On computational methods for the valuation of credit derivatives

by

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Abstract

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A credit derivative is a financial instrument whose value depends on the credit risk of an underlying asset or assets. Credit risk is the possibility that the obligor fails to honor any payment obligation. This thesis proposes four new computational methods for the valuation of credit derivatives.

Compared with synthetic collateralized debt obligations (CDOs) or basket default swaps (BDS), the value of which depends on the defaults of a prescribed underlying portfolio, a forward-starting CDO or BDS has a random underlying portfolio, as some "names" may default before the CDO or BDS starts. We develop an approach to convert a forward product to an equivalent standard one. Therefore, we avoid having to consider the default combinations in the period between the start of the forward contract and the start of the associated CDO or BDS. In addition, we propose a hybrid method combining Monte Carlo simulation with an analytical method to obtain an effective method for pricing forward-starting BDS.

Current factor copula models are static and fail to calibrate consistently against market quotes. To overcome this deficiency, we develop a novel chaining technique to build a multi-period factor copula model from several one-period factor copula models. This allows the default correlations to be time-dependent, thereby allowing the model to fit market quotes consistently. Previously developed multi-period factor copula models require multi-dimensional integration, usually computed by Monte Carlo simulation,

which makes the calibration extremely time consuming. Our chaining method, on the other hand, possesses the Markov property. This allows us to compute the portfolio loss distribution of a completely homogeneous pool analytically.

The multi-period factor copula is a discrete-time dynamic model. As a first step towards developing a continuous-time dynamic model, we model the default of an underlying by the first hitting time of a Wiener process, which starts from a random initial state. We find an explicit relation between the default distribution and the initial state distribution of the Wiener process. Furthermore, conditions on the existence of such a relation are discussed. This approach allows us to match market quotes consistently.

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Chapter 1

Introduction

A credit derivative is a financial instrument whose value depends on the credit risk of an underlying asset or assets. Credit risk is the possibility that the obligor fails to honor any payment obligation [47]. Credit derivatives are one of the most important financial innovations of the last two decades. According to the survey by