Fitting random stable solar systems to Titius-Bode laws

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Abstract

Simple "solar systems" are generated with planetary orbital radii r distributed uniformly random in $\log r$ between 0.2 and 50 AU. A conservative stability criterion is imposed by requiring that adjacent planets are separated by a minimum distance of k Hill radii, for values of k ranging from 1 to 8. Least-squares fits of these systems to generalized Bode laws are performed, and compared to the fit of our own Solar System. We find that this stability criterion, and other "radius-exclusion" laws, generally produce approximately geometrically spaced planets that fit a Titius-Bode law about as well as our own Solar System. We then allow the random systems the same exceptions that have historically been applied to our own Solar System. Namely, one gap may be inserted, similar to the gap between Mars and Jupiter, and up to 3 planets may be "ignored", similar to how some forms of Bode's law ignore Mercury, Neptune, and Pluto. With these particular exceptions, we find that our Solar System fits significantly better than the random ones. However, we believe that this choice of exceptions, designed specifically to give our own Solar System a better fit, gives it an unfair advantage that would be lost if other exception rules were used. We conclude that the significance of Bode's law is simply that stable planetary systems tend to be regularly spaced; this conclusion could be strengthened by the use of more stringent methods of rejecting unstable solar systems, such as long-term orbit integrations.

"For a statistician, fitting a three-parameter curve of uncertain form to ten points with three exceptions certainly brings one to the far edge of the known world."
— Bradley Efron (1971)

1 Introduction

The Titius-Bode "law",

$$r_i = 0.4 + 0.15 \times 2^i, \quad i = -\infty, 1, \ldots, 8, \tag{1}$$

roughly describes the planetary semi-major axes in astronomical units, with Mercury assigned $i = -\infty$, Venus $i = 1$, Earth $i = 2$, *etc.* Usually the asteroid belt is counted as $i = 4$. The law fits the planets Venus through Uranus quite well, and successfully predicted the existence and locations of Uranus and the asteroids. However, (i) the law breaks down badly for Neptune and Pluto; (ii) there is no reason why Mercury should have $i = -\infty$ rather than $i = 0$, except that it fits better that way; (iii) the total mass of the asteroid belt is far smaller than the mass of any planet, so it is not clear that it should be counted as one. The question of the significance of Bode's law has taken on increased interest with discoveries of extra-solar planets, and is also worth re-examination because computer speeds now permit more powerful statistical tests than were previously possible.

A fairly comprehensive history of the law and attempts to explain it up to the year 1971 can be found in Nieto (1972). Most modern arguments concerning the validity of Bode's law can be assigned to one of three broad classes:

1. Attempts to elucidate the physical processes leading to Bode's law. These are based on a variety of mechanisms, including dynamical instabilities in the protoplanetary disk (Graner & Dubrulle 1994;

Dubrulle & Graner 1994; Li *et al.* 1995), gravitational interactions between planetesimals (Lecar 1973), or long-term instabilities of the planetary orbits (Hills 1970; Llibre & Piñol 1987; Conway & Elsner 1988). We shall not comment on these explanations, except to say that we find none of them entirely convincing.

2. Discussions that ignore physics but try to assess whether the success of Bode's law is statistically significant. Good (1969) performs a likelihood test under the null hypothesis that the planet distances should be distributed uniformly random in $\log r$. He includes the asteroid belt, but ignores Mercury, Neptune, and Pluto, subjectively assigning (*i.e.,* guessing) a factor of 5 penalty to his likelihood ratio for ignoring these planets. He concludes that there is a likelihood ratio of 300–700 in favour of Bode's law being "real" rather than artifactual. Efron (1971) attacks Good's analysis, in particular his choice of null hypothesis (Good's and Efron's articles are followed by over a dozen extended "comments" from other statisticians). Efron notes that the difference between semi-major axes of adjacent planets is an increasing function of distance for all adjacent planet pairs except Neptune-Pluto. He proposes, without physical basis, that this *law of increasing differences* is a better null hypothesis, the only reason cited being that Bode's law "predicts" increasing differences. Duplicating Good's analysis with this new null hypothesis, he computes a likelihood ratio in favour of Bode's law of only 8:5, and concludes that "there is no compelling evidence for believing that Bode's law is not artifactual." Conway & Zelenka (1988) repeat Efron's analysis using the law of increasing differences, this time ignoring only Pluto, and computing a more realistic penalty for doing so. They compute a likelihood ratio of approximately unity, also concluding that Bode's law is artifactual. We believe these analyses are flawed because there is no physical basis for the law of increasing differences; in fact later we will show that systems that are stable according to our criteria only rarely satisfy the law of increasing differences.

3. Discussions of other laws that may influence the spacing of the planets. Many of these involve resonances between the mean motions of the planets, such as Molchanov (1968; but see Hénon 1969), Birn (1973), and Patterson (1987). A promising development is the recognition that planets are capable of migrating significant distances after their formation (Fernández & Ip 1984; Wetherill 1988; Ipatov 1993; Lin, Bodenheimer & Richardson 1996). For example, this process can explain the resonant relationship between Neptune and Pluto (Malhotra 1993, 1995) and may explain the spacing of the terrestrial planets (Laskar 1997).

The present paper combines the first two of these approaches: we generate a broad range of possible model solar systems and exclude those that are known to be dynamically unstable. We then ask which of the remaining ones satisfy laws similar to Bode's.

2 Method

2.1 Radius-exclusion laws

A necessary, but not sufficient, condition for the stability of a solar system is that its planets never get "too close to each other" (Lecar 1973). This can be formalized into several "radius-exclusion laws".

