# Optimal Quadratic and Cubic Spline Collocation on Non-uniform Partitions

Christina C. Christara and Kit Sun Ng
Department of Computer Science
University of Toronto
Toronto, Ontario M5S 3G4, Canada
{ccc,ngkit}@cs.utoronto.ca

July 2005

**Abstract**

We develop optimal quadratic and cubic spline collocation methods for solving linear second-order two-point boundary value problems on non-uniform partitions. To develop optimal non-uniform partition methods, we use a mapping function from uniform to non-uniform partitions and develop expansions of the error at the non-uniform collocation points of some appropriately defined spline interpolants. The existence and uniqueness of the spline collocation approximations are shown, under some conditions. Optimal global and local orders of convergence of the spline collocation approximations and derivatives are derived, similar to those of the respective methods for uniform partitions. Numerical results on a variety of problems, including a boundary layer problem, and a non-linear problem, verify the optimal convergence of the methods, even under more relaxed conditions than those assumed by theory.

**AMS Subject Classification**: 65L10, 65L20, 65L50, 65L60, 65L70, 65D05, 65D07.

**Key words**: spline collocation, second-order two-point boundary value problem, error bounds, optimal order of convergence, spline interpolation.

## 1 Introduction

Consider a linear two-point Boundary Value Problem (BVP) described by the operator equation

$$\mathbf{L}u \equiv ru'' + pu' + qu = g \quad \text{in} \quad \Omega \equiv (0,1), \tag{1}$$

where $r, p, q$ and $g$ are given functions of $x$, and $u$ is the unknown function of $x$, subject to boundary conditions on the boundary $\partial\Omega$ of $\Omega$, described by

$$\mathbf{B}u \equiv \{\alpha_0 u(0) + \beta_0 u'(0) = \gamma_0, \alpha_1 u(1) + \beta_1 u'(1) = \gamma_1\}, \tag{2}$$

where $\alpha_i$, $\beta_i$ and $\gamma_i$, $i = 0, 1$ are given scalars. We focus the discussion to the above BVP, but some of the ideas and methods considered are useful for higher order and multi-dimensional BVPs.

Piecewise polynomial collocation is a commonly used method for the solution of (1)-(2). Let $\Delta$ be a partition for $\Omega$. Based on $\Delta$, a piecewise polynomial space $\mathcal{P}$ is defined. A set $T$ of data points, called *collocation* points, is prescribed in $\overline{\Omega} \equiv \Omega \cup \partial\Omega$. The number of collocation points depends on the choice of the piecewise polynomial space. Let $T_{\mathbf{B}} \equiv T \cap \partial\Omega$ be the set of boundary collocation points. Let $T_{\mathbf{L}}$ the set of points at which (1) is collocated. Usually $T_{\mathbf{L}} \equiv T - T_{\mathbf{B}}$, but in some cases $T_{\mathbf{L}} \equiv T$, that is, the operator equation (1) is collocated on the boundary as well.

In its standard formulation, piecewise polynomial collocation seeks an approximation $u_\Delta \in \mathcal{P}$ to the solution $u$ of (1)-(2), such that

$$\mathbf{L}u_\Delta = g \;\text{ in } T_{\mathbf{L}} \quad \text{and} \quad \mathbf{B}u_\Delta = \gamma \;\text{ on } T_{\mathbf{B}}. \tag{3}$$

The choice of collocation points is critical for the order of convergence of the arising method.

De Boor and Swartz [8] introduced and analyzed Gaussian collocation for one-dimensional $m$th order non-linear ODEs with $m$ linear side conditions. For the second order BVP described by (1)-(2), Gaussian collocation determines a $\mathbf{C}^1$ piecewise $(k+1)$-degree polynomial approximation $u_\Delta$ by collocating at the $k$ Gaussian points in each partition interval, and the two linear side conditions. Under some conditions, the global error in $u_\Delta$ is shown to be $O(H^{k+2})$, where $H$ is the maximum partition length. Furthermore, at the nodes, the approximation and the first derivative have error $O(H^{2k})$, that is, they exhibit superconvergence for $k > 2$ and $k \geq 2$, respectively.

Splines are piecewise polynomials of degree $k$ and continuity $\mathbf{C}^{k-1}$, that is, they exhibit the maximum possible smoothness. Spline collocation uses only one collocation point per subinterval and hence the linear system arising from (3) is the smallest possible among all types of piecewise polynomial collocation for the same number of partition points. Typical choices of spline collocation points are the nodes of the partition, if the degree of the splines is odd, and the midpoints, if the degree of the splines is even. It is worth noting that, when extending Gaussian collocation to two-dimensional domains through a tensor product approach, the number of collocation points per subrectangle is $k^2$, while for spline collocation it remains one. Thus the efficiency advantage of spline over Gaussian collocation becomes more apparent in two- and higher-dimensional problems.

However, the standard quadratic and cubic spline collocation methods applied to (1)-(2) produce $O(H^2)$ approximations even on uniform partitions. In contrast, quadratic spline interpolation at the midpoints gives an $O(H^3)$ global error bound, while, at the nodes of a uniform partition, it gives an $O(H^4)$ error bound. Moreover, cubic spline interpolation at the nodes gives an $O(H^4)$ global error bound.

Several researchers were concerned with developing optimal spline collocation methods, that is, methods that exhibit the same orders of convergence as interpolation in the same spline space. Optimal spline collocation methods on uniform partitions have been constructed based on acceleration techniques using appropriate perturbations $P_{\mathbf{L}}$ and $P_{\mathbf{B}}$ of the operators $\mathbf{L}$ and $\mathbf{B}$, respectively. The formulae that define these perturbation operators are derived by Taylor expansions at the collocation points of the errors between the exact solution $u$ and an appropriately defined spline interpolant of $u$.

Two types of methods were formulated. The first type of method, called *one-step* or *extrapolated* method, involves determining a spline $u_\Delta$ such that

$$\mathbf{L}u_\Delta + P_{\mathbf{L}}u_\Delta = g \;\text{ in } T_{\mathbf{L}} \;\text{ and }\; \mathbf{B}u_\Delta + P_{\mathbf{B}}u_\Delta = \gamma \;\text{ on } T_{\mathbf{B}}. \tag{4}$$

In this method, collocation is applied to a perturbed differential equation.

The second type of method, called *two-step* or *deferred-correction* method, involves determining first a spline $u_{\Delta[1]}$ that satisfies (3), and then a spline $u_\Delta$, such that

$$\mathbf{L}u_\Delta = g - P_\mathbf{L}u_{\Delta[1]} \text{ in } T_\mathbf{L} \text{ and } \mathbf{B}u_\Delta = \gamma - P_\mathbf{B}u_{\Delta[1]} \text{ on } T_\mathbf{B}. \tag{5}$$

Both methods are optimal. When the methods are extended to multi-dimensional BVPs, the deferred-correction method is more efficient in terms of time and memory requirements.

The first optimal spline collocation methods proposed to solve BVPs were based on cubic splines. For one-dimensional second-order BVPs and uniform partitions, Fyfe [9] proposed a deferred-correction cubic spline method, while [1] and [7] developed and analyzed an extrapolated cubic spline method. Extrapolated and deferred-correction quadratic spline methods, using the midpoints of the uniform partition intervals as collocation points, were proposed and analyzed in [10]. These optimal cubic and quadratic spline collocation methods were extended to two-dimensional second-order elliptic BVPs for rectangular domains in [12] and [5], respectively. Optimal quintic and quartic spline collocation methods [13], [17] were developed for one-dimensional fourth-order BVPs on uniform partitions. The orders of convergence of some optimal spline collocation methods and some Gaussian piecewise polynomial collocation methods are displayed in Tables 1 and 2.

| $d$ | 2 | | | | 3 | | | |
|---|---|---|---|---|---|---|---|---|
| $m = 2$ | global | mid | nodal | Gauss | global | mid | nodal | Gauss |
| $u - u_\Delta$ | 3 | 4 | 4 | | 4 | | | |
| $u' - u'_\Delta$ | 2 | | | 3 | 3 | 4 | 4 | |
| $u'' - u''_\Delta$ | 1 | 2 | | | 2 | | | 3 |
| $u^{(3)} - u^{(3)}_\Delta$ | N/A | N/A | N/A | N/A | 1 | 2 | | |
| $d$ | 4 | | | | 5 | | | |
| $m = 4$ | global | mid | nodal | Gauss | global | mid | nodal | Gauss |
| $u - u_\Delta$ | 5 | 6 | 6 | | 6 | | | |
| $u' - u'_\Delta$ | 4 | | | | 5 | | 6 | |
| $u'' - u''_\Delta$ | 3 | 4 | 4 | | 4 | | | |
| $u^{(3)} - u^{(3)}_\Delta$ | 2 | | | 3 | 3 | | 4 | |
| $u^{(4)} - u^{(4)}_\Delta$ | 1 | 2 | | | 2 | | | |

Table 1: Order of convergence of optimal spline collocation methods. Here, $d$ is the degree of the spline ($d - 1$ continuity), and $m$ the order of the BVP. Blank entries denote that the global convergence results hold (no local superconvergence result was found in the literature).

None of the above optimal spline collocation methods has been extended to non-uniform partitions. The applicability of a discretization method to non-uniform partitions is essential for problems with rough solution behaviour, layers, etc, which are usually solved by adaptive techniques, and for multi-dimensional problems in non-rectangular domains. In the rest of the paper, we develop optimal non-uniform partition quadratic spline collocation (QSC) and cubic spline collocation (CSC) methods for linear one-dimensional second-order BVPs. In [6], we develop adaptive techniques for spline collocation.

| | $m=2, k=2, d=3$ | | $m=2, k=4, d=5$ | | $m=4, k=6, d=9$ | |
|---|---|---|---|---|---|---|
| | global | nodal | global | nodal | global | nodal |
| $u-u_\Delta$ | 4 | | 6 | 8 | 10 | 12 |
| $u'-u'_\Delta$ | 3 | 4 | 5 | 8 | 9 | 12 |
| $u''-u''_\Delta$ | 2 | | 4 | | 8 | 12 |
| $u^{(3)}-u^{(3)}_\Delta$ | 1 | | 3 | | 7 | 12 |

Table 2: Order of convergence of degree $d = m + k - 1$ piecewise polynomial collocation ($m - 1$ continuity) for a $m$th order BVP at $k$ Gaussian points in each subinterval. Blank entries denote that the global convergence results hold (no local superconvergence result was found in the literature).

The outline of this paper is as follows. In Section 2, we present a technique of deriving optimal spline collocation methods for non-uniform partitions by mapping uniform to non-uniform partitions. We apply the technique to derive two QSC methods on non-uniform partitions, the two-step and the one-step methods. We emphasize that, at least numerically, this technique is *not* equivalent to applying a transformation of variables to the problem and obtaining a problem which can be solved effectively on a uniform grid. We prove the existence and uniqueness of the QSC approximation for a certain class of problems, and derive optimal global and local error bounds. In Section 3, we derive and analyze an optimal two-step CSC method on non-uniform partitions. The derivation and analysis of the method again assumes that the non-uniform partition is the image of a uniform partition under an appropriate mapping. However, the final formulation and implementation of the method is mapping-free, in the sense that, once the location of the grid points is given, no mapping needs to be computed, in contrast to the QSC method. In Section 4, we present numerical results that demonstrate the behaviour of the QSC and CSC methods on a variety of problems, including a boundary layer problem, and a non-linear problem, and indicate that the conditions assumed for the development and analysis of the methods are only sufficient and not necessary. In the last section, we make some concluding remarks and briefly mention possible extensions of this work.

# 2 Non-uniform Partition Quadratic Spline Collocation

Consider the two-point BVP (1)-(2). Let $\Delta \equiv \{x_0 = 0 < x_1 < \ldots < x_N = 1\}$ be a uniform partition of $\overline{\Omega}$ with stepsize $h \equiv 1/N$ and $T \equiv \{\tau_0 \equiv x_0, \tau_i \equiv (x_i + x_{i-1})/2; i = 1, \ldots, N, \tau_{N+1} \equiv x_N\}$ be a set of data points. Furthermore, define the Gaussian points $\delta_{ij} \equiv x_i - \lambda_j h$; $j = 1, 2$, $i = 1, \ldots, N$, where $\lambda_1 \equiv (3 - \sqrt{3})/6$ and $\lambda_2 \equiv (3 + \sqrt{3})/6$.

Let $w(x) : \overline{\Omega} \to \overline{\Omega}$ be a bijective function in $\mathbf{C}^3$, with $w'(x) > 0$, $\forall x \in \overline{\Omega}$. Let $\Delta_w \equiv \{s_i \equiv w(x_i), i = 0, \ldots, N\}$ be the partition of $\overline{\Omega}$ with respect to which the splines will be defined, and $T_w \equiv \{w_i \equiv w(\tau_i), i = 0, \ldots, N+1\}$ be the set of collocation points. That is, for the initial development of the methods, we assume that the non-uniform grid points and collocation points are images of uniform grid points and collocation points, respectively. Let also $\sigma_{ij} \equiv w(\delta_{ij})$, $j = 1, 2$, $i = 1, \ldots, N$. Adopt the following notation: $h_i \equiv w_{i+1} - w_i$, $i = 1, \ldots, N-1$, $h_i^a \equiv s_i - w_i$, $i = 1, \ldots, N$, $h_i^b \equiv w_{i+1} - s_i$ $i = 0, \ldots, N-1$, $H_i \equiv s_{i+1} - s_i$, $i = 0, \ldots, N-1$, and $H \equiv \max\{\max_{i=1,\ldots,N}\{h_i^a\}, \max_{i=0,\ldots,N-1}\{h_i^b\}\}$. Figure 1 shows the notation for the uniform grid and its non-uniform image for $N + 1$ grid points. For any given $w(x)$, we denote by $S_{\Delta_w}$ the quadratic spline space with respect to the partition $\Delta_w$. Note that the second derivative of a quadratic spline is not continuous on $s_i, i = 0, \dots, N$, but, whenever we need its value, we define it by right (without loss of generality) continuity for $s_i, i = 0, \dots, N-1$, and by left continuity for $s_N$.

