01	1.94	3.4-01	2.00	3.4-01	2.00	5.1-03	3.00
	64	4.2-02	1.97	8.6-02	2.00	8.6-02	2.00	6.4-04	3.00
	128	1.1-02	1.99	2.1-02	2.00	2.1-02	2.00	8.1-05	3.00
	8	6.6+00		2.5+02		2.5+02		1.4+02	
	16	1.7+00	1.99	1.3+02	0.96	1.3+02	0.96	7.4+01	0.95
$u^{(4)}_\Delta$	32	4.2-01	1.98	6.5+01	0.98	6.5+01	0.98	3.7+01	0.98
	64	1.1-01	1.99	3.3+01	0.99	3.3+01	0.99	1.9+01	0.99
	128	2.7-02	1.99	1.6+01	1.00	1.6+01	1.00	9.5+00	0.99

Finally we apply the one-step collocation method to Problem 3 with extra boundary conditions (3.52) instead of (3.3). The functions $f(x)$, g_1 and g_2 are chosen so that $u(x) = e^x \sin(\pi x)$. Table 6.30 shows the errors and the orders of convergence.

Comparing the results in Table 6.30 with those in Table 6.15 obtained by the two-step collocation method, we can see that the numerical results by the one-step collocation method are slightly better than those by the two-step collocation method, specifically, for the approximation to $u(x)$ at the midpoints, the grid points, the global points and Gauss points.

We also applied the one-step method to more problems among those listed in Section 6.1. From the results obtained, we can say that, in most cases, the results by the two-step quartic spline collocation method are slightly better than those by the one-step collocation method in accuracy and orders of convergence.

Table 6.30: Errors and orders of convergence for Problem 3 where $u(x) = e^x \sin(\pi x)$ for the one-step method.

	N	τ_i		x_i		∞		λ_i	
u_Δ	8	1.3-04		1.3-04		1.3-04		1.3-04	
	16	2.4-06	5.76	2.4-06	5.74	2.5-06	5.69	2.5-06	5.69
	32	3.6-08	6.07	3.6-08	6.06	4.0-08	6.00	4.0-08	6.00
	64	5.1-10	6.14	5.2-10	6.13	6.3-10	5.99	6.3-10	5.99
	128	7.4-12	6.12	7.5-12	6.11	1.1-11	5.82	1.1-11	5.82
u'_Δ	8	1.7-04		2.8-04		2.8-04		1.5-04	
	16	1.3-05	3.71	1.4-05	4.30	1.4-05	4.30	3.5-06	5.44
	32	7.8-07	4.02	8.8-07	4.00	8.8-07	4.01	1.6-07	4.51
	64	4.9-08	4.01	5.5-08	3.99	5.5-08	3.99	9.3-09	4.07
	128	3.0-09	4.00	3.5-09	4.00	3.5-09	4.00	5.8-10	4.01
u''_Δ	8	2.5-03		3.4-03		1.1-02		1.1-02	
	16	1.8-04	3.79	2.3-04	3.91	1.3-03	3.03	1.3-03	3.03
	32	1.1-05	3.97	1.4-05	4.06	1.6-04	3.01	1.6-04	3.01
	64	7.2-07	3.99	8.4-07	4.03	2.0-05	3.00	2.0-05	3.00
	128	4.5-08	3.99	5.2-08	4.01	2.6-06	3.00	2.6-06	3.00
$u^{(3)}_\Delta$	8	4.4-01		8.7-01		8.7-01		9.6-02	
	16	1.1-01	2.01	2.2-01	2.00	2.1-01	2.02	1.3-02	2.85
	32	2.7-02	2.00	5.4-02	2.00	5.3-02	2.02	1.6-03	3.02
	64	6.8-03	2.00	1.4-02	2.00	1.3-02	2.04	2.1-04	3.01
	128	1.7-03	2.00	3.4-03	2.00	3.3-03	1.95	2.6-05	3.00
$u^{(4)}_\Delta$	8	1.9+00		4.1+01		4.1+01		2.4+01	
	16	5.3-01	1.82	2.1+01	0.98	2.1+01	1.00	1.2+01	1.01
	32	1.3-01	1.98	1.0+01	1.00	1.0+01	1.01	6.0+00	1.00
	64	3.4-02	1.99	5.2+00	1.00	5.1+00	1.00	3.0+00	1.00
	128	8.5-03	1.99	2.6+00	1.00	2.6+00	0.99	1.5+00	1.00