1. Simple scaling arguments for near-circular, coplanar orbits suggest that a test particle on a stable orbit cannot approach a planet more closely than k Hill radii for some k, using the Hill radius h as defined by Lissauer (1987, 1993),

$$h = H_M r, \qquad H_M = \left(\frac{M}{3M_\odot}\right)^{\frac{1}{3}} \tag{2}$$

for a planet of mass M, semi-major axis r, and *fractional Hill radius* H_M. We shall extend this criterion to two adjacent planets with non-zero mass by summing their respective Hill radii. There are other plausible ways to combine adjacent planets: it might be more physically reasonable to use the sum of the masses to define a single combined Hill radius, although it is not clear that this is preferred when more than two planets are involved. Exponents other than 1/3 may also be reasonable (Wisdom 1980; Chambers *et al.* 1996). However, the difference between these approaches is probably unimportant given the uncertainty in k, as discussed below.

2. For non-circular orbits, we also expect that the aphelion distance of the inner planet is less than the perihelion distance of the outer one. In other words, if the i^{th} planet has semi-major axis r_i and eccentricity e_i, we expect that $r_i(1+e_i) < r_{i+1}(1-e_{i+1})$. Taking this further, we may demand that the planets are separated by a Hill radius even at their closest possible approach, giving

$$r_i(1+e_i+H_{M_i}) < r_{i+1}(1-e_{i+1}-H_{M_{i+1}}).$$

3. Several authors have argued that boundaries between stable and unstable orbits occur at resonances of the form $j:(j+1)$ (Birn 1973; Wisdom 1980; Weidenschilling & Davis 1985; Patterson 1987; Holman & Murray 1996). Weidenschilling & Davis (1985) argue that two planets are unlikely to form closer than their mutual 2:3 resonance (because small solid bodies are trapped in the outer $j:(j+1)$ resonances of a protoplanet due to gas-induced drag; once trapped, their eccentricities are pumped up, causing orbit crossing). For the 2:3 resonance, we can define $H_{2:3}$ by using Kepler's third law to define $R_{2:3}=(3/2)^{2/3}$, and splitting the distance between two adjacent planets by solving $R_{2:3}=1+H_{2:3}+R_{2:3}H_{2:3}$.

Combining all three of Hill radii, eccentricities, and the 2:3 resonance, we obtain

$$r_i(1+V_i) < r_{i+1}(1-V_{i+1}), \qquad V_i = \max(H_{2:3}, e_i + H_{M_i}).$$

Clearly our results will be highly dependent upon the extent of radius exclusion. For the Hill radius of equation (2), which was derived for the case of two small planets orbiting a massive central object, a value of k=2–4 is believed to leave the two planets in permanently stable orbits (Wetherill & Cox 1984, 1985; Lissauer 1987; Wetherill 1988; Gladman 1993). For more than two planets, recent work by Chambers *et al.* (1996) suggests that *no* value of k gives permanent stability. Instead, the stability timescale grows exponentially with increasing orbit separation, with billion-year stability for our solar system requiring $k \gtrsim 13$. Furthermore, simulations of the stability of test particles in the current Solar System (Holman 1997) seem to show that there remain few stable orbits in the outer Solar System other than those near Trojan points. This provides circumstantial evidence that a small value of k is not enough to separate stable orbits, since the outer planets are separated from each other by more than 15 Hill radii. For these reasons, our experiments use several radius-exclusion laws, including various combinations of Hill radii, 2:3 resonances, eccentricities, and k.

It is easy to see why radius-exclusion laws tend to produce planetary distances that approximately follow a geometric progression. If a fixed fractional radius exclusion V is used for every planet, and planets are packed as tightly as possible according the radius-exclusion law, then the physical extent of radius exclusion at distance r is rV, and the resulting planetary separations would follow an exact geometric progression with semi-major axis ratio $(1+V)/(1-V)$. If the planets are packed less tightly, then the progression will be only approximately geometric.

2.2 Generating and fitting solar systems

We generate 9 planet distances $r_i, i=0,\ldots,8$, distributed uniformly random in $\log r$ between 0.2 and 50 AU, constrained so that the exclusion radii of adjacent planets do not overlap. We generated 4096 samples for each of the radius-exclusion laws listed in Table 1. For e_i, we use the maximum eccentricity for each planet over the past 3 million years (Quinn *et al.* 1991). Table 1 lists the average number of Monte-Carlo trials required to find a list of distances satisfying the radius-exclusion law. If a list of distances did not satisfy the exclusion criterion, the entire list was discarded. The mean number of trials needed to build a "valid" system that satisfies the radius-exclusion criterion increases as the exclusion radii get larger.

For each sample that satisfies the relevant radius-exclusion, we perform a nonlinear least-squares fit of the distances r_i to

$$a + bc^i, \tag{3}$$

which we call a "generalized Bode law". The fit is performed by minimizing the objective function

$$\chi^2 = \sum_{i=0}^{8} \left(\frac{\log(a+bc^i) - \log r_i}{\sigma_i} \right)^2, \tag{4}$$

Name	Description	trials	Excl $\Rightarrow$ LID	LID $\Rightarrow$ Excl
M_0	Each planet has zero mass	1	0.4%	100%
$1H_i$	Hill radii corresponding to planets of our Solar System	1.92	0.8%	93.3%
$2H_i$	Like $1H_i$, except radius exclusion of 2 Hill radii	3.94	1.6%	87.2%
M_J	All planets have Jupiter's fractional Hill radius	7.88	1.9%	49.3%
$4H_i$	Like $1H_i$, except radius exclusion of 4 Hill radii	19.3	4.5%	57.6%
$H_i + e_i$	Adjacent planets no closer than eccentricity + 1 Hill radius	44.1	3.4%	23.1%
$H_{2:3}$	Adjacent planets no closer than the 2:3 resonance	230	10.5%	24.8%
max	Adjacent planets no closer than $\max(H_i + e_i, H_{2:3})$	479	9.7%	16.9%
$8H_i$	Like $1H_i$, except radius exclusion of 8 Hill radii	2820	11.7%	1.7%