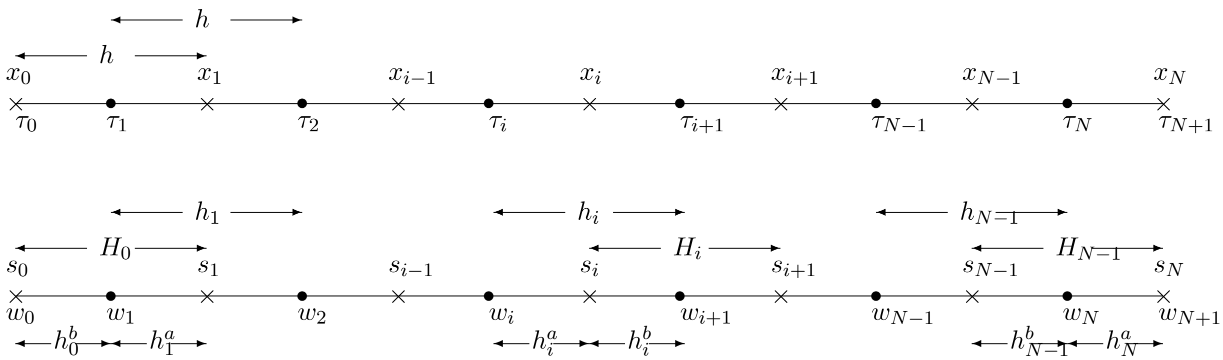


Figure 1: The uniform grid (above) and its non-uniform image (below, drawn as uniform for convenience).

Let $S \in S_{\Delta_w}$ be the quadratic spline interpolant of $u$ such that

$$\begin{array}{ll} S''(w_1) & = u''(w_1) - \frac{1}{24}\{4(h_1^a - h_0^b)u^{(3)}(w_1) + (h_1^a + h_0^b)^2 u^{(4)}(w_1)\}, \\ S(w_i) & = u(w_i), \quad i = 1, \dots, N, \\ S''(w_N) & = u''(w_N) - \frac{1}{24}\{4(h_N^a - h_{N-1}^b)u^{(3)}(w_N) + (h_N^a + h_{N-1}^b)^2 u^{(4)}(w_N)\}. \end{array} \tag{6}$$

Note that the second set of relations in (6) is typical for a quadratic spline interpolant of $u$, while the first and third relations are a set of end-conditions that make the quadratic spline interpolant unique.

For simplicity, adopt the following notation: $S_i \equiv S(w_i)$, $S_i' \equiv S'(w_i)$ and $S_i'' \equiv S''(w_i)$. Notice that, when an integer subscript is attributed to the function $w$ or its derivatives, it denotes values of the function or its derivatives, respectively, at collocation points of a uniform partition, i.e. $w_i \equiv w(\tau_i)$, $w_i' \equiv w'(\tau_i)$ and $w_i'' \equiv w''(\tau_i)$, while, when it is attributed to other functions, it denotes values of the function at images through $w$ of collocation points of a uniform partition, i.e. $u_i \equiv u(w_i) = u(w(\tau_i))$, etc. Furthermore, let $e(x) \equiv u(x) - S(x)$, thus $e_i \equiv e(w_i)$. We will develop expansions of the errors of the derivatives of $S$ at the collocation points.

## 2.1 Expansions of the errors of a quadratic spline interpolant

We begin by considering the second derivative error at the midpoints. The symbols $\alpha_{ki}$, $\beta_{ki}$, $\gamma_{ki}$, $k = 0, 1, 2$, $i = 2, \dots, N-1$, denote scalars defined within the subsection. By setting up the equations,

$$\alpha_{2i} S_{i-1}'' + \beta_{2i} S_i'' + \gamma_{2i} S_{i+1}'' = \alpha_{0i} S_{i-1} + \beta_{0i} S_i + \gamma_{0i} S_{i+1} \tag{7}$$

for $i = 2, \dots, N-1$, and solving for $\alpha_{ki}$, $\beta_{ki}$, $\gamma_{ki}$, $k = 0, 2$, with Maple we have

$$\begin{array}{ll} \alpha_{0i} = -2h_i, & \alpha_{2i} = -h_i(h_{i-1}^a)^2, \\ \beta_{0i} = 2(h_i + h_{i-1}), & \beta_{2i} = h_i(h_{i-1}^a)^2 + h_{i-1}(h_i^b)^2 - (h_i + h_{i-1})h_i h_{i-1}, \\ \gamma_{0i} = -2h_{i-1}, & \gamma_{2i} = -h_{i-1}(h_i^b)^2. \end{array} \tag{8}$$

It is worth noting that relations (7) hold for any quadratic spline $S \in S_{\Delta_w}$.
For later convenience, let $\wedge u_i \equiv \frac{1}{-(h_i+h_{i-1})h_ih_{i-1}}(\alpha_{2i}u_{i-1} + \beta_{2i}u_i + \gamma_{2i}u_{i+1})$, $i = 2, \dots, N-1$.

**LEMMA 1** *If $w(x) \in \mathbf{C}^3[\overline{\Omega}]$, then*

$$\begin{array}{ll} h_i = hw'_i + \frac{h^2}{2}w''_i + O(h^3), & h_{i-1} = hw'_i - \frac{h^2}{2}w''_i + O(h^3), \\ h^a_i = \frac{h}{2}w'_i + \frac{h^2}{8}w''_i + O(h^3), & h^b_i = \frac{h}{2}w'_i + \frac{3h^2}{8}w''_i + O(h^3), \\ h^b_{i-1} = \frac{h}{2}w'_i - \frac{h^2}{8}w''_i + O(h^3), & h^a_{i-1} = \frac{h}{2}w'_i - \frac{3h^2}{8}w''_i + O(h^3), \end{array} \tag{9}$$

*where the index $i$ takes all possible values consistent with the definition of the stepsizes.*

PROOF
Using the definitions of $h_i, h^a_i$ and $h^b_i$, and Taylor expansions, the proof of this lemma is trivial. $\diamondsuit$

In the following, the symbol $\circ$ denotes composition of functions.

**THEOREM 1** *If $u \in \mathbf{C}^6[\overline{\Omega}]$, $w(x) : \overline{\Omega} \to \overline{\Omega}$ is a bijective function in $\mathbf{C}^3$, with $w'(x) > 0$, $\forall x \in \overline{\Omega}$, $v = -\frac{1}{24}\{h^2(w'' \circ w^{-1})u^{(3)} + h^2(w' \circ w^{-1})^2u^{(4)}\} \in \mathbf{C}^2[\overline{\Omega}]$, and $S$ is defined by (6), then*

$$S''_i = u''_i - \frac{1}{24}\{4(h^a_i - h^b_{i-1})u^{(3)}_i + (h^a_i + h^b_{i-1})^2u^{(4)}_i\} + O(h^4), \tag{10}$$

*for $i = 1, \dots, N$.*

PROOF
Since $S_i = u_i$, $i = 1, \dots, N$, and $u \in \mathbf{C}^6$, the right-hand side of (7) after applying Taylor expansion to $u_{i\pm1}$ is equivalent to

$$\begin{aligned} \alpha_{0i}u_{i-1} + \beta_{0i}u_i + \gamma_{0i}u_{i+1} \;\; &= \;\; -h_{i-1}h_i(h_i + h_{i-1})[u''_i + \frac{1}{3}(h_i - h_{i-1})u^{(3)}_i + \frac{1}{12}(h^2_i - h_ih_{i-1} + h^2_{i-1})u^{(4)}_i] \\ &+ \;\; O(H^6), \quad i = 2, \dots, N-1. \end{aligned} \tag{11}$$

Moreover, using Taylor expansion of a function $f \in \mathbf{C}^4$ we can show

$$\begin{aligned} \alpha_{2i}f_{i-1} + \beta_{2i}f_i \;\; &+ \;\; \gamma_{2i}f_{i+1} \\ &= \;\; -h_ih_{i-1}(h_i + h_{i-1})[f_i - \frac{(h^a_{i-1} + h^b_i)(h^a_{i-1} - h^b_i)}{h_i + h_{i-1}}f'_i + \frac{h_i(h^b_i)^2 + h_{i-1}(h^a_{i-1})^2}{2(h_i + h_{i-1})}f''_i] \\ &+ \;\; O(H^6), \quad i = 2, \dots, N-1 \end{aligned} \tag{12}$$

By using Lemma 1 to expand $h_i, h^a_i$ and $h^b_i$, relation (11) implies

$$\wedge S''_i = u''_i + \frac{1}{3}h^2w''_iu^{(3)}_i + \frac{1}{12}h^2(w'_i)^2u^{(4)}_i + O(h^4), \quad i = 2, \dots, N-1 \tag{13}$$

and (12) implies

$$\wedge f_i = f_i + \frac{3}{8}h^2w''_if'_i + \frac{1}{8}h^2(w'_i)^2f''_i + O(h^4), \quad i = 2, \dots, N-1. \tag{14}$$

Let $v = -\frac{1}{24}\{h^2(w''\circ w^{-1})u^{(3)} + h^2(w'\circ w^{-1})^2u^{(4)}\}$ and $f = u'' + v$. Then $f_i = u_i'' - \frac{1}{24}\{h^2 w_i'' u_i^{(3)} + h^2(w_i')^2u_i^{(4)}\} = u_i'' - \frac{1}{24}\{4(h_i^a - h_{i-1}^b)u_i^{(3)} + (h_i^a + h_{i-1}^b)^2u_i^{(4)}\} + O(h^4)$. As $u \in \mathbf{C}^6$, we have

$$\wedge f_i = u_i'' + \frac{1}{3}h^2 w_i'' u_i^{(3)} + \frac{1}{12}h^2(w_i')^2u_i^{(4)} + \frac{3}{8}h^4 w_i'' v_i' + \frac{1}{8}h^4(w_i')^2 v''(\xi_i) + O(h^4), \quad i = 2,\ldots,N-1, \tag{15}$$

where $\xi_i \in (w_{i-1}, w_{i+1})$. As $v \in \mathbf{C}^2$ and $w \in \mathbf{C}^3$ we can regroup the $h^4$ terms as just $O(h^4)$. Hence,

$$\wedge(S_i'' - f_i) = O(h^4), \quad i = 2,\ldots,N-1. \tag{16}$$

Moreover, from the end-conditions in (6), we have

$$S_1'' - f_1 = O(h^4) \tag{17}$$

and

$$S_N'' - f_N = O(h^4). \tag{18}$$

Relations (17), (16) and (18) (in that order) form a system of equations that is strictly diagonally dominant. More specifically, using Lemma 1, and the fact that $w'(x) > 0$, $\forall x \in \overline{\Omega}$, we can show that $\frac{|\beta_{2i}|-|\alpha_{2i}|-|\gamma_{2i}|}{|-(h_i+h_{i-1})h_ih_{i-1}|} = \frac{1}{2} + O(h^2)$, which, for sufficiently small $h$, is greater than $\frac{1}{3}$. Therefore, the infinity norm of the inverse of the matrix of equations (17), (16) and (18) is bounded by 3. This implies that

$$S_i'' - f_i = O(h^4), \quad i = 1,\ldots,N. \qquad \diamondsuit$$

**Remark 1** In the proof we assumed that $v \in \mathbf{C}^2[\overline{\Omega}]$ in order to bound the $O(h^4)$ terms in (15). We can substitute this assumption with the assumptions that $(w''\circ w^{-1})v' \in \mathbf{C}[\overline{\Omega}]$ and $(w'\circ w^{-1})^2 v'' \in \mathbf{C}[\overline{\Omega}]$, which, sometimes, are less restrictive. (See also the numerical results on Problem 2.)

**Remark 2** The assumption that $w'(x) > 0$, $\forall x \in \overline{\Omega}$, together with Lemma 1 guarantee that the partition generated by $w$ satisfies $H/\min\{\min_{i=1,\ldots,N}\{h_i^a\}, \min_{i=0,\ldots,N-1}\{h_i^b\}\} \leq C$, for some positive constant $C$, independent of $h$.

**Remark 3** Theorem 1 shows that, under the assumptions, $S_i'' - u_i''$ has a smooth expansion, since $S_i'' - u_i'' = -\frac{1}{24}\{h^2 w_i'' u_i^{(3)} + h^2(w_i')^2u_i^{(4)}\}$.

We now prove the counterpart of Theorem 1 for the first derivative error at the midpoints. Using Maple we can show the following relations for any $S \in S_{\Delta_w}$:

$$\begin{aligned}
\alpha_{1i}S_{i-1}' + \beta_{1i}S_i' + \gamma_{1i}S_{i+1}' &= \alpha_{0i}S_{i-1} + \beta_{0i}S_i + \gamma_{0i}S_{i+1}, \quad i = 2,\ldots,N-1 && (19)\\
S_i' - S_{i-1}' &= h_{i-1}^b S_i'' + h_{i-1}^a S_{i-1}'', \qquad i = 2,\ldots,N && (20)\\
S_{i+1}' - S_i' &= h_i^b S_{i+1}'' + h_i^a S_i'', \qquad i = 1,\ldots,N-1, && (21)
\end{aligned}$$

where

$$\begin{array}{ll}
\alpha_{0i} = -2h_ih_i^a & \alpha_{1i} = h_ih_i^ah_{i-1}^a \\
\beta_{0i} = 2(h_ih_i^a - h_{i-1}h_{i-1}^b) & \beta_{1i} = h_ih_{i-1}(h_i^a + h_{i-1}^2) + h_i^ah_{i-1}^b(h_i + h_{i-1}) \\
\gamma_{0i} = 2h_{i-1}h_{i-1}^b & \gamma_{1i} = h_{i-1}h_{i-1}^bh_i^b.
\end{array}$$

Note that $\alpha_{0i}, \beta_{0i}, \gamma_{0i}$ are different from those in (8).