Chapter 7

Conclusions and Future Work

In this thesis, we present some optimal and some non-optimal quartic spline collocation methods for the solution of the general linear second-order two-point boundary value problems. We utilize the definition of the quartic spline interpolant and some asymptotic relations for the error of the interpolant in [16] to derive the collocation methods. We introduce extra boundary conditions in order to uniquely determine the quartic spline approximation. We formulate two optimal quartic spline collocation methods, the extrapolated (one-step) and the deferred-correction (two-step) methods. We also formulate three non-optimal collocation methods by choosing different extra boundary conditions. For a certain class of problems, we study the properties of the matrices arising from the two optimal methods. The convergence analysis and derivation of apriori error bounds are carried out by the Green's functions approach. The analysis shows that the global orders of convergence for the k-th derivative of the approximation, $k = 0, 1, 2$, are $O(h^{5-k})$, the orders of convergence for the approximation at the midpoints and grid points are $O(h^6)$, and the orders of convergence for the j-th derivative of the approximation, $j = 1, 2$, are $O(h^4)$ at the midpoints. Numerically we also show that the orders of convergence for the third derivative of the approximation at Gauss points are $O(h^3)$. We also derive a technique of grid refinement for layer problems. Our approach indicates that the error and grid size estimation techniques are reliable, and the technique of refinement is effective for several problems. Numerical results on a variety of problems verify the theoretical behavior of the optimal methods, and the technique of refinement for the layer problems. The numerical results show that C^6 continuity for $u(x)$ is necessary to obtain optimal orders of convergence. This indicates that the condition of C^{10} continuity shown in theory is only a sufficient and not a necessary condition.

A natural extension of the methods presented in this thesis is an optimal bi-quartic spline collocation method for elliptic partial differential equations. The method for two-point BVPs that uses extra boundary conditions at x_1 and x_{N-1} can be easily extended to two dimensions, while the method that uses x_0 and x_N may require careful development of extra boundary conditions at the four corners of the two-dimensional domain. The development of fast solvers, such as Fast Fourier Transform solvers and preconditioners, or multigrid solvers, for the linear system arising from the bi-quartic spline collocation methods is another interesting and useful piece of future work. The treatment of the layer problems in Chapter 5 is relatively primitive. It would be more effective to develop optimal quartic spline collocation methods on a non-uniform grid, as well as adaptive techniques for the placement of discretization points. The development of appropriate perturbations of the differential, boundary and extra boundary operators for quartic splines on non-uniform grids is a difficult and interesting task.

Bibliography

[1] C.de Boor. *A Practical Guide to Splines*. Springer-Verlag, 1978.

[2] C.de Boor and B.Swartz. Collocation at gaussian points. *SIAM J.Numer. Anal.,*, 10(4):582–606, 1973.

[3] C.C.Christara and Kit Sun Ng. Adaptive techniques for spline collocation. *Computing*, 76:259–277, 2006.

[4] C.C.Christara and Kit Sun Ng. Optimal quadratic and cubic spline collocation on nonuniform partitions. *Computing*, 76:227–257, 2006.

[5] E.N.Houstis, C.C.Christara, and J.R.Rice. Quadratic spline collocation methods for two-point boundary value problems. *International Journal for Numerical Methods in Engineering*, 26:935–952, 1988.

[6] G.Bader and U.Ascher. A new basis implementation for a mixed order boundary value ode solver. *SIAM J. Sci. Stat. Comp.*, 8(4):483–500, 1987.

[7] J.W.Daniel and B.K.Swartz. Extrapolated collocation for two-point boundary value problems using cubic splines. *J.Inst.Maths Applics*, 16:161–174, 1975.

[8] M.Irodotou-Ellina and E.N.Houstis. An O(h^6) quintic spline collocation method for fourth order two-point boundary value problems. *BIT*, 28:288–301, 1988.

[9] M.Irodotou-Ellina, E.N.Houstis, and S.B.Kim. An O(h^6) quintic spline collocation method for second order two-point boundary value problems. *Tech. Rep. CSD-TR-904, Purdue University*, 1989.

[10] P.M.Prenter. *Splines and Variational Methods*. John Wiley and Sons, 1975.

[11] R.A.Usmani. The use of quartic splines in the numerical solution of a fourth-order boundary value problem. *Journal of Computational and Applied Mathematics*, 44:187–199, 1992.

[12] R.D.Russell and L.F.Shampine. A collocation method for boundary value problems. *Numer. Math.*, 19:1–28, 1971.

[13] U.Ascher, J.Christiansen, and R.D.Russell. A collocation solver for mixed order systems of boundary value problems. *Math. Comp.*, 33(146):659–679, 1997.

[14] W.D.Hoskins and D.S.Meek. Linear dependence relations for polynomial splines at mid-knots. *BIT*, 15:272–276, 1975.

[15] W.J.Kammerer, G.W.Reddien, and R.S.Varga. Quadratic interpolatory splines. *Numer. Math.*, 22:241–259, 1974.

[16] Ying Zhu. Quartic-spline collocation methods for fourth-order two-point boundary value problems. Master's thesis, University of Toronto, 2001.