Table 1: The various radius-exclusion laws we used. The table is ordered by the "trials" column, which is the 4096-sample average number of Monte-Carlo trials required from a log-uniform distribution to find a sample that satisfies the corresponding radius exclusion criterion. The last two columns (see §3.2) list the percentage of samples for which the law of increasing differences (LID) agrees with exclusion laws for a 9-planet system. For example, every sample satisfies M_0 exclusion, but only 0.4% of those satisfy LID.

constrained so that $a, b > 0$ and $c > 1$. We fit on $\log r$ rather than r because we want the fractional error of each planet to be weighted equally. In the Appendix, we derive the standard deviations σ_i analytically. The initial guess for the parameters in the objective function is

$$c_0 = \frac{1}{8} \sum_{i=0}^{7} \frac{r_{i+1}}{r_i},$$

$$b_0 = r_8 / c_0^8,$$

$$a_0 = \max(0, r_0 - b_0).$$

To more accurately reflect the various forms of Bode's law, we also attempt fits that "ignore" 1, 2, and 3 planets. We do this by performing fits on all $\binom{9}{j}$, $j = 1, 2, 3$, possible combinations of ignoring j out of 9 planets, and choosing the best fit for each j. At this point we have 5 "fits" for each solar system: an initial guess, and a fit that ignores 0, 1, 2, and 3 planets. We then repeat the entire procedure, allowing one gap to be inserted between the two adjacent planets with the largest r_{i+1}/r_i ratio, to mimic the gap between Mars and Jupiter. This gives us ten fits out of $2\sum_{j=0}^{3}\binom{9}{j} = 260$ trials for each solar system.

3 Results

3.1 Fits

Results of all the fits for each type of system are presented in Figures 1, 2, and 3. Not surprisingly, the best fit for our own Solar System occurs when a gap is added between Mars and Jupiter, while Mercury, Neptune, and Pluto are ignored. Even for our own Solar System, the best fit depends slightly on the radius-exclusion law, since this affects the denominator σ_i (see Appendix); for the three cases in which the radius exclusion for each planet is identical in $\log r$ (M_0, M_J, 2:3), the fit for our Solar System is identically

$$0.450 + 0.132 \times 2.032^i, \quad i = 0, 1, \ldots, 8,$$

with χ^2 values of 0.003, 0.005, and 0.009, respectively. This result can be compared to the original Bode's law, equation (1).

Consider the case where no gaps are allowed, and up to 3 planets may be ignored (Figure 1). Note first that the values of χ^2 for the initial guess (top solid curve) are roughly equal to 6, which is the expected value since there are 6 degrees of freedom (9 planets minus the 3 parameters of Bode's law). The only exceptions are the cases $4H_i$ and $8H_i$; this is probably because the σ_i in equation (4) are only approximate when the radius exclusions are different for each planet (as is the case for all the kH_i systems, $k = 1, 2, 4, 8$), and the approximation worsens as the radius exclusion increases.

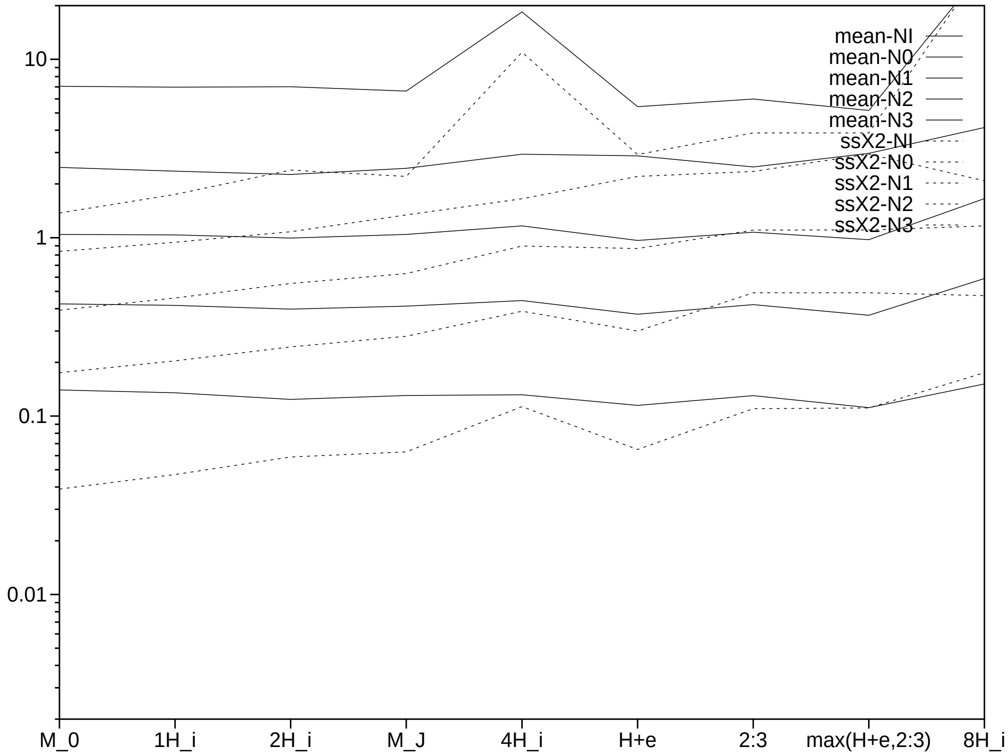

Figure 1: χ^2 fits, using equation (4), when no gap between planets (analogous to the asteroid belt) is allowed (label 'N'). The χ^2 of our own Solar System is shown by dashed lines, and the means of 4096 random ones are shown by solid lines. The labels on the horizontal axis correspond to the first column of Table 1. The labels I, 0, 1, 2, and 3 denote the fit for the initial guess, and 0, 1, 2, 3 planets ignored, respectively.