**THEOREM 2** *Under the assumptions of Theorem 1,*

$$S_i' = u_i' + \frac{1}{24}(h_i^a + h_{i-1}^b)^2 u_i^{(3)} + O(h^4), \tag{22}$$

*for* $i = 1, \ldots, N$.

PROOF

Computing $\alpha_{1i} \times (20) - \gamma_{1i} \times (21) + (19)$ we have

$$\begin{aligned} & [\alpha_{0i}S_{i-1} + \beta_{0i}S_i + \gamma_{0i}S_{i+1}] + [\alpha_{1i}(h_{i-1}^b S_i'' + h_{i-1}^a S_{i-1}'') - \gamma_{1i}(h_i^b S_{i+1}'' + h_i^a S_i'')] \\ = \; & (\alpha_{1i} + \beta_{1i} + \gamma_{1i})S_i', \quad i = 2, \ldots, N-1. \end{aligned} \tag{23}$$

Now,

$$\begin{aligned} & \alpha_{0i}S_{i-1} + \beta_{0i}S_i + \gamma_{0i}S_{i+1} = \alpha_{0i}u_{i-1} + \beta_{0i}u_i + \gamma_{0i}u_{i+1} \\ = \; & h_i h_{i-1}[2(h_i^a + h_{i-1}^b)u_i' + (h_i h_{i-1}^b - h_{i-1}h_i^a)u_i'' + \tfrac{1}{3}(h_i^2 h_{i-1}^b + h_{i-1}^2 h_i^a)u_i^{(3)} - \tfrac{1}{12}(h_i^3 h_{i-1}^b + h_{i-1}^3 h_i^a)u_i^{(4)}] \\ + \; & O(H^7), \quad i = 2, \ldots, N-1. \end{aligned} \tag{24}$$

Using Lemma 1 we have

$$\alpha_{1i} + \beta_{1i} + \gamma_{1i} = 2h_i h_{i-1}(h_i^a + h_{i-1}^b) = 2(w_i')^3 h^3 + O(h^5). \tag{25}$$

By using (24), (25) and Lemma 1 we have

$$\frac{1}{\alpha_{1i} + \beta_{1i} + \gamma_{1i}}(\alpha_{0i}S_{i-1} + \beta_{0i}S_i + \gamma_{0i}S_{i+1}) = u_i' + \frac{1}{8}h^2 w_i'' u_i'' + \frac{1}{6}h^2 (w_i')^2 u_i^{(3)} + O(h^4), \; i = 2, \ldots, N-1. \tag{26}$$

Moreover, by Theorem 1, Lemma 1 and the use of Maple we can show that

$$\begin{aligned} & \frac{1}{\alpha_{1i} + \beta_{1i} + \gamma_{1i}}[\alpha_{1i}(h_{i-1}^b S_i'' + h_{i-1}^a S_{i-1}'') - \gamma_{1i}(h_i^b S_{i+1}'' + h_i^a S_i'')] \\ = \; & \frac{-\frac{1}{4}h^5 (w_i')^3 w_i'' u_i'' - \frac{1}{4}h^5 (w_i')^5 u_i^{(3)} + O(h^7)}{2(w_i')^3 h^3 + O(h^5)} \\ = \; & -\frac{1}{8}h^2 w_i'' u_i'' - \frac{1}{8}h^2 (w_i')^2 u_i^{(3)} + O(h^4), \quad i = 2, \ldots, N-1. \end{aligned} \tag{27}$$

So combining (23), (26) and (27) we have

$$S_i' = u_i' + \frac{1}{24}h^2 (w_i')^2 u_i^{(3)} + O(h^4), \quad i = 2, \ldots, N-1. \tag{28}$$

Furthermore, we can use (20), (21), (28), Lemma 1 and Theorem 1, and simplify with Maple to get

$$\begin{aligned} S_1' &= S_2' - h_1^b S_2'' - h_1^a S_1'' && = u_1' + \tfrac{1}{24}h^2 (w_1')^2 u_1^{(3)} + O(h^4) \\ S_N' &= S_{N-1}' + h_{N-1}^b S_N'' + h_{N-1}^a S_{N-1}'' && = u_N' + \tfrac{1}{24}h^2 (w_N')^2 u_N^{(3)} + O(h^4). \end{aligned}$$

But $h^2 (w_i')^2 u_i^{(3)} = (h_i^a + h_{i-1}^b)^2 u_i^{(3)} + O(h^4)$. ◊

Next, we develop expansions of the error of $S$ at the nodes.

**THEOREM 3** *Under the assumptions of Theorem 1,*

$$S(s_i) = u(s_i) - \frac{1}{16}(h_i^a - h_{i-1}^b)(h_i^a + h_{i-1}^b)^2 u^{(3)}(s_i) - \frac{1}{128}(h_i^a + h_{i-1}^b)^4 u^{(4)}(s_i) + O(h^5), \tag{29}$$

*for* $i = 1, \ldots, N$, *and*

$$S(s_i) = u(s_i) - \frac{1}{16}(h_{i+1}^a - h_i^b)(h_{i+1}^a + h_i^b)^2 u^{(3)}(s_i) - \frac{1}{128}(h_{i+1}^a + h_i^b)^4 u^{(4)}(s_i) + O(h^5) \tag{30}$$

*for* $i = 0, \ldots, N-1$.

PROOF
By Taylor expansion

$$e(s_i) = e_i + h_i^a e_i' + \frac{1}{2}(h_i^a)^2 e_i'' + \frac{1}{6}(h_i^a)^3 u_i^{(3)} + \frac{1}{24}(h_i^a)^4 u_i^{(4)} + O((h_i^a)^5), \;\; i = 1, \ldots, N. \tag{31}$$

By noting that $e_i = 0$, and by using Theorems 2 and 1 for $e_i'$ and $e_i''$, respectively, and Lemma 1, (31) becomes

$$\begin{aligned}
e(s_i) &= \{\frac{h}{2}w_i' + \frac{h^2}{8}w_i'' + O(h^3)\}\{-\frac{h^2}{24}(w_i')^2 u_i^{(3)} + O(h^4)\} \\
&+ \frac{1}{2}\{\frac{h^2}{4}(w_i')^2 + O(h^3)\}\{\frac{h^2}{24}w_i'' u_i^{(3)} + \frac{h^2}{24}(w_i')^2 u_i^{(4)} + O(h^4)\} \\
&+ \frac{1}{6}\{\frac{h^3}{8}(w_i')^3 + \frac{h^4}{32}(w_i')^2 w_i'' + \frac{h^4}{16}(w_i')^2 w_i'' + O(h^5)\}u_i^{(3)} + \frac{1}{24}\frac{h^4}{16}(w_i')^4 u_i^{(4)} + O(h^5) \\
&= \frac{h^4}{64}(w_i')^2 w_i'' u_i^{(3)} + \frac{h^4}{128}(w_i')^4 u_i^{(4)} + O(h^5) \\
&= \frac{h^4}{64}(w_i')^2 w_i'' u^{(3)}(s_i) + \frac{h^4}{128}(w_i')^4 u^{(4)}(s_i) + O(h^5) \\
&= \frac{1}{16}(h_i^a + h_{i-1}^b)(h_i^a - h_{i-1}^b)^2 u^{(3)}(s_i) + \frac{1}{128}(h_i^a + h_{i-1}^b)^4 u^{(4)}(s_i) + O(h^5).
\end{aligned}$$

Similarly,

$$e(s_i) = \frac{1}{16}(h_{i+1}^a - h_i^b)(h_{i+1}^a + h_i^b)^2 u^{(3)}(s_i) + \frac{1}{128}(h_{i+1}^a + h_i^b)^4 u^{(4)}(s_i) + O(h^5), \quad i = 0, \ldots, N-1. \;\; \diamondsuit$$

Finally, we develop expansions of the error of the first derivative of $S$ at the nodes.

**THEOREM 4** *Under the assumptions of Theorem 1,*

$$S'(s_i) = u'(s_i) - \frac{1}{12}\{(h_i^a + h_{i-1}^b)^2 + 4(h_i^a - h_{i-1}^b)(h_i^a + h_{i-1}^b)\}u^{(3)}(s_i) + O(h^4) \tag{32}$$

*for* $i = 1, \ldots, N$, *and*

$$S'(s_i) = u'(s_i) - \frac{1}{12}\{(h_{i+1}^a + h_i^b)^2 - 4(h_{i+1}^a - h_i^b)(h_{i+1}^a + h_i^b)\}u^{(3)}(s_i) + O(h^4) \tag{33}$$

*for* $i = 0, \ldots, N-1$.

PROOF
By Taylor expansion

$$e'(s_i) = e'_i + h^a_i e''_i + \frac{1}{2}(h^a_i)^2 u^{(3)}_i + \frac{1}{6}(h^a_i)^3 u^{(4)}_i + O((h^a_i)^4), \ \ i = 1, \ldots, N. \tag{34}$$

By using Theorems 2 and 1 for $e'_i$ and $e''_i$, respectively, Lemma 1, and the Taylor relations $u^{(3)}_i = u^{(3)}(s_i) - h^a_i u^{(4)}(s_i) + O((h^a_i)^2)$ and $u^{(4)}_i = u^{(4)}(s_i) + O(h^a_i)$, (34) becomes

$$\begin{aligned}
e'(s_i) &= -\frac{h^2}{24}(w'_i)^2 u^{(3)}_i + O(h^4) \\
&+ \{\frac{h}{2}w'_i + \frac{h^2}{8}w''_i + O(h^3)\}\{\frac{h^2}{24}w''_i u^{(3)}_i + \frac{h^2}{24}(w'_i)^2 u^{(4)}_i + O(h^4)\} \\
&+ \frac{1}{2}\{\frac{h}{2}w'_i + \frac{h^2}{8}w''_i + O(h^3)\}^2 u^{(3)}_i + \frac{1}{6}\{\frac{h}{2}w'_i + \frac{h^2}{8}w''_i + O(h^3)\}^3 u^{(4)}_i + O(h^4) \\
&= \frac{1}{12}h^2(w'_i)^2 u^{(3)}_i + \frac{1}{12}h^3 w'_i w''_i u^{(3)}_i + \frac{1}{24}h^3(w'_i)^3 u^{(4)}_i + O(h^4) \\
&= \frac{1}{12}h^2(w'_i)^2\{u^{(3)}(s_i) - h^a_i u^{(4)}(s_i) + O((h^a_i)^2)\} \\
&+ \frac{1}{12}h^3 w'_i w''_i\{u^{(3)}(s_i) - h^a_i u^{(4)}(s_i) + O((h^a_i)^2)\} + \frac{1}{24}h^3(w'_i)^3\{u^{(4)}(s_i) + O(h^a_i)\} + O(h^4) \\
&= \frac{1}{12}h^2(w'_i)^2 u^{(3)}(s_i) + \frac{1}{12}h^3 w'_i w''_i u^{(3)}(s_i) + O(h^4) \\
&= \frac{1}{12}(h^a_i + h^b_{i-1})^2 u^{(3)}(s_i) + \frac{1}{3}(h^a_i - h^b_{i-1})(h^a_i + h^b_{i-1})u^{(3)}(s_i) + O(h^4).
\end{aligned}$$

Similarly,

$$e'(s_i) = \frac{1}{12}(h^a_{i+1} + h^b_i)^2 u^{(3)}(s_i) - \frac{1}{3}(h^a_{i+1} - h^b_i)(h^a_{i+1} + h^b_i)u^{(3)}(s_i) + O(h^4), \quad i = 0, \ldots, N-1. \ \ \diamondsuit$$

**THEOREM 5** *Under the assumptions of Theorem 1,*

$$\begin{aligned}
|e(s_i)| &= O(h^4), & (35)\\
|e'(\sigma_{ij})| &= O(h^3), & (36)\\
|e''(w_i)| &= O(h^2), \quad \textit{and} & (37)\\
\|e^{(k)}\|_\infty &= O(h^{3-k}), \ \ k = 0, 1, 2. & (38)
\end{aligned}$$

PROOF
Equations (35) and (37) are direct results of Theorems 3 and 1, respectively. To prove (36), we first consider some point $\sigma_i \equiv w(\tau_i + \mu h) = w_i + \mu h w'_i + O(h^2)$, for some constant $\mu$. Then by Taylor expansion

$$e'(w_i + \mu h w'_i + O(h^2)) = e'_i + \mu h w'_i e''_i + \frac{1}{2}\mu^2 h^2 (w'_i)^2 u^{(3)}_i + O(h^3). \tag{39}$$

By using Theorems 2 and 1 for $e'_i$ and $e''_i$, respectively, (39) becomes

$$\begin{aligned}
e'(\sigma_i) &= -\frac{1}{24}h^2(w'_i)^2 u^{(3)}_i + \frac{1}{2}\mu^2 h^2 (w'_i)^2 u^{(3)}_i + O(h^3) \\
&= \{-\frac{1}{24} + \frac{1}{2}\mu^2\}h^2(w'_i)^2 u^{(3)}_i + O(h^3).
\end{aligned}$$

With $\mu = \pm\sqrt{3}/6$, the $h^2$ term vanishes. By noting that $\tau_i + \mu h = x_i - \lambda h$ when $\lambda = (3 \pm \sqrt{3})/6$, we are done.