When there is no radius-exclusion law (left edge of the Figure), the χ^2 for the Solar System is always substantially less (by a factor 3–6) than the mean of the random systems with the same number of ignored planets; this is consistent with the conclusion that the Solar System satisfies a generalized Bode law significantly better than random systems. However, as we apply more stringent radius-exclusion laws (moving right in the Figure), the χ^2 values for the Solar System become quite similar to the mean of the random systems, indicating that the Solar System is no closer to a generalized Bode law than random systems. This situation changes in Figure 2, which shows the case where a gap is allowed. For every radius-exclusion law, our Solar System's best χ^2 value with three planets ignored and one gap (bottom dashed line in Figure 2) is 1–1.5 orders of magnitude smaller than the mean of the random systems with the same number of exceptions. We suggest that this is because the particular exceptions we investigated were historically designed specifically to make our Solar System fit better.

Figure 3 shows the Solar System's quantile—the fraction of random systems with the same exceptions that have a χ^2 smaller than the Solar System. The quantile is not exceptional if no gap is allowed (upper five curves), ranging from about 0.15 to 0.7, and only mildly exceptional if a gap is allowed ($\gtrsim 0.04$), and generally becomes less exceptional as the random solar systems are chosen with increasingly stringent radius-exclusion laws (moving right in the Figure).

To compare our results to those of Good (1969), Efron (1971), and Conway & Zelenka (1988), we have used the same—though rather arbitrary—procedure advocated by Efron: we compute the "likelihood ratio" L of Bode's law being "real", as opposed to artifactual, using the formula

$$L = \frac{1}{4Q}\left(1 - \tfrac{1}{3}\right) \tag{5}$$

where Q is the Solar System's quantile, the $(1 - \frac{1}{3})$ factor is meant to penalize us for restricting ourselves to

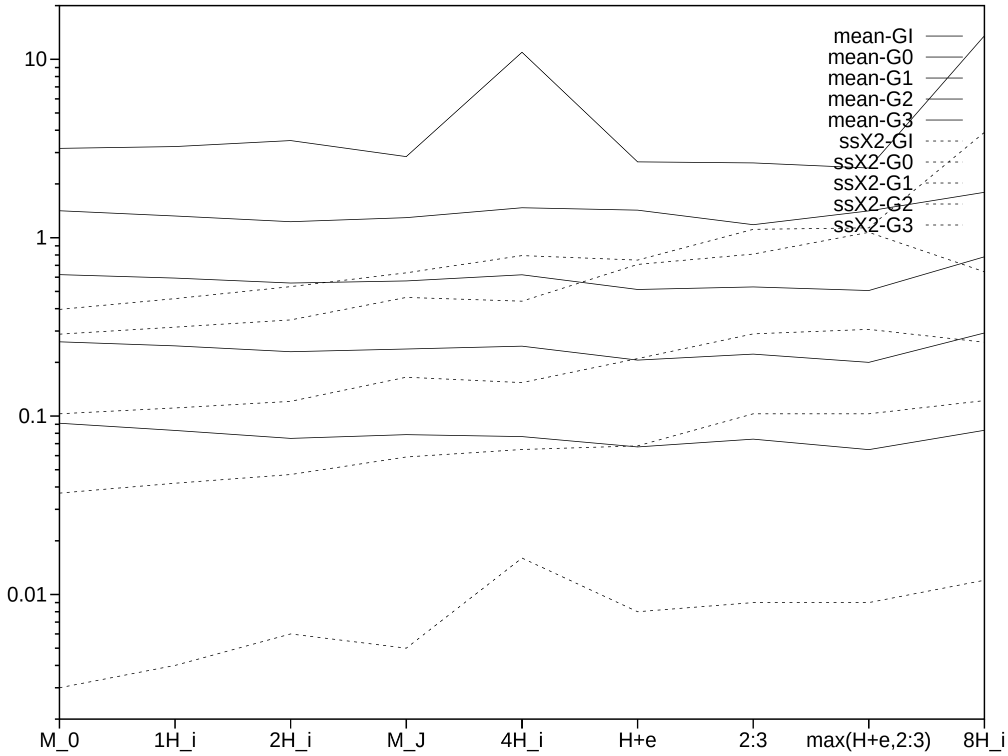

Figure 2: Same as Figure 1, but with a gap allowed (label 'G').

a geometric law (as opposed, say, to one based on n^2), and the $\frac{1}{4}$ factor is designed to normalize the ratio so that anything within about one standard deviation of the mean has a likelihood ratio less than 1. The results of this calculation are presented in Figure 4. The highest likelihood ratios occur for the M_0 case, since the random systems are least constrained when there is no radius exclusion, but even these are all $\lesssim 5$, which is not particularly significant.

One could also argue that the main asteroid belt should "count" as an object. In Figure 5, we display the Solar System's quantile and likelihood ratio for this case, with no exceptions allowed. In this case, the Solar System's χ^2 value is generally in the top 2 to 10 percentile, with a likelihood ranging from unity up to 8. (We ignore the $8H_i$ case, which is probably badly skewed due to the approximate nature of the σ_i.) This likelihood ratio is also not particularly significant, given that the case we are examining is, to some extent, tailored to the properties of our Solar System.

The value of the parameter a in equation (3) is zero in about 25% of the samples, representing an exact geometric progression, while the remainder distribute uniformly up to about 0.5 AU. The value of b is roughly bimodal, with peaks near 0 and 0.2; the first tends to occur when a is non-zero and the second when a is near zero. The value of c clusters around 1.5–1.8 when no planets are ignored, and slightly higher if some are. There was no observable correlation of a, b, or c with χ^2.

A histogram of which planet gets ignored in the $(M_0, N1)$ systems is shown in Figure 6. The innermost and outermost planets are ignored most often, which is expected since a planet with only one neighbor is less constrained than those with two. The placement of the gap is approximately uniform between all planet pairs, for all types of systems. Adding a gap was observed to produce a better fit in about 65% of all cases.

It is prudent to show that our results are not strongly dependent upon the assumption that the underlying distribution is uniform in $\log r$. If instead we assume a disk surface density that scales as $r^{-3/2}$, which roughly corresponds to the expected density in the protoplanetary disk (Lissauer 1993), then if all the planets are equally massive, they should be uniformly distributed in $\sqrt{r}$. We therefore performed our entire suite of experiments again, this time trying to fit an $a + bc^i$ law to solar systems with an underlying random distribution that is uniform in $\sqrt{r}$. We find that most of the above results are qualitatively unchanged. For

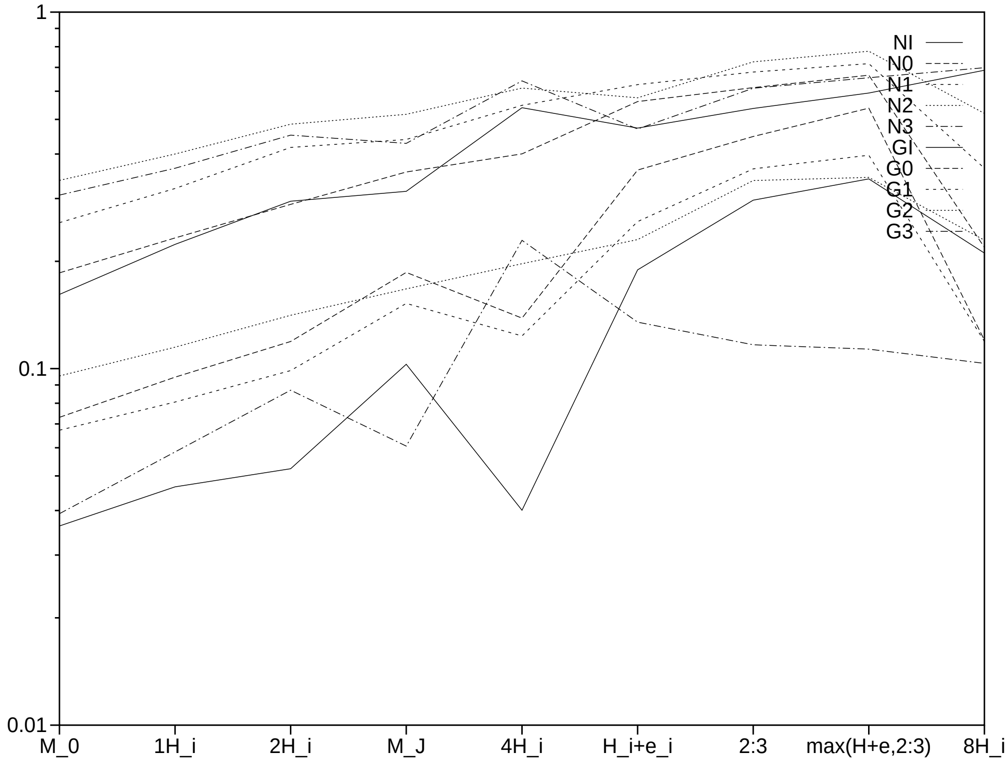

Figure 3: The quantile of the Solar System's χ^2, compared to the 4096 random ones, including systems both allowing a gap ('G'), and not ('N'). The labels are the same as in Figure 1.

example, in the $N0$ case, the fit of the random systems worsens, so that our Solar System's quantile moves down, but only by about 0.05, to 0.15. In the $G3$ case, the Solar System's quantile is almost the same in the $\sqrt{r}$ distribution as in the $\log r$ distribution. Furthermore, as radius exclusion increases, the effect of the underlying distribution is suppressed because radius exclusion is biased towards accepting solar systems that follow a roughly geometric progression. We conclude that our comparisons are not strongly affected by the assumption that the underlying distribution is uniform in $\log r$.

3.2 The law of increasing differences

Efron (1971) and Conway & Zelenka (1988) have noted that the distance between planets in the Solar System is an increasing function of distance for all adjacent pairs except Neptune-Pluto. They propose that this is a reasonable law on which to base a null hypothesis concerning the statistical significance of Bode's law. They note that a pure log-uniform distribution produces increasing differences only a small percentage of the time, and that if we assume the law of increasing differences then the success of Bode's law is unsurprising.

We are uncomfortable with this law because it has no physical basis. This concern has prompted us to examine the relation between radius-exclusion laws and the law of increasing differences. Table 1 shows the occurrences of agreement between the law of increasing differences, and radius exclusions. As Table 1 shows, (i) a system that satisfies radius exclusion rarely satisfies the law of increasing differences; (ii) one that satisfies the law of increasing differences will often satisfy all but the most stringent exclusion laws. We also observed that (iii) the number of trials required to find a random log-uniform sample that satisfies the law of increasing differences is 1 to 2 orders of magnitude larger than that for radius exclusion; (iv) random solar systems that satisfy the law of increasing differences have χ^2 values that are 1.3 to 3 times smaller than ones generated using radius exclusion. Thus, the law of increasing differences is a much more restrictive assumption than radius-exclusion laws, and in the absence of any physical justification, it does not form a sound basis from which to judge the validity of Bode's law.