Now, $e(x) = e_i + cH_{i-1}e'_i + \frac{1}{2}(cH_{i-1})^2 e''_i + O(H_{i-1}^3)$ for $x \in [s_{i-1}, s_i]$ with $-1 < c < 1$. Thus, by using Lemma 1 and the bounds for the local errors we have

$$\begin{aligned}\|e(x)\|_\infty &= O(h^4) + O(h)O(h^2) + O(h^2)O(h) + O(h^3) \\ &= O(h^3).\end{aligned}$$

The other global errors follow similarly. ♢

## 2.2 Approximations to the high derivatives

We now develop approximations to the high derivatives of $u$ at the collocation points using values of $S$, $S'$ and $S''$ at the collocation points.

**THEOREM 6** *Under the assumptions of Theorem 1,*

$$u_i^{(k)} = \frac{2h_i S_{i-1}^{(k-2)} - 2(h_{i-1} + h_i) S_i^{(k-2)} + 2h_{i-1} S_{i+1}^{(k-2)}}{h_{i-1}(h_{i-1} + h_i)h_i} + O(h^2), \tag{40}$$

*for* $k = 3, 4$ *and* $i = 2, \ldots, N-1$.

PROOF
Equation (40) for $k = 3$ follows from the relation $u_i^{(3)} = \frac{2h_i u'_{i-1} - 2(h_{i-1}+h_i)u'_i + 2h_{i-1}u'_{i+1}}{h_{i-1}(h_{i-1}+h_i)h_i} + O(h_i - h_{i-1})$, (22), and the use of Lemma 1 for $h_i - h_{i-1}$ and $h_i^a + h_{i-1}^b$. Similarly, equation (40) for $k = 4$ follows from the relation $u_i^{(4)} = \frac{2h_i u''_{i-1} - 2(h_{i-1}+h_i)u''_i + 2h_{i-1}u''_{i+1}}{h_{i-1}(h_{i-1}+h_i)h_i} + O(h_i - h_{i-1})$, (10), and the use of Lemma 1 for $h_i - h_{i-1}$, $h_i^a - h_{i-1}^b$ and $h_i^a + h_{i-1}^b$. ♢

For convenience, we adopt the notation $\sqcap u_i \equiv \frac{2h_i u_{i-1} - 2(h_{i-1}+h_i)u_i + 2h_{i-1}u_{i+1}}{h_{i-1}(h_{i-1}+h_i)h_i}$. From Theorem 6 we have $u_i^{(4)} = \sqcap S''_i + O(h^2)$ and $u_i^{(3)} = \sqcap S'_i + O(h^2)$ for $i = 2, \ldots, N-1$.

**COROLLARY 1** *Under the assumptions of Theorem 1, for* $k = 3, 4$*,*

$$u_0^{(k)} = \frac{(h_0^b + h_1)(h_1 + h_2) - h_0^b h_2}{h_1 h_2} \sqcap S_2^{(k-2)} - \frac{(h_0^b + h_1)}{h_2} \sqcap S_3^{(k-2)} + O(h^2), \tag{41}$$

$$u_1^{(k)} = \frac{(h_1 + h_2) \sqcap S_2^{(k-2)} - h_1 \sqcap S_3^{(k-2)}}{h_2} + O(h^2), \tag{42}$$

$$u_N^{(k)} = \frac{(h_{N-1} + h_{N-2}) \sqcap S_{N-1}^{(k-2)} - h_{N-1} \sqcap S_{N-2}^{(k-2)}}{h_{N-2}} + O(h^2), \tag{43}$$

$$u_{N+1}^{(k)} = \frac{(h_N^a + h_{N-1})(h_{N-1} + h_{N-2}) - h_N^a h_{N-2}}{h_{N-1} h_{N-2}} \sqcap S_{N-1}^{(k-2)} - \frac{(h_N^a + h_{N-1})}{h_{N-2}} \sqcap S_{N-2}^{(k-2)} + O(h^2) \tag{44}$$

PROOF
Equations (41) and (42) come from the relations $u_0^{(k)} = \frac{(h_0^b + h_1)u_1^{(k)} - h_0^b u_2^{(k)}}{h_1} + O(h^2)$, $u_1^{(k)} = \frac{(h_1 + h_2)u_2^{(k)} - h_1 u_3^{(k)}}{h_2} + O(h^2)$ and Theorem 6. Equations (44) and (43) follow similarly. ♢

## 2.3 The Method of Optimal QSC

Consider solving the BVP (1)-(2). Based on the relations from Theorems 1-4 we observe that the interpolant $S$ of $u$ satisfies the relations

$$\mathbf{L}S_i = g_i - \frac{r_i}{24}\{4(h_i^a - h_{i-1}^b)u_i^{(3)} + (h_i^a + h_{i-1}^b)^2 u_i^{(4)}\} + \frac{p_i}{24}(h_i^a + h_{i-1}^b)^2 u_i^{(3)} + O(h^4), \;\; i = 1, \ldots, N \quad (45)$$

and

$$\begin{aligned} \mathbf{B}S_0 &= \gamma_0 - \frac{\beta_0}{12}\{(h_1^a + h_0^b)^2 - 4(h_1^a - h_0^b)(h_1^a + h_0^b)\}u_0^{(3)} + O(h^4), \\ \mathbf{B}S_{N+1} &= \gamma_{N+1} - \frac{\beta_{N+1}}{12}\{(h_N^a + h_{N-1}^b)^2 + 4(h_N^a - h_{N-1}^b)(h_N^a + h_{N-1}^b)\}u_{N+1}^{(3)} + O(h^4). \quad (46) \end{aligned}$$

Notice that due to Lemma 1, Theorem 6 and Corollary 1, relations (45) and (46) can be written as

$$\mathbf{L}S_i = g_i - P_{\mathbf{L}}S_i + O(h^4), \;\; i = 1, \ldots, N \quad (47)$$

and

$$\mathbf{B}S_i = \gamma_i - P_{\mathbf{B}}S_i + O(h^4), \;\; i = 0, N+1, \quad (48)$$

where

$$\begin{aligned} P_{\mathbf{L}}S_1 &= \frac{r_1}{24}\{4(h_1^a - h_0^b)\frac{(h_1 + h_2) \sqcap S_2' - h_1 \sqcap S_3'}{h_2} + H_0^2 \frac{(h_1 + h_2) \sqcap S_2'' - h_1 \sqcap S_3''}{h_2}\} \\ &- \frac{p_1}{24} H_0^2 \frac{(h_1 + h_2) \sqcap S_2' - h_1 \sqcap S_3'}{h_2}, \quad (49) \\ P_{\mathbf{L}}S_i &= \frac{r_i}{24}\{(h_i - h_{i-1}) \sqcap S_i' + H_{i-1}^2 \sqcap S_i''\} - \frac{p_i}{24} H_{i-1}^2 \sqcap S_i', \quad i = 2, \ldots, N-1 \quad (50) \\ P_{\mathbf{L}}S_N &= \frac{r_N}{24}\{4(h_N^a - h_{N-1}^b)\frac{(h_{N-1} + h_{N-2}) \sqcap S_{N-1}' - h_{N-1} \sqcap S_{N-2}'}{h_{N-2}} \\ &\quad + H_{N-1}^2 \frac{(h_{N-1} + h_{N-2}) \sqcap S_{N-1}'' - h_{N-1} \sqcap S_{N-2}''}{h_{N-2}}\} \\ &- \frac{p_N}{24} H_{N-1}^2 \frac{(h_{N-1} + h_{N-2}) \sqcap S_{N-1}' - h_{N-1} \sqcap S_{N-2}'}{h_{N-2}}, \quad (51) \\ P_{\mathbf{B}}S_0 &= \frac{\beta_0}{12}\{H_0^2 - 4(h_1^a - h_0^b)H_0\}\{\frac{(h_0^b + h_1)(h_1 + h_2) - h_0^b h_2}{h_1 h_2} \sqcap S_2' - \frac{(h_0^b + h_1)}{h_2} \sqcap S_3'\}, \quad (52) \\ P_{\mathbf{B}}S_{N+1} &= \frac{\beta_{N+1}}{12}\{H_{N-1}^2 + 4(h_N^a - h_{N-1}^b)H_{N-1}\} \\ &\quad \{\frac{(h_N^a + h_{N-1})(h_{N-1} + h_{N-2}) - h_N^a h_{N-2}}{h_{N-1}h_{N-2}} \sqcap S_{N-1}' - \frac{(h_N^a + h_{N-1})}{h_{N-2}} \sqcap S_{N-2}'\}. \quad (53) \end{aligned}$$

Let $T_{w\mathbf{L}} \equiv \{w_i, i = 1, \ldots, N\}$ and $T_{w\mathbf{B}} \equiv \{s_0, s_N\}$. We now present an optimal *two-step QSC method* to determine an approximation $u_\Delta \in S_{\Delta_w}$ of the solution of the BVP (1)-(2).

**Step 1:** Determine $u_{\Delta[1]} \in S_{\Delta_w}$ by forcing it to satisfy

$$\begin{aligned} \mathbf{L}u_{\Delta[1]} &= g \;\text{ in }\; T_{w\mathbf{L}}, \quad (54) \\ \mathbf{B}u_{\Delta[1]} &= \gamma \;\text{ on }\; T_{w\mathbf{B}}. \quad (55) \end{aligned}$$

**Step 2:** Determine $u_\Delta \in S_{\Delta_w}$ by forcing it to satisfy

$$\begin{aligned} \mathbf{L}u_\Delta &= g - P_\mathbf{L} u_{\Delta[1]} \quad \text{in} \quad T_{w\mathbf{L}}, && (56)\\ \mathbf{B}u_\Delta &= \gamma - P_\mathbf{B} u_{\Delta[1]} \quad \text{on} \quad T_{w\mathbf{B}}. && (57)\end{aligned}$$

We also present an *optimal one-step QSC method* which determines an approximation $\widehat{u_\Delta} \in S_{\Delta_w}$ of the solution of the BVP (1)-(2) by the equations

$$\begin{aligned} \tilde{\mathbf{L}}\widehat{u_\Delta} \equiv \mathbf{L}\widehat{u_\Delta} + P_\mathbf{L}\widehat{u_\Delta} &= g \quad \text{in} \quad T_{w\mathbf{L}}, && (58)\\ \tilde{\mathbf{B}}\widehat{u_\Delta} \equiv \mathbf{B}\widehat{u_\Delta} + P_\mathbf{B}\widehat{u_\Delta} &= \gamma \quad \text{on} \quad T_{w\mathbf{B}}. && (59)\end{aligned}$$

From relations (47)-(48), we get an intuition why $u_\Delta$ and $\widehat{u_\Delta}$ are optimal approximations to $u$. A formal proof, though it is very similar to the one in [10], is given in the next section for completeness.

## 2.4 Convergence Analysis and Error Bounds

The convergence analysis of the QSC methods for non-uniform partitions is similar to that for uniform partitions [10], but we present it here mainly for completeness. For this section we will assume that $r(x) = 1$ in (1) and that, without loss of generality, we have homogeneous boundary conditions, that is $\mathbf{B}u = 0$ in (2). In order to analyze the QSC methods, we adopt the approach used by many [16], [15], [7], [10], [13] for proving existence, uniqueness and error bounds for piecewise polynomial collocation. According to this approach, we introduce an integral representation of equations (54), (56), (58) and of the differential equation (1). For this purpose, we assume that the BVP $u'' = 0, \mathbf{B}u = 0$ has a unique solution. This implies that there is a Green's function $G(x,t)$ for this problem. If we assume that $u, u_{\Delta[1]}$, $u_\Delta$ and $\widehat{u_\Delta}$ satisfy homogeneous boundary conditions, then $u, u_{\Delta[1]}$ and $u_\Delta$ can be obtained via the Green's function $G(x,t)$ and the respective second derivatives. That is, we have

$$\begin{aligned} u(x) &= \int_0^1 G(x,t)u''(t)dt, & u'(x) &= \int_0^1 G_x(x,t)u''(t)dt,\\ u_{\Delta[1]}(x) &= \int_0^1 G(x,t)u''_{\Delta[1]}(t)dt, & u'_{\Delta[1]}(x) &= \int_0^1 G_x(x,t)u''_{\Delta[1]}(t)dt,\\ u_\Delta(x) &= \int_0^1 G(x,t)u''_\Delta(t)dt, & u'_\Delta(x) &= \int_0^1 G_x(x,t)u''_\Delta(t)dt,\\ \widehat{u_\Delta}(x) &= \int_0^1 G(x,t)\widehat{u_\Delta}''(t)dt, & \widehat{u_\Delta}'(x) &= \int_0^1 G_x(x,t)\widehat{u_\Delta}''(t)dt.\end{aligned}$$

Let $L$ be the set of bounded functions in $\overline{\Omega}$, that are continuous except possibly at the nodes $\{s_i\}_0^N$ of any partition, where their values are determined by right (without loss of generality) continuity for $s_i, i = 0, \ldots, N-1$, and by left continuity for $s_N$, and where we impose the additional condition that, for each $f \in L$, the limits $\lim_{x \to s_i -} f(x)$ exist for all $i = 1, \ldots, N-1$. Note that although $\lim_{x \to s_i -} f(x)$ exists, it may be different than $f(s_i)$. Note also that $L$ includes all continuous functions in $\overline{\Omega}$, and all piecewise constant functions with respect to any partition. We assume that the coefficient functions $p(x)$ and $q(x)$ in (1) are continuous, and introduce the compact operator $K$ that maps $L$ to $\mathbf{C}[\overline{\Omega}]$ by

$$Kf(x) = p(x)\int_0^1 G_x(x,t)f(t)dt + q(x)\int_0^1 G(x,t)f(t)dt. \qquad (60)$$

Equation (1) can now be written as

$$u'' + Ku'' = g \tag{61}$$

and equations (54) and (56) as

$$\begin{aligned} (u''_{\Delta[1]} + Ku''_{\Delta[1]})_i &= g_i, & (62)\\ (u''_{\Delta} + Ku''_{\Delta})_i &= \tilde{g}_i, & (63) \end{aligned}$$

respectively, where $\tilde{g} \equiv g - P_{\mathbf{L}} u_{\Delta[1]}$.