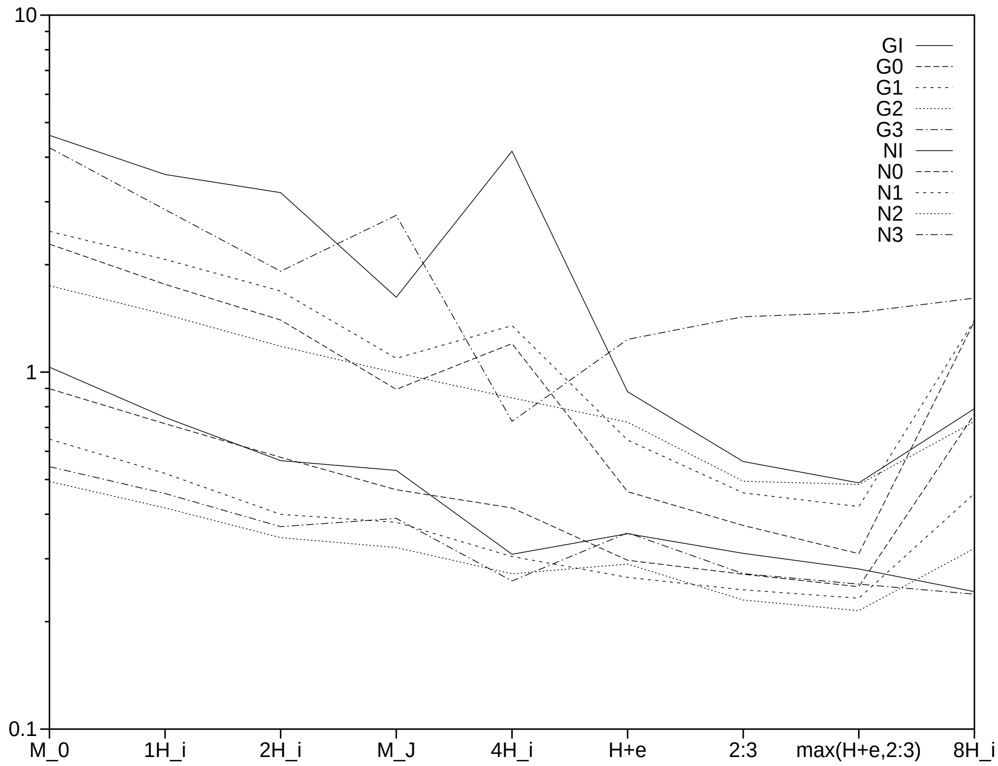

Figure 4: The "likelihood ratio" of Bode's law being "real", using the formula of Good, Efron, and Conway & Zelenka (eq. 5). The labels are the same as in Figure 3.

4 Discussion and conclusions

We have measured the deviation from Bode's law of solar systems whose planetary distances are distributed uniformly random in $\log r$, subject to radius exclusion constraints. We find that, as radius exclusion becomes more stringent, the systems tend to fit Bode's law better. We compare these fits to that of our own Solar System. We find that, when no exceptions or gaps are allowed, our Solar System fits marginally better than random systems that follow weak radius-exclusion laws, but fits no better, or even worse, than those that satisfy more stringent but still reasonable radius exclusions. If one gap is allowed to be added, and up to 3 planets are ignored, then our Solar System fits significantly better than random ones built with weak radius exclusions, and marginally better than ones with strong radius exclusions; however, this modest success for Bode's law probably arises because these rules (3 planets removed, 1 gap added) were designed specifically in earlier centuries to make our Solar System fit better.

Even though our underlying distance distribution, uniform in $\log r$, is scale invariant, the analysis of Graner and Dubrulle (1994) and Dubrulle and Graner (1994) does not apply to most of our cases, since the distribution of planetary masses and Hill radii is not scale-invariant. We found that, even in the cases where the radius-exclusion law *is* scale-invariant (M_0, M_J, 2:3), the best-fitting generalized Bode law has $a \neq 0$ in 36% of the cases, and hence is *not* scale-invariant.

Our approach to Bode's law is very simplistic. We ignore all of the details of planet formation, and use simple stability criteria to discard unstable systems. The natural next step is to replace the approximate stability criteria we have used by actual orbit integrations. We conjecture that if we repeat the experiments of the present paper using a direct integration of several Gyr (*eg.,* Wisdom & Holman 1991) to discard unstable systems, we would find that the surviving systems fit a generalized Bode law better than the ones in this paper and approximately as well as our own Solar system, showing that the significance of Bode's law is simply that stable planetary systems tend to be regularly spaced.