Furthermore, we introduce the operator $P_\Delta$ which maps functions in $L$ to step functions via piecewise constant interpolation at the midpoints $\{w_i\}_1^N$. That is, for $f \in L$, $P_\Delta f = f(w_i)$ for $x \in [s_{i-1}, s_i)$, $i = 1, \ldots, N$ and $P_\Delta f(s_N) = f(w_N)$. Based on the notation introduced, and taking into account the uniqueness of the piecewise constant interpolant for a given set of midpoint values, we can rewrite equations (62) and (63) in the form

$$\begin{aligned} u''_{\Delta[1]} + P_\Delta K u''_{\Delta[1]} &= P_\Delta g, & (64)\\ u''_{\Delta} + P_\Delta K u''_{\Delta} &= P_\Delta \tilde{g}, & (65) \end{aligned}$$

respectively, since $P_\Delta u''_{\Delta[1]} = u''_{\Delta[1]}$ and $P_\Delta u''_\Delta = u''_\Delta$.

By the definition of $P_\Delta$, $\|f - P_\Delta f\|_\infty \to 0$ as $h \to 0$ for all $f \in L$, that is, the sequence of operators $P_\Delta$ converges strongly to the identity operator $I : L \to L$. Moreover, $\|P_\Delta f\|_\infty \le \|f\|_\infty$. The compactness of $K$, and the strong convergence and uniform boundedness of $P_\Delta$ imply that $\|K - P_\Delta K\|_\infty \to 0$ as $h \to 0$.

**THEOREM 7** *If*
*(A1) the coefficients* $p$ *and* $q$*, and the right-hand side* $g$ *are in* $\mathbf{C}[\Omega]$*,*
*(A2) the BVP* $\mathbf{L}u = g, \mathbf{B}u = 0$ *has a unique solution,*
*(A3) the BVP* $u'' = 0, \mathbf{B}u = 0$ *has a unique solution,*
*(A4) the assumptions of Theorem 1 hold,*
*then* $u_{\Delta[1]} \in S_{\Delta_w}$ *defined by (54)-(55) exists, is unique, and satisfies the global error estimates*

$$\|u - u_{\Delta[1]}\|_\infty = O(h^2),\ \|u' - u'_{\Delta[1]}\|_\infty = O(h^2),\ \text{ and }\ \|u'' - u''_{\Delta[1]}\|_\infty = O(h), \tag{66}$$

*and the local error estimates*

$$|(u - u_{\Delta[1]})^{(k)}(w_i)| = O(h^2),\ k = 0, 1, 2,\ \text{ and }\ i = 1, \ldots, N. \tag{67}$$

PROOF
Assumption (A2) and the equivalence of (1) and (61) imply that $(I + K)^{-1}$ exists and is bounded. Since $\|K - P_\Delta K\|_\infty \to 0$ as $h \to 0$, Neumann's theorem [15] implies that $(I + P_\Delta K)^{-1}$ exists and is uniformly bounded for sufficiently small $h$. The unique solvability of (64) follows from the existence and uniform boundedness of $(I + P_\Delta K)^{-1}$. The existence and uniqueness of $u''_{\Delta[1]}$ and the use of the Green's function $G(x, t)$ in association with (55) lead to the existence and uniqueness of $u_{\Delta[1]}$.

To establish the error bounds (66) and (67), simplify relations (47) and (48) to

$$\begin{aligned} \mathbf{L}S_i &= g_i + O(h^2),\ \ i = 1, \ldots, N \text{ and} & (68)\\ \mathbf{B}S_i &= O(h^2),\ \ i = 0, N+1. & (69) \end{aligned}$$

From (54), (55), (68) and (69), we obtain

$$\begin{aligned} \mathbf{L}(S - u_{\Delta[1]})_i &= O(h^2), \ \ i = 1, \ldots, N \text{ and} \quad (70) \\ \mathbf{B}(S - u_{\Delta[1]})_i &= O(h^2), \ \ i = 0, N+1. \quad (71) \end{aligned}$$

Notice that, because of assumption (A3), there is a linear function $l$ such that $\mathbf{B}(S - u_{\Delta[1]})_j = \mathbf{B}l_j = O(h^2)$, $j = 0, N+1$, with $\|l\|_\infty = O(h^2)$, $\|l'\|_\infty = O(h^2)$, and thus $\mathbf{L}l_i = O(h^2)$, $i = 1, \ldots, N$. From $\mathbf{L}(S - u_{\Delta[1]} - l)_i = O(h^2)$ and $\mathbf{B}(S - u_{\Delta[1]} - l)_j = 0$, we conclude that

$$(I + P_\Delta K)(S'' - u''_{\Delta[1]} - l'') = O(h^2).$$

The uniform boundedness of $(I + P_\Delta K)^{-1}$ yields

$$\|S'' - u''_{\Delta[1]} - l''\|_\infty = O(h^2). \quad (72)$$

Since $\mathbf{B}(S - u_{\Delta[1]} - l)_j = 0$, $j = 0, N+1$, we have

$$(S - u_{\Delta[1]} - l)^{(k)}(x) = \int_0^1 G_x^{(k)}(x,t)(S'' - u''_{\Delta[1]} - l'')(t)dt.$$

This implies that

$$\|S - u_{\Delta[1]} - l\|_\infty = O(h^2), \ \|S' - u'_{\Delta[1]} - l'\|_\infty = O(h^2). \quad (73)$$

The error bounds (66) and (67) now follow from Theorem 5, the definition of $l$, the triangle inequality and relations (72) and (73). ♢

**REMARK 4** Assumption (A4) implies that $u \in \mathbf{C}^6[\overline{\Omega}]$. However, we can relax this condition and assume that $u \in \mathbf{C}^4[\overline{\Omega}]$, and still obtain the results of Theorem 7, since the simplified relations $\mathbf{L}S_i = g_i + O(h^2)$ and $\mathbf{B}S_j = O(h^2)$ arising from (45) and (46), respectively, can be proved under the relaxed assumption.

The next theorem gives the optimal error bounds that $u_\Delta$ satisfies. To proceed with the proof, we make the assumption that
(A5) $S'_i - (u'_{\Delta[1]})_i$ and $S''_i - (u''_{\Delta[1]})_i$, $i = 0, \ldots, N+1$, have smooth expansions.
We do not provide a mathematical proof of this statement, however, extensive numerical experiments have shown that the statement holds. Of course, we can equivalently assume that $u'_i - (u'_{\Delta[1]})_i$ and $u''_i - (u''_{\Delta[1]})_i$, have smooth expansions, since, by Theorems 1, 2, and 4, we know that $S'_i - u'_i$ and $S''_i - u''_i$ have smooth expansions.

**THEOREM 8** *Under the assumptions of Theorem 7, and assumption (A5), $u_\Delta \in S_{\Delta_w}$ defined by (56)-(57) exists, is unique, and satisfies the global error estimates*

$$\|(u - u_\Delta)^{(k)}\|_\infty = O(h^{3-k}), \ k = 0, 1, 2, \quad (74)$$

*and the local error estimates*

$$|(u - u_\Delta)(x)| = O(h^4) \textit{ for } x = s_i \textit{ and } w_i, \ |(u - u_\Delta)'(\sigma_{ij})| = O(h^3), \ |(u - u_\Delta)''(w_i)| = O(h^2). \quad (75)$$

PROOF
As in the proof of Theorem 7, the existence and uniform boundedness of $(I + P_\Delta K)^{-1}$ imply existence and uniqueness of $u''_\Delta$ in (65), which in turn implies existence and uniqueness of $u_\Delta$.

We now establish the optimal error bounds (74) and (75). By (A5), we conclude that $\sqcap S_i^{(k)} = \sqcap (u_{\Delta[1]}^{(k)})_i + O(h^2)$, $k = 1, 2$. Therefore relations (47) and (48) imply

$$
\begin{aligned}
\mathbf{L}S_i &= \tilde{g}_i + O(h^4), \ \ i = 1, \ldots, N \text{ and} &(76)\\
\mathbf{B}S_i &= \tilde{\gamma}_i + O(h^4), \ \ i = 0, N+1, &(77)
\end{aligned}
$$

where $\tilde{\gamma} \equiv \gamma - P_{\mathbf{B}} u_{\Delta[1]}$. From (56), (57), (76) and (77), we obtain

$$
\begin{aligned}
\mathbf{L}(S - u_\Delta)_i &= O(h^4), \ \ i = 1, \ldots, N \text{ and} &(78)\\
\mathbf{B}(S - u_\Delta)_i &= O(h^4), \ \ i = 0, N+1. &(79)
\end{aligned}
$$

Notice that there is a linear function $l$ such that $\mathbf{B}(S - u_\Delta)_j = \mathbf{B}l_j = O(h^4)$, $j = 0, N+1$, with $\|l\|_\infty = O(h^4)$, $\|l'\|_\infty = O(h^4)$, and thus $\mathbf{L}l_i = O(h^4)$, $i = 1, \ldots, N$. From $\mathbf{L}(S - u_\Delta - l)_i = O(h^4)$ and $\mathbf{B}(S - u_\Delta - l)_j = 0$, we conclude that

$$(I + P_\Delta K)(S'' - u''_\Delta - l'') = O(h^4).$$

Applying the arguments used in the proof of Theorem 7 and with $\sigma_{ij} = w(x_i - \lambda_j h)$, we obtain the optimal results (74) and (75). $\diamondsuit$

We now turn to the analysis of the one-step QSC method. In order to write (58) in integral form, we introduce some more notations. Let $D_\Delta : L \to \mathbf{R}^N$ be defined by $(D_\Delta f)_i = f(\tau_i)$ for $i = 1, \ldots, N$, $E_p = \text{diag}(p(\tau_i))$, $E_q = \text{diag}(q(\tau_i))$ be $N \times N$ diagonal matrices and $M_\Delta$ be the operator which maps $\mathbf{R}^N$ to step functions via piecewise constant interpolation at the midpoints $\{w_i\}_1^N$.

We note that from Theorem 6 and Lemma 1 we have

$$u_i^{(k)} = \frac{S_{i-1}^{(k-2)} - 2S_i^{(k-2)} + S_{i+1}^{(k-2)}}{h^2 (w'_i)^2} + \frac{O(h) S_{i-1}^{(k-2)} + O(h^2) S_i^{(k-2)} + O(h) S_{i+1}^{(k-2)}}{h^2 (w'_i)^2} + O(h^2),$$

for $k = 3, 4$ and $i = 2, \ldots, N-1$. Close to the boundary we have

$$
\begin{aligned}
u_1^{(k)} &= \frac{2S_1^{(k-2)} - 5S_2^{(k-2)} + 4S_3^{(k-2)} - S_4^{(k-2)}}{h^2 (w'_1)^2}\\
&+ \frac{O(h) S_1^{(k-2)} + O(h) S_2^{(k-2)} + O(h) S_3^{(k-2)} + O(h) S_4^{(k-2)}}{h^2 (w'_1)^2} + O(h^2),\\
u_N^{(k)} &= \frac{2S_N^{(k-2)} - 5S_{N-1}^{(k-2)} + 4S_{N-2}^{(k-2)} - S_{N-3}^{(k-2)}}{h^2 (w'_N)^2}\\
&+ \frac{O(h) S_N^{(k-2)} + O(h) S_{N-1}^{(k-2)} + O(h) S_{N-2}^{(k-2)} + O(h) S_{N-3}^{(k-2)}}{h^2 (w'_N)^2} + O(h^2),
\end{aligned}
$$

where each $O(h)$ term involves the value of at least one of the first three derivatives of $w$. Hence, from Theorems 1 and 2, we have

$$u''_i = \frac{1}{24}(S''_{i-1} + 22S''_i + S''_{i+1}) + O(h) S''_{i-1} + O(h^2) S''_i + O(h) S''_{i+1} \quad (80)$$

$$
\begin{aligned}
&+ \quad \frac{w_i''}{24(w_i')^2}(S_{i-1}' - 2S_i' + S_{i+1}') + O(h)S_{i-1}' + O(h^2)S_i' + O(h)S_{i+1}' + O(h^4) \qquad (81)\\
u_i' &= \frac{1}{24}(-S_{i-1}' + 26S_i' - S_{i+1}') + O(h)S_{i-1}' + O(h^2)S_i' + O(h)S_{i+1}' + O(h^4), \qquad (82)
\end{aligned}
$$

for $i = 2, \ldots, N-1$. It is important to note that there is at most a constant number (actually two) of $O(h)$ terms in each of (80), (81) and (82).

We have similar results for $u_i''$ and $u_i'$ for $i = 1, N$. Let $\Psi$ and $\Phi$ be the almost tridiagonal $N \times N$ matrices defined in [10], that is,
$\Psi_{i,i} = 22/24$, $\Psi_{i-1,i} = \Psi_{i+1,i} = 1/24$, $i = 2, \ldots, N-1$,
$\Psi_{1,1} = \Psi_{N,N} = 26/24$, $\Psi_{1,2} = \Psi_{N,N-1} = -5/24$, $\Psi_{1,3} = \Psi_{N,N-2} = 4/24$, $\Psi_{1,4} = \Psi_{N,N-3} = -1/24$,
$\Phi_{i,i} = 26/24$, $\Phi_{i-1,i} = \Phi_{i+1,i} = -1/24$, $i = 2, \ldots, N-1$,
$\Phi_{1,1} = \Phi_{N,N} = 22/24$, $\Phi_{1,2} = \Phi_{N,N-1} = 5/24$, $\Phi_{1,3} = \Phi_{N,N-2} = -4/24$, $\Phi_{1,4} = \Phi_{N,N-3} = 1/24$, with the rest of the entries of $\Psi$ and $\Phi$ equal to 0. Let also $\Gamma$ be a matrix which differs from $\Psi$ only on the diagonal, more specifically, $\Gamma_{i,i} = -2/24$, $i = 2, \ldots, N-1$, and $\Gamma_{1,1} = \Gamma_{N,N} = 2/24$. Relations (80) for $i = 2, \ldots, N-1$, and the respective boundary relations give rise to the matrix $\tilde{\Psi} = \Psi + h\Sigma$, where $h\Sigma$ is an $N \times N$ matrix that represents the $O(h)$ and $O(h^2)$ terms. Similarly, relations (81) for $i = 2, \ldots, N-1$, and the respective boundary relations give rise to the matrix $\tilde{\Gamma} = \Theta\Gamma + h\mathrm{Z}$, where $\Theta = \mathrm{diag}(\frac{w_i''}{24(w_i')^2})$, and $h\mathrm{Z}$ is an $N \times N$ matrix representing the $O(h)$ and $O(h^2)$ terms. Finally, relations (82) for $i = 2, \ldots, N-1$, and the respective boundary relations give rise to the matrix $\tilde{\Phi} = \Phi + h\Upsilon$, where $h\Upsilon$ is an $N \times N$ matrix representing the $O(h)$ and $O(h^2)$ terms. Note that the matrices $\Sigma$, $\mathrm{Z}$ and $\Upsilon$ have at most 4 non-zero entries per row, and their entries involve values of the first 3 derivatives of $w$. Thus, assuming $w \in \mathbf{C}^3$, $||\Sigma||_\infty$, $||\mathrm{Z}||_\infty$ and $||\Upsilon||_\infty$ are bounded. Since $w'(x) > 0$, $\forall x \in \overline{\Omega}$, $||\Theta||_\infty$ is bounded as well. Also, from the boundedness of $||\Psi^{-1}||_\infty$ and the fact that $\tilde{\Psi} = \Psi(\mathbf{I} + h\Psi^{-1}\Sigma)$, we have the boundedness of $||\tilde{\Psi}^{-1}||_\infty$, for sufficiently small $h$.
We now rewrite (58) equivalently as

$$
\widehat{u_\Delta}'' + R_\Delta \widehat{u_\Delta}'' = M_\Delta \tilde{\Psi}^{-1} D_\Delta g, \tag{83}
$$

where $R_\Delta$ is the integral operator defined by

$$
\begin{aligned}
R_\Delta f \;\equiv\; & M_\Delta \tilde{\Psi}^{-1}\tilde{\Gamma} D_\Delta \int_0^1 G_x(x,t) f(t) dt + M_\Delta \tilde{\Psi}^{-1} E_p \tilde{\Phi} D_\Delta \int_0^1 G_x(x,t) f(t) dt \\
+ \; & M_\Delta \tilde{\Psi}^{-1} E_q D_\Delta \int_0^1 G(x,t) f(t) dt.
\end{aligned}
$$

**LEMMA 2** *The sequence of operators $R_\Delta$ converges strongly to the integral operator $K$ in $L$.*

PROOF
Consider the convergence of $\|R_\Delta f - M_\Delta D_\Delta K f\|_\infty$ for $f \in L$. According to the definition of $R_\Delta$ and the use of the triangular inequality we obtain

$$
\begin{aligned}
\|R_\Delta f - M_\Delta D_\Delta K f\|_\infty \;\leq\; & \|M_\Delta \tilde{\Psi}^{-1}\tilde{\Gamma} D_\Delta \textstyle\int_0^1 G_x f dt\|_\infty \\
+ \; & \|M_\Delta \tilde{\Psi}^{-1} E_p \tilde{\Phi} D_\Delta \textstyle\int_0^1 G_x f dt - M_\Delta \tilde{\Psi}^{-1}\tilde{\Psi} E_p D_\Delta \int_0^1 G_x f dt\|_\infty \\
+ \; & \|M_\Delta \tilde{\Psi}^{-1} E_q D_\Delta \textstyle\int_0^1 G f dt - M_\Delta \tilde{\Psi}^{-1}\tilde{\Psi} E_q D_\Delta \int_0^1 G f dt\|_\infty.
\end{aligned}
$$

From the boundedness of $\|M_\Delta\|_\infty$, $\|\tilde{\Psi}^{-1}\|_\infty$, $\|\mathrm{Z}\|_\infty$, $\|\Sigma\|_\infty$ and $\|\Upsilon\|_\infty$, and the definitions of $\tilde{\Gamma}$, $\tilde{\Psi}$ and $\tilde{\Phi}$, we have

$$\begin{array}{rl} \|R_\Delta f - M_\Delta D_\Delta K f\|_\infty \le C & \{\|\Theta\Gamma D_\Delta \int_0^1 G_x f dt\|_\infty \\ & + \|E_p \Phi D_\Delta \int_0^1 G_x f dt - \Psi E_p D_\Delta \int_0^1 G_x f dt\|_\infty \\ & + \|E_q D_\Delta \int_0^1 G f dt - \Psi E_q D_\Delta \int_0^1 G f dt\|_\infty\} + O(h), \end{array} \tag{84}$$

for some constant $C > 0$. Notice that $\Gamma D_\Delta \int_0^1 G_x f dt$ is bounded by the modulus of continuity over a $3h$ interval. Hence, the first term of (84) is $O(h)$. The treatment of the other two terms and the rest of the proof of the lemma is identical to that in [10]. ♢

With techniques identical to those in [10], we can now show that, under the assumptions of Theorem 7, $\widehat{u_\Delta}$ exists, is unique, and satisfies the same error bounds as $u_\Delta$.

# 3 Non-uniform Partition Cubic Spline Collocation

In this section, we develop an optimal CSC method for second-order two-point BVPs on non-uniform partitions. Our results reduce to those in [7], if the partition is uniform. For CSC, the partition points (nodes) of $\Delta_w$ will also be the collocation points.

For any given $w(x)$, we denote by $S^3_{\Delta_w}$ the cubic spline space with respect to the partition $\Delta_w$, that is the space of piecewise cubic polynomials which are globally in $\mathbf{C}^2$. Let $S \in S^3_{\Delta_w}$ be the cubic spline interpolant of $u$ such that

$$\begin{array}{ll} S''(s_0) & = u''(s_0) - \frac{1}{24} H_0 (5H_0 - 4H_1 + H_2) u^{(4)}(s_0), \\ S(s_i) & = u(s_i), \quad i = 0, \ldots, N, \\ S''(s_N) & = u''(s_N) - \frac{1}{24} H_{N-1}(5H_{N-1} - 4H_{N-2} + H_{N-3}) u^{(4)}(s_N). \end{array} \tag{85}$$

The symbols $\alpha_{ki}$, $\beta_{ki}$, $\gamma_{ki}$, $k = 0, 2$, $i = 1, \ldots, N-1$, denote scalars defined within the section and different from those in Section 2.

We will develop expansions of the errors of the derivatives of $S$ at the collocation points. We consider the second derivative error at the grid points. By setting up the equations,

$$\alpha_{2i} S''(s_{i-1}) + \beta_{2i} S''(s_i) + \gamma_{2i} S''(s_{i+1}) = \alpha_{0i} S(s_{i-1}) + \beta_{0i} S(s_i) + \gamma_{0i} S(s_{i+1}) \tag{86}$$

for $i = 1, \ldots, N-1$, and solving for $\alpha_{0i}$, $\beta_{0i}$, $\gamma_{0i}$, $\alpha_{2i}$, $\beta_{2i}$ and $\gamma_{2i}$ with Maple we have

$$\begin{array}{ll} \alpha_{0i} = \frac{6}{H_{i-1}}, & \alpha_{2i} = H_{i-1}, \\ \beta_{0i} = -\frac{6(H_i + H_{i-1})}{H_{i-1} H_i}, & \beta_{2i} = 2(H_i + H_{i-1}), \\ \gamma_{0i} = \frac{6}{H_i}, & \gamma_{2i} = H_i. \end{array} \tag{87}$$

The proof of the following lemma is trivial and is therefore omitted.

**LEMMA 3** *If $w(x) \in \mathbf{C}^3[\overline{\Omega}]$, then*

$$H_i = h w'(x_i) + \frac{h^2}{2} w''(x_i) + O(h^3), \quad H_{i-1} = h w'(x_i) - \frac{h^2}{2} w''(x_i) + O(h^3). \tag{88}$$

*where the index $i$ takes all possible values consistent with the definition of the stepsizes.*

**THEOREM 9** *If* $u \in \mathbf{C}^6[\overline{\Omega}]$*,* $w(x) : \overline{\Omega} \to \overline{\Omega}$ *is a bijective function in* $\mathbf{C}^3$*, with* $w'(x) > 0$*,* $\forall x \in \overline{\Omega}$*,* $w^{-1} \in \mathbf{C}^1[\overline{\Omega}]$*, and* $S$ *is defined by (85), then*

$$S''(s_i) = u''(s_i) - \frac{1}{12} H_i H_{i-1} u^{(4)}(s_i) + O(h^4) \tag{89}$$

*for* $i = 1, \ldots, N-1$.

PROOF

Since $S(s_i) = u(s_i)$, $i = 0, \ldots, N$ and $u \in \mathbf{C}^6$, the right-hand side of (86) after applying Taylor expansion to $u(s_{i\pm 1})$ is equivalent to

$$\begin{aligned} \alpha_{0i} u(s_{i-1}) + \beta_{0i} u(s_i) + \gamma_{0i} u(s_{i+1}) = &\ 3(H_i + H_{i-1})[u''(s_i) + \frac{1}{3}(H_i - H_{i-1}) u^{(3)}(s_i) \\ + &\ \frac{1}{12} H_i H_{i-1} u^{(4)}(s_i)] + O(H^4),\ i = 1, \ldots, N-1. \end{aligned} \tag{90}$$

Moreover, using Taylor expansion of a function $f \in \mathbf{C}^4$ we can show

$$\begin{aligned} \alpha_{2i} f(s_{i-1}) + \beta_{2i} f(s_i) + \gamma_{2i} f(s_{i+1}) = &\ 3(H_i + H_{i-1})[f(s_i) + \frac{1}{3}(H_i - H_{i-1}) f'(s_i) \\ + &\ \frac{1}{6} H_i H_{i-1} f''(s_i)] + O(H^4),\ i = 1, \ldots, N-1. \end{aligned} \tag{91}$$

Let $\wedge u(s_i) \equiv \frac{1}{3(H_i + H_{i-1})}(\alpha_{2i} u(s_{i-1}) + \beta_{2i} u(s_i) + \gamma_{2i} u(s_{i+1}))$, $i = 1, \ldots, N-1$. Then by using Lemma 3 to expand the truncation errors of (90) and (91), we have

$$\wedge S''(s_i) = u''(s_i) + \frac{1}{3}(H_i - H_{i-1}) u^{(3)}(s_i) + \frac{1}{12} H_i H_{i-1} u^{(4)}(s_i) + O(h^4),\ i = 1, \ldots, N-1.$$

Using the definition of $\wedge$, relation (91) becomes

$$\begin{aligned} \wedge f(s_i) = &\ f(s_i) + \frac{1}{3}(H_i - H_{i-1}) f'(s_i) \\ + &\ \frac{1}{6}(H_i^2 - H_i H_{i-1} + H_{i-1}^2) f''(s_i) + O(h^4),\ i = 1, \ldots, N-1. \end{aligned}$$

Let $v = h^2 w' \circ w^{-1}$. By using Lemma 3 and Taylor expansion, we have

$$v(s_i) = H_i H_{i-1} + O(h^4),\ i = 1, \ldots, N-1, \tag{92}$$

$$v(s_i) = H_i(5H_i - 4H_{i+1} + H_{i+2})/2 + O(h^4),\ i = 0, \ldots, N-3, \tag{93}$$

$$v(s_i) = H_{i-1}(5H_{i-1} - 4H_{i-2} + H_{i-3})/2 + O(h^4),\ i = 3, \ldots, N. \tag{94}$$

Now, by taking $f = u'' - \frac{1}{12} v u^{(4)}$ and using (92), we have

$$\wedge(S''(s_i) - f(s_i)) = O(h^4),\ i = 1, \ldots, N-1. \tag{95}$$

Moreover, from the end conditions in (85) and relations (93) and (94), we have

$$S''(s_0) - f(s_0) = O(h^4) \tag{96}$$

and

$$S''(s_N) - f(s_N) = O(h^4). \tag{97}$$

Relations (96), (95) and (97) (in that order) form a system of equations that is strictly diagonally dominant. More specifically, $\frac{|\beta_{2i}|-|\alpha_{2i}|-|\gamma_{2i}|}{3(H_i+H_{i-1})} = \frac{1}{3}$. Therefore, the infinity norm of the inverse of the matrix of equations (96), (95) and (97) is bounded by 3. This implies that

$$S''(s_i) - f(s_i) = O(h^4), \quad i = 0, \ldots, N. \qquad \diamondsuit$$

The proof of the following theorem is similar to the proofs of Theorems 4 and 5, and is therefore omitted.

**THEOREM 10** *Under the assumptions of Theorem 9,*

$$|e'(w_i)| = O(h^4), \tag{98}$$

$$|e'(s_i)| = O(h^4), \tag{99}$$

$$|e''(\sigma_{ij})| = O(h^3), \tag{100}$$

$$|e^{(3)}(w_i)| = O(h^2), \quad \textit{and} \tag{101}$$

$$\|e^{(k)}\|_\infty = O(h^{4-k}), \quad k = 0, \ldots, 3. \tag{102}$$

The following theorem is the cubic spline counterpart of Theorem 6 for quadratic splines and its proof is similar to that proof.