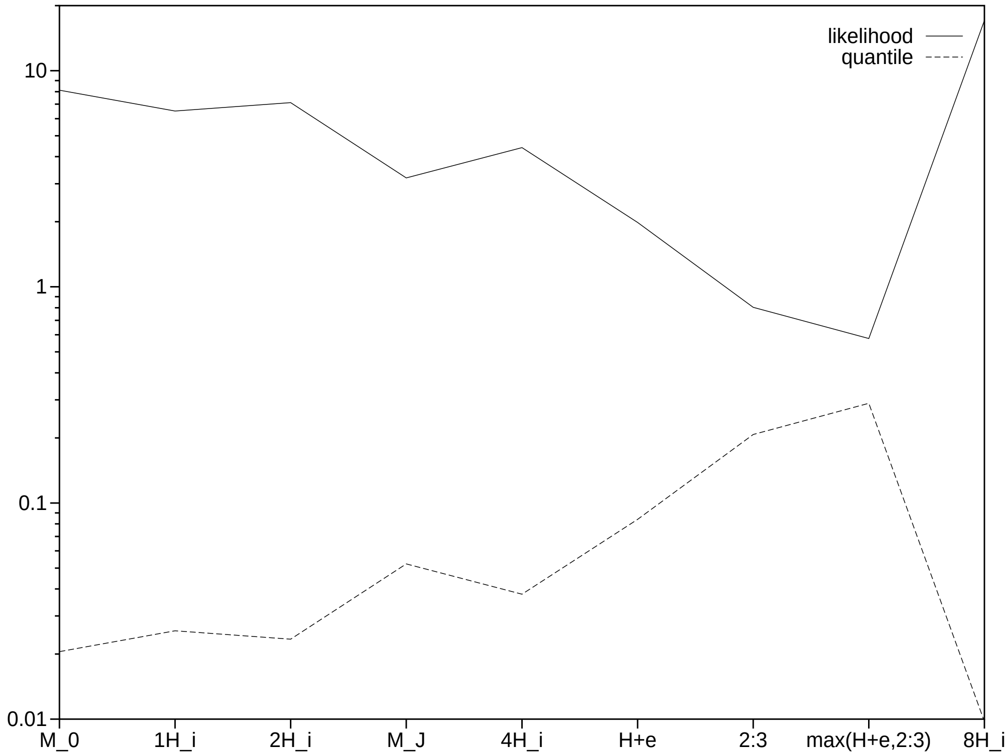

Figure 5: The quantile and likelihood ratio of the Solar System, compared to 4096 random ones, when no exceptions are allowed, but the main asteroid belt is included as a "planet" at radius 2.8 AU.

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Appendix: mean, variance, compression of uniform distributions

If n numbers are chosen from the uniform random interval $U[0,1]$ and sorted into non-decreasing order, then the mean and variance of the i^{th} one X_i are (Rice 1988, problem 4.15)

$$\bar{X}_i = i/(n+1), \qquad \sigma^2_{X_i} = \frac{i(n-i+1)}{(n+1)^2(n+2)}. \tag{6}$$

which we then scale to the interval $[\log(0.2), \log(50)]$ used in this paper. Radius exclusion makes the allowed intervals between planets smaller, thus compressing the uniform standard deviations (6) by a factor C_i. To compute C_i, assume $r_i = 1$ and that all the planets have the same fractional Hill radius H and perfectly fit a Bode's law with exponent $c = \beta$ and $a = 0$. It is easy to show that

$$C_i = \frac{\log(\beta)}{\log(\beta) + \log(1-H) - \log(1+H)}, \tag{7}$$

finally giving

$$\sigma_i^2 = (\sigma_{X_i}/C_i)^2, \tag{8}$$

which we substitute into equation (4). When distance is measured in $\log r$ and all planets have the same radius exclusion, equation (7) is exact; otherwise it is only approximate.

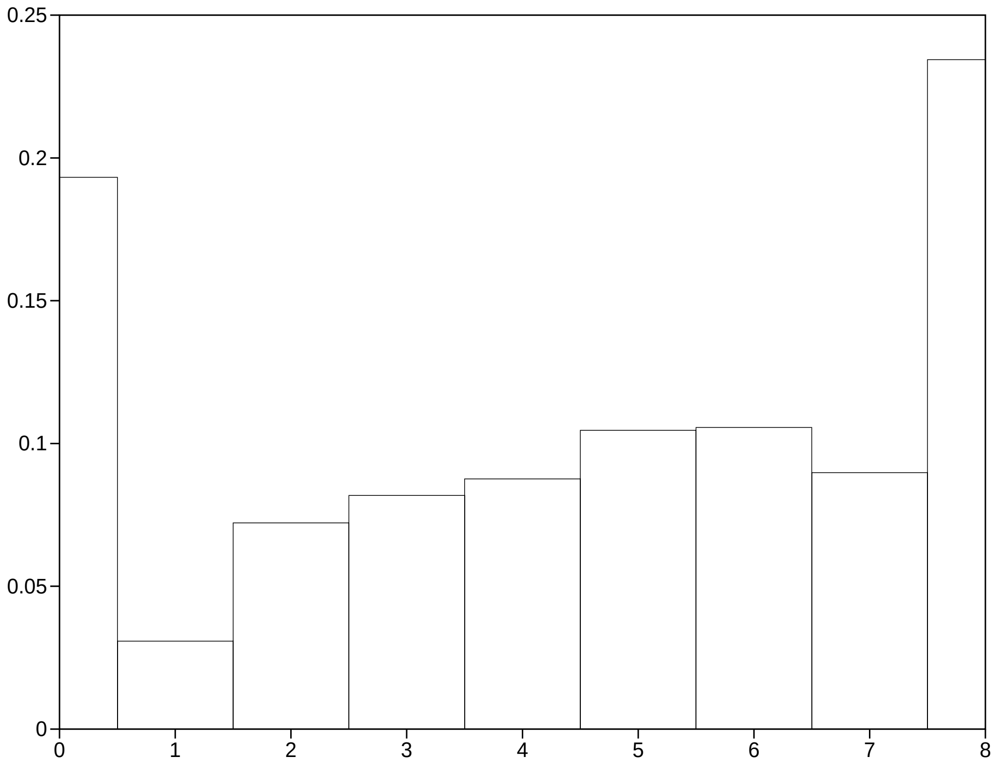

Figure 6: A histogram showing how often each planet is chosen to be ignored, as a fraction of the total, when ignoring one planet in the M_0 system. The distribution does not change significantly in other systems.

References

[1] J. Birn. On the Stability of the Planetary System. *Astronomy and Astrophysics*, 24:283–293, 1973.

[2] J. E. Chambers, G. W. Wetherill, and A. P. Boss. The Stability of Multi-Planet Systems. *Icarus*, 119:261–268, 1996.