**THEOREM 11** *Under the assumptions of Theorem 9,*

$$u^{(4)}(s_0) = \frac{(H_0 + H_1) \sqcap S''(s_1) - H_0 \sqcap S''(s_2)}{H_1} + O(h^2), \tag{103}$$

$$u^{(4)}(s_i) = \sqcap S''(s_i) + O(h^2), \quad i = 1, \ldots, N-1, \tag{104}$$

$$u^{(4)}(s_N) = \frac{(H_{N-1} + H_{N-2}) \sqcap S''(s_{N-1}) - H_{N-1} \sqcap S''(s_{N-2})}{H_{N-2}} + O(h^2), \tag{105}$$

*where* $\sqcap u(s_i) \equiv \frac{2H_i u(s_{i-1}) - 2(H_{i-1}+H_i)u(s_i) + 2H_{i-1}u(s_{i+1})}{H_{i-1}(H_{i-1}+H_i)H_i}$.

Consider solving the BVP (1)-(2). Based on the relations from Theorems 9-10, we observe that the interpolant $S$ of $u$ satisfies the relations

$$\begin{aligned}
\mathbf{L}S(s_0) &= g(s_0) - \frac{r(s_0)}{24} H_0(5H_0 - 4H_1 + H_2)u^{(4)}(s_0) + O(h^4), \\
\mathbf{L}S(s_i) &= g(s_i) - \frac{r(s_i)}{12} H_i H_{i-1} u^{(4)}(s_i) + O(h^4), \quad i = 1, \ldots, N-1, \\
\mathbf{L}S(s_N) &= g(s_N) - \frac{r(s_N)}{24} H_{N-1}(5H_{N-1} - 4H_{N-2} + H_{N-3})u^{(4)}(s_N) + O(h^4),
\end{aligned} \tag{106}$$

and

$$\mathbf{B}S(s_i) = \gamma(s_i) + O(h^4), \quad i = 0, N. \tag{107}$$

Notice that due to Theorem 11, relations (106) can be written as

$$\mathbf{L}S(s_i) = g(s_i) - P_{\mathbf{L}}S(s_i) + O(h^4), \quad i = 0, \ldots, N, \tag{108}$$

where

$$P_{\mathbf{L}}S(s_0) = \frac{r(s_0)}{24}H_0(5H_0 - 4H_1 + H_2)\frac{(H_0 + H_1) \sqcap S''(s_1) - H_0 \sqcap S''(s_2)}{H_1}, \tag{109}$$

$$P_{\mathbf{L}}S(s_i) = \frac{r(s_i)}{12}H_i H_{i-1} \sqcap S''(s_i), \;\; i = 1, \ldots, N-1, \tag{110}$$

$$P_{\mathbf{L}}S(s_N) = \frac{r(s_N)}{24}H_{N-1}(5H_{N-1} - 4H_{N-2} + H_{N-3}) \frac{(H_{N-1} + H_{N-2}) \sqcap S''(s_{N-1}) - H_{N-1} \sqcap S''(s_{N-2})}{H_{N-2}}. \tag{111}$$

We are now ready to present an *optimal two-step CSC method* to determine an approximation $u^3_\Delta \in S^3_{\Delta_w}$ of the solution of the BVP (1)-(2).
**Step 1:** Determine $u^3_{\Delta[1]} \in S^3_{\Delta_w}$ by forcing it to satisfy

$$\mathbf{L}u^3_{\Delta[1]} = g \;\; \text{in} \;\; \Delta_w, \tag{112}$$

$$\mathbf{B}u^3_{\Delta[1]} = \gamma \;\; \text{on} \;\; T_{w\mathbf{B}}. \tag{113}$$

**Step 2:** Determine $u^3_\Delta \in S^3_{\Delta_w}$ by forcing it to satisfy

$$\mathbf{L}u^3_\Delta = g - P_{\mathbf{L}}u^3_{\Delta[1]} \;\; \text{in} \;\; \Delta_w, \tag{114}$$

$$\mathbf{B}u^3_\Delta = \gamma \;\; \text{on} \;\; T_{w\mathbf{B}}. \tag{115}$$

Similarly as for QSC, we can also define an *optimal one-step CSC method*. Notice that the implementation of the CSC methods is essentially mapping-free, since, once the grid points are given, the collocation equations can be set up, in contrast to the QSC methods, in which both the grid points and the "midpoints" are needed.
Using techniques similar to those in Section 2, we can show the following theorems.

**THEOREM 12** *If*
*(A1) the coefficients $p$ and $q$, and the right-hand side $g$ are in $\mathbf{C}[\Omega]$,*
*(A2) the BVP $\mathbf{L}u = g, \mathbf{B}u = 0$ has a unique solution,*
*(A3) the BVP $u'' = 0, \mathbf{B}u = 0$ has a unique solution,*
*(A4) the assumptions of Theorem 9 hold,*
*then $u^3_{\Delta[1]} \in S^3_{\Delta_w}$ defined by (112)-(113) exists, is unique, and satisfies the global error estimates*

$$\|(u - u^3_{\Delta[1]})^{(k)}\|_\infty = O(h^2), \; k = 0, 1, 2, \;\; \textit{and} \;\; \|(u - u^3_{\Delta[1]})^{(3)}\|_\infty = O(h), \tag{116}$$

*and the local error estimates*

$$|(u - u^3_{\Delta[1]})^{(3)}(w_i)| = O(h^2), \; i = 1, \ldots, N. \tag{117}$$

**THEOREM 13** *Under the assumptions of Theorem 12, and the assumption that $u'' - u^{3''}_{\Delta[1]}$ has a smooth expansion at the collocation points, $u^3_\Delta \in S^3_{\Delta_w}$ defined by (114)-(115) exists, is unique, and satisfies the global error estimates*

$$\|(u - u^3_\Delta)^{(k)}\|_\infty = O(h^{4-k}), \; k = 0, 1, 2, 3, \tag{118}$$

*and the local error estimates*

$$|(u - u^3_\Delta)'(x)| = O(h^4) \; \textit{for} \; x = s_i \; \textit{and} \; w_i, \;\; |(u - u^3_\Delta)''(\sigma_{ij})| = O(h^3), \;\; |(u - u^3_\Delta)'''(w_i)| = O(h^2). \tag{119}$$

# 4 Numerical Results

In this section, we present numerical results to demonstrate the convergence of the two-step QSC and two-step CSC methods for BVPs with non-uniform grids. All computations were carried out in double precision. For the implementation of the QSC and CSC methods we used non-uniform B-spline basis functions, the exact form of which is given in [6]. The linear systems arising from the QSC and CSC methods were solved by Gauss elimination using the backslash operator or the $lu$ function in MATLAB. In all tables, the notation $x.y \pm z$ means $x.y \times 10^{\pm z}$. The observed errors are denoted by $\epsilon$ and $\epsilon^3$, for QSC and CSC, respectively. The uniform norm $\| \cdot \|_\infty$ is approximated by the maximum absolute value on a constant grid of 1001 evaluation points, independently of the discretization grid.

The first problem is used to test the convergence of the QSC and CSC methods with a predefined $w$. The operator has variable coefficients and the boundary conditions are mixed.

**PROBLEM 1** $$\exp(x)u'' + \sin(x)u' - \tfrac{1}{2+x}u = g \quad \text{in} \quad (0,1),$$

$$u(0) - u'(0) = \gamma(0), \;\; u(1) + u'(1) = \gamma(1).$$

The function $g$ is chosen so that $u(x) = \sin(x)$ is the solution. The mapping function is $w(x) = \frac{\exp(x)-1}{\exp(1)-1}$, and it is intentionally chosen so that it does not have any particular properties that fit this problem. Table 3 shows that the QSC and CSC methods have optimal convergence both globally and locally, for general differential equations with general mixed boundary conditions, and an "arbitrary" $w$.

| $N$ | error | order | error | order | error | order | error | order |
|---|---|---|---|---|---|---|---|---|
| QSC | $\|\epsilon(x)\|_\infty$ | | $\lvert\epsilon(s_i)\rvert$ | | $\lvert\epsilon'(\sigma_{ij})\rvert$ | | $\lvert\epsilon''(w_i)\rvert$ | |
| 32 | 6.80-7 | | 1.65-7 | | 5.67-7 | | 4.63-5 | |
| 64 | 7.48-8 | 3.2 | 9.91-9 | 4.1 | 7.41-8 | 2.9 | 1.21-5 | 1.9 |
| 128 | 8.72-9 | 3.1 | 6.04-10 | 4.0 | 9.60-9 | 2.9 | 3.11-6 | 2.0 |
| 256 | 1.06-9 | 3.0 | 3.73-11 | 4.0 | 1.22-9 | 3.0 | 7.86-7 | 2.0 |
| CSC | $\|\epsilon^3(x)\|_\infty$ | | $\lvert\epsilon^3(s_i)\rvert$ | | $\lvert\epsilon^{3'}(s_i)\rvert$ | | $\lvert\epsilon^{3''}(\sigma_{ij})\rvert$ | |
| 32 | 3.57-8 | | 3.57-8 | | 3.57-8 | | 2.82-6 | |
| 64 | 2.06-9 | 4.1 | 2.06-9 | 4.1 | 2.06-9 | 4.1 | 3.39-7 | 3.1 |
| 128 | 1.23-10 | 4.1 | 1.23-10 | 4.1 | 1.23-10 | 4.1 | 4.15-8 | 3.0 |
| 256 | 7.48-12 | 4.0 | 7.48-12 | 4.0 | 7.35-12 | 4.1 | 5.13-9 | 3.0 |

Table 3: Observed errors and respective orders of convergence corresponding to Problem 1 solved by QSC and CSC with mapping function $w(x) = \frac{\exp(x)-1}{\exp(1)-1}$.

The next problem is designed to analyze the effect of the smoothness of $w$ on the convergence of QSC.

**PROBLEM 2** $$u'' + u' - u = g \quad \text{in} \quad (0,1), \quad u(0) = 0, \;\; u(1) = 1.$$

The function $g$ is chosen so that $u(x) = x^q$, $q > 0$, is the solution to the problem. We note that for $q \in \mathbf{I}^+$, $q$ should be greater than 2, otherwise the approximate and the exact solution would be the same. The mapping function is chosen to be of the form $w(x) = x^p$, $p > 0$. In order to satisfy the condition $v \in \mathbf{C}^2$ of Theorem 1 we need $q \geq 4 + 2/p$, while to satisfy the (looser) conditions of Remark 1, we need

$q \geq 2 + 4/p$, thus if $q \geq 3$ then $p \geq 4$. Table 4 shows that when $q = 3$, we have optimal convergence even for $p = 1.5$. Moreover, it shows that, even for a non-smooth function $u(x) = x^{1.5}$, there are mapping functions, such as $w(x) = x^3$, that map the uniform grid to non-uniform grids that produce optimal results. One advantage of QSC over some other spline collocation methods is that it evaluates the second derivatives at midpoints instead of grid points and thus avoids potential singularities at the end points. Table 5 shows that a non-smooth mapping function $w(x) = x^{0.5}$ ($w'(0)$ is unbounded) can still give rise to a non-uniform grid with optimal convergence provided that the exact solution $u$ has a sufficiently high degree $q$. Also notice that the fact that, for $p > 1$, $w'(0) = 0$, did not affect the optimal convergence. These results indicate that the conditions of the theorems are only sufficient and not necessary.

| QSC | $q = 3, p = 1.5$ | | $q = 3, p = 4$ | | $q = 1.5, p = 1$ | | $q = 1.5, p = 2$ | | $q = 1.5, p = 3$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| $N$ | error | order | error | order | error | order | error | order | error | order |
| 32 | 1.49-7 | | 1.55-5 | | 1.28-4 | | 4.71-6 | | 1.42-6 | |
| 64 | 9.37-9 | 4.0 | 1.02-6 | 4.0 | 4.60-5 | 1.4 | 6.32-7 | 2.9 | 9.07-8 | 4.0 |
| 128 | 5.87-10 | 4.0 | 6.53-8 | 4.0 | 1.63-5 | 1.5 | 8.27-8 | 2.9 | 5.75-9 | 4.0 |

Table 4: Observed midpoint errors and respective orders of convergence corresponding to Problem 2 solved by QSC.

Problem 3 was taken from [4]. Its solution has a boundary layer at the left endpoint. The parameter $\eta$ controls the sharpness of the boundary layer.

**PROBLEM 3** $$\{(1 + \eta x)u'\}' = 0 \quad \text{in} \quad (0, 1), \quad u(0) = 0, \;\; u(1) = 1.$$

The analytical solution to this problem is $u(x) = \frac{\log(1+\eta x)}{\log(1+\eta)}$. Figure 2 plots $u$ for various values of $\eta$. The QSC solutions for this BVP with different parameters $\eta$ and mapping functions $w_I(x) = x$ and $w_{x3}(x) = x^3$ are analyzed. The function $w_I$ maps the uniform grid to itself, while $w_{x3}$ maps the uniform grid to a grid with more points near $x = 0$ and fewer near $x = 1$. Table 6 shows that for $\eta = 1$, both grids $\Delta_{w_I}$ and $\Delta_{w_{x3}}$ resulting from $w_I$ and $w_{x3}$, respectively, produce optimal results. However, the numerical results worsen for the grid $\Delta_{w_I}$ as $\eta$ increases, but remain optimal for the grid $\Delta_{w_{x3}}$. Asymptotically, we expect both the uniform and non-uniform grids to give optimal convergence, but, for this problem, the uniform grid will require a very large $N$ to reach the optimal asymptotic behaviour. Figure 3 shows that the approximate solution arising from $w_{x3}$ is visibly more accurate than that arising from $w_I$, even for a moderate $\eta$.