[3] Bruce A. Conway and Timothy J. Elsner. Dynamical Evolution of Planetary Systems and the Significance of Bode's Law. In Roy [30], pages 3–12.

[4] Bruce A. Conway and Richard E. Zelenka. Further Numerical Investigations into the significance of Bode's Law. In Roy [30], pages 13–20.

[5] B. Dubrulle and F. Graner. Titius-Bode laws in the Solar System II. Build your own law from disk models. *Astronomy and Astrophysics*, 282:269–276, 1994.

[6] Bradley Efron. Does an Observed Sequence of Numbers Follow a Simple Rule? (Another Look at Bode's Law). *Journal of the American Statistical Association*, 66(335):552–559, September 1971.

[7] J.A. Fernández and W.-H. Ip. Some dynamical aspects of the accretion of Uranus and Neptune — The exchange of orbital angular momentum with planetesimals. *Icarus*, 58:109–120, April 1984.

[8] B. Gladman. Dynamics of systems of two close planets. *Icarus*, 106:247–263, 1993.

[9] I.J. Good. A Subjective Evaluation of Bode's Law and an 'Objective' Test for Approximate Numerical Rationality. *Journal of the American Statistical Association*, 64:23–66, March 1969.

[10] F. Graner and B. Dubrulle. Titius-Bode laws in the Solar System I. Scale Invariance explains everything. *Astronomy and Astrophysics*, 282:262–268, 1994.

[11] M. Hénon. A comment on "The Resonant Structure of the Solar System" by A. M. Molchanov. *Icarus*, 11:93–94, 1969.

[12] J. G. Hills. Dynamic Relaxation of Planetary Systems and Bode's Law. *Nature*, 225:840–842, 1970.

[13] Matthew J. Holman. The Distribution of Mass in the Kuiper Belt. Preprint, Canadian Institute of Theoretical Astrophysics, 1997.

[14] Matthew J. Holman and Norman W. Murray. Chaos in High-Order Mean Motion Resonances in the Outer Asteroid Belt. *The Astronomical Journal*, 112(3):1278–1293, September 1996.

[15] S. I. Ipatov. Migration of bodies in the accretion of planets. *Solar System Research*, 27:65–79, 1993.

[16] J. Laskar. Large scale chaos and the spacing of the inner planets. *Astronomy and Astrophysics*, 317:L75–L78, 1997.

[17] M. Lecar. Bode's Law. *Nature*, 242:318–319, March 1973.

[18] X.Q. Li, H. Zhang, and Q.B. Li. Self-similar collapse in nebular disk and the Titius-Bode law. *Astronomy and Astrophysics*, 304:617–621, 1995.

[19] D.N.C Lin, P. Bodenheimer, and D.C. Richardson. Orbital migration of the planetary companion of 51 Pegasi to its present location. *Nature*, 380:606–607, 1996.

[20] Jack J. Lissauer. Timescales for Planetary Accretion and the Structure of the Protoplanetary Disk. *Icarus*, 69:249–265, 1987.

[21] Jack J. Lissauer. Planet Formation. *Annual Review of Astronomy and Astrophysics*, 31:129–174, 1993.

[22] Jaume Llibre and Conchita Piñol. A Gravitational Approach to the Titius-Bode Law. *The Astronomical Journal*, 93(5):1272–1279, May 1987.

[23] R. Malhotra. The origin of Pluto's peculiar orbit. *Nature*, 365(6499):819–821, October 1993.

[24] Renu Malhotra. The Origin of Pluto's Orbit: Implications for the Solar System Beyond Neptune. *The Astronomical Journal*, 110:420, 1995.

[25] A. M. Molchanov. The Resonant Structure of the Solar System: The Law of Planetary Distances. *Icarus*, 8:203–215, 1968.

[26] Michael Martin Nieto. *The Titius-Bode law of planetary distances: its history and theory.* Oxford, New York, Pergamon Press, 1972.

[27] Chris W. Patterson. Resonance Capture and the Evolution of the Planets. *Icarus*, 70:319–333, 1987.

[28] Thomas R. Quinn, Scott Tremaine, and Martin Duncan. A Three Million Year Integration of the Earth's Orbit. *The Astronomical Journal*, 101(6):2287–2305, June 1991.

[29] John A. Rice. *Mathematical Statistics and Data Analysis.* Wadsworth & Brooks/Cole Advanced Books & Software, 1988.

[30] A. E. Roy, editor. *Long-Term Dynamical Behaviour of Natural and Artificial N-body Systems.* Klewer Academic, 1988.

[31] Stuart J. Weidenschilling and Donald R. Davis. Orbital Resonance in the Solar Nebula: Implications for Planetary Accretion. *Icarus*, 62:16–29, 1985.

[32] G. W. Wetherill. Accumulation of Mercury from Planetesimals. In F. Vilas, C.R. Chapman, and M.S. Matthews, editors, *Mercury*, pages 670–691. Univ. of Arizona Press, Tucson, 1988.

[33] George W. Wetherill and Larry P. Cox. The Range of Validity of the Two-Body Approximation in Models of Terrestrial Planet Accumulation II. Gravitational Cross Sections and Runaway Accretion. *Icarus*, 63:290–303, 1985.

[34] G.W. Wetherill and Larry P. Cox. The Range of Validity of the Two-Body Approximation in Models of Terrestrial Planet Accumulation I. Gravitational Perturbations. *Icarus*, 60:40–55, 1984.

[35] Jack Wisdom. The Resonance Overlap Criterion and the Onset of Stochastic Behavior in the Restricted Three-Body Problem. *The Astronomical Journal*, 85(8):1122–1133, August 1980.

[36] Jack Wisdom and Matthew Holman. Symplectic maps for the n-body problem. *The Astronomical Journal*, 102:1528–1538, October 1991.