Finally, we present results from the application of CSC to a non-linear problem, taken from [16].

| QSC | $q = 5$ | | $q = 6$ | | $q = 7$ | |
|---|---|---|---|---|---|---|
| $N$ | error | order | error | order | error | order |
| 32 | 1.31-4 | | 9.97-5 | | 1.03-5 | |
| 64 | 2.40-5 | 2.4 | 1.26-5 | 3.0 | 6.41-7 | 4.0 |
| 128 | 4.48-6 | 2.4 | 1.58-6 | 3.0 | 3.74-8 | 4.0 |

Table 5: Observed midpoint errors and respective orders of convergence corresponding to Problem 2 solved by QSC with $p = 0.5$.

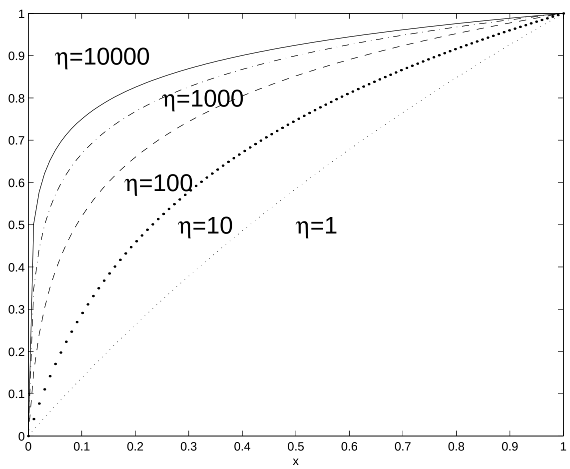


Figure 2: Exact solution $u$ of Problem 3 with different $\eta$ constants.

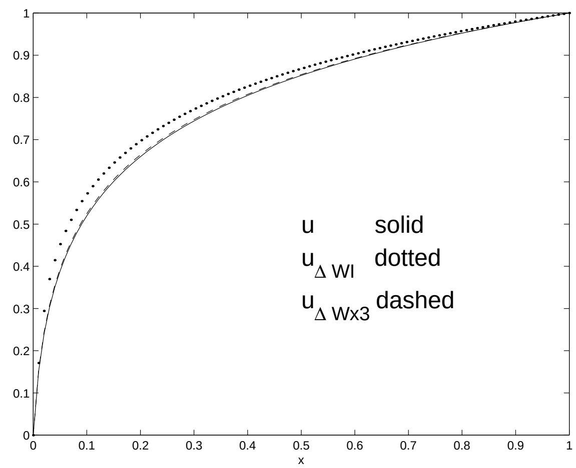


Figure 3: Exact solution $u$ of Problem 3 with $\eta = 100$ and approximate solutions computed by QSC based on mapping functions $w_I$ and $w_{x3}$ for $N = 32$ grid.

| QSC | $w_I$ | | | | $w_{x3}$ | | | |
|---|---|---|---|---|---|---|---|---|
| | $\|\epsilon(x)\|_\infty$ | | $\|\epsilon(w_i)\|$ | | $\|\epsilon(x)\|_\infty$ | | $\|\epsilon(w_i)\|$ | |
| $N$ | error | order | error | order | error | order | error | order |
| | $\eta = 1$ | | | | | | | |
| 16 | 5.42-6 | | 9.38-7 | | 1.95-5 | | 2.26-6 | |
| 32 | 6.48-7 | 3.1 | 6.42-8 | 3.8 | 2.36-6 | 3.0 | 6.01-8 | 5.2 |
| 64 | 8.30-8 | 3.0 | 4.14-9 | 3.9 | 2.95-7 | 3.0 | 2.84-9 | 4.4 |
| 128 | 1.05-8 | 3.0 | 2.61-10 | 4.0 | 3.49-8 | 3.1 | 1.64-10 | 4.1 |
| | $\eta = 100$ | | | | | | | |
| 16 | 6.99-2 | | 6.57-2 | | 4.38-3 | | 4.19-3 | |
| 32 | 2.40-2 | 1.5 | 2.26-2 | 1.5 | 2.47-4 | 4.1 | 2.36-4 | 4.1 |
| 64 | 5.29-3 | 2.0 | 5.29-3 | 2.0 | 1.73-5 | 3.8 | 1.43-5 | 4.0 |
| 128 | 9.85-4 | 2.4 | 9.82-4 | 2.4 | 1.08-6 | 4.0 | 8.91-7 | 4.0 |
| | $\eta = 10,000$ | | | | | | | |
| 16 | 5.64-1 | | 5.57-1 | | 1.52+1 | | 1.37+1 | |
| 32 | 6.89-1 | -0.2 | 6.51-1 | -0.2 | 3.49-1 | 5.4 | 3.39-1 | 5.4 |
| 64 | 7.79-1 | -0.1 | 7.46-1 | -0.1 | 7.93-3 | 5.4 | 7.87-3 | 5.4 |
| 128 | 7.82-1 | -0.0 | 7.58-1 | -0.0 | 3.94-4 | 4.2 | 4.02-4 | 4.3 |
| 256 | 6.51-1 | 0.2 | 6.40-1 | 0.2 | 2.42-5 | 4.0 | 2.40-5 | 4.0 |

Table 6: Observed errors and respective orders of convergence corresponding to Problem 3 solved by QSC with different $\eta$ constants and mapping functions $w_I$ and $w_{x3}$.

**PROBLEM 4** $$u'' - \exp(u) = 0 \ \text{ in } \ (0,1), \ \ u(0) = u(1) = 0.$$

The unique solution to this problem is $u = 2\log(\zeta \sec(\zeta(x-0.5)/2)) - \log 2$, where $\zeta = 1.3360556949061$. (The value of $\zeta$ was calculated as the unique root of $z - \sqrt{2}\cos(z/4) = 0$ [8]). We solve this problem using the two-step CSC method and each of the following two iteration schemes: the simple scheme [16] $u^{[k+1]''} - u^{[k+1]} = \exp(u^{[k]}) - u^{[k]}$ and the scheme arising from Newton's method [8], $u^{[k+1]''} - \exp(u^{[k]})u^{[k+1]} = \exp(u^{[k]}) - \exp(u^{[k]})u^{[k]}$. We apply the non-linear iteration first to the first step of CSC until convergence and then to the second step of CSC until convergence. The non-linear iteration of the first step uses the zero vector as initial guess, while that of the second step uses the last solution of the first step. The stopping criterion is the infinity norm of the difference of the degrees of freedom vectors between two consecutive iterations, and the non-linear iteration tolerance (grid dependent) is $10^{-2}h^2$ for the first step and $10^{-2}h^4$ for the second. Table 7 gives the errors of CSC for this problem. It is also worth noting that, in several cases, Newton's method requires fewer iterations, but also requires the CSC matrix to be formed at each iteration. Moreover, Newton's method is applicable to other non-linear problems, while the particular simple iteration scheme is designed for this problem.

We also present the results from Hermite piecewise cubic collocation (which we refer to as HPCC) in COLSYS (COLNEW) [2, 3], which uses Newton's method. The relaxation factor for the damping of Newton's iteration in COLSYS was equal to 1, so no damping was used. Both methods used uniform grids. Since COLSYS always doubles the grid size at least once, and because we did not want to make changes to the COLSYS code, to obtain the results for grid size $N$, we set the tolerance to something easily reached, and let the code run with starting grid size $N/2$. In all occasions, COLSYS did 2 iterations at grid size $N/2$ and 1 at $N$. We present the total number of iterations, though we acknowledge that this cannot be used to directly compare HPCC with CSC.

| | simple iteration | | | | | | Newton's | | | | | | Newton's | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| CSC | Step 1 | | | Step 2 | | | Step 1 | | | Step 2 | | | HPCC-COLSYS | | |
| $N$ | $itnl$ | error | order | $itnl$ | error | order | $itnl$ | error | order | $itnl$ | error | order | $itnl$ | error | order |
| 16 | 3 | 3.19-5 | | 3 | 1.45-8 | | 3 | 3.20-5 | | 2 | 1.47-8 | | 3 | 4.63-8 | |
| 32 | 3 | 7.95-6 | 2.01 | 3 | 9.81-10 | 3.88 | 3 | 7.99-6 | 2.00 | 2 | 9.86-10 | 3.90 | 3 | 2.95-9 | 3.97 |
| 64 | 4 | 2.00-6 | 1.99 | 3 | 6.13-11 | 4.00 | 3 | 2.00-6 | 2.00 | 2 | 6.28-11 | 3.97 | 3 | 1.86-10 | 3.99 |
| 128 | 4 | 4.99-7 | 2.00 | 3 | 3.57-12 | 4.10 | 3 | 4.99-7 | 2.00 | 2 | 3.94-12 | 4.00 | 3 | 1.17-11 | 3.99 |

Table 7: Observed number of non-linear iterations $itnl$, grid point errors and respective orders of convergence for both steps of CSC and for HPPC on Problem 4.

# 5 Final Remarks and Future Work

One may consider that the technique of mapping uniform to non-uniform points and solving the problem on the non-uniform partition is mathematically equivalent to applying a transformation of variables to the problem and obtaining a problem which can be solved effectively on a uniform grid. We emphasize that, at least numerically, this is not true. Our technique computes an approximation to $w(x)$ and to the location of the non-uniform grid points and collocation points, then applies collocation on the non-uniform partition. To implement the transformation of variables technique, we need to compute, an approximation to $w(x)$, $w'(x)$ and $w''(x)$, then apply collocation to the transformed problem (which involves $w$, $w'$ and $w''$) on the uniform partition. When we attempted to do that, we found that our numerical results were significantly affected. More specifically, the order of convergence was hardly above 2, and the actual errors significantly larger. Therefore, we did not choose (and do not recommend) this approach.

The technique used in this paper to extend the optimal quadratic and cubic spline collocation methods from uniform to non-uniform partitions can be used for other (smooth) spline collocation methods and higher order BVPs (e.g. quartic or quintic splines and fourth-order BVPs). However, the development of appropriate spline interpolants (i.e. the derivation of the boundary relations the spline interpolant has to satisfy) and the derivation of the expansions of the derivative errors of the interpolant at the non-uniform collocation points is quite cumbersome and not straightforward. The generalization of the spline interpolant definition and derivative error expansions to higher degree splines is a challenging piece of research.

The spline collocation methods described in this paper can be relatively easily extended to two-dimensional (and higher-dimensional) rectangular domains, if the domain discretization is also rectangular. However, a rectangular discretization is usually not the most effective one. Moreover, the analysis of the methods for multi-dimensional domains is not straightforward. In [14], quadratic and cubic spline collocation methods are developed for L- and T-shaped domains. More work is also needed to extend the methods to general non-rectangular domains, for example, following the approach of [11], and to general quadrilateral discretizations. The development of efficient linear solvers for the resulting system of equations is another interesting extension of this work.

## Acknowledgements

The authors wish to thank the referees for the thorough reading of the paper and their helpful suggestions. This research was supported by NSERC (National Science and Engineering Research Council of Canada) and OGS (Ontario Graduate Scholarship).

## References

[1] D. Archer. An $O(h^4)$ cubic spline collocation method for quasilinear parabolic equations. *SIAM J. Numer. Anal.*, 14(4):620–637, 1977.

[2] U. Ascher, J. Christiansen, and R. D. Russell. A collocation solver for mixed order systems of boundary value problems. *Math. Comp.*, 33(146):659–679, 1979.

[3] G. Bader and U. Ascher. A new basis implementation for a mixed order boundary value ODE solver. *SIAM J. Sci. Stat. Comp.*, 8(4):483–500, 1987.

[4] M. A. Celia and W. G. Gray. *Numerical Methods for Differential Equations*. Prentice Hall, 1992.

[5] C. C. Christara. Quadratic spline collocation methods for elliptic partial differential equations. *BIT*, 34:33–61, 1994.

[6] C. C. Christara and K. S. Ng. Adaptive techniques for spline collocation. To appear in *Computing*, 2005.

[7] J. W. Daniel and B. K. Swartz. Extrapolated collocation for two-point boundary-value problems using cubic splines. *J. Inst. Maths Applics*, 16:161–174, 1975.

[8] C. de Boor and B. Swartz. Collocation at Gaussian points. *SIAM J. Numer. Anal.*, 10(4):582–606, 1973.

[9] D. J. Fyfe. The use of cubic splines in the solution of two-point boundary value problems. *Comput. J.*, 12:188–192, 1969.

[10] E. N. Houstis, C. C. Christara, and J. R. Rice. Quadratic-spline collocation methods for two-point boundary value problems. *International Journal for Numerical Methods in Engineering*, 26:935–952, 1988.

[11] E. N. Houstis, W. F. Mitchell, and J. R. Rice. Collocation software for second-order elliptic partial differential equations. *ACM Trans. Math. Soft.*, 11(4):379–412, 1985.

[12] E. N. Houstis, E. A. Vavalis, and J. R. Rice. Convergence of an $O(h^4)$ cubic spline collocation method for elliptic partial differential equations. *SIAM J. Numer. Anal.*, 25(1):54–74, 1988.

[13] M. Irodotou-Ellina and E. N. Houstis. An $O(h^6)$ quintic spline collocation method for fourth order two-point boundary value problems. *BIT*, 28:288–301, 1988.

[14] K. S. Ng. *Spline Collocation on Adaptive Grids and Non-Rectangular Regions*. PhD thesis, Department of Computer Science, University of Toronto, Toronto, Ontario, Canada, 2005. http://www.cs.toronto.edu/pub/reports/na/ccc/ngkit-05-phd.ps.gz.

[15] P. M. Prenter. *Splines and Variational Methods*. Wiley, 1975.

[16] R. D. Russell and L. F. Shampine. A collocation method for boundary value problems. *Numer. Math.*, 19:1–28, 1971.

[17] Y. Zhu. Optimal quartic spline collocation methods for fourth order two-point boundary value problems. M.Sc. Thesis, Department of Computer Science, University of Toronto, Toronto, Ontario, Canada, http://www.cs.toronto.edu/pub/reports/na/ccc/yz-01-msc.ps.gz, April 2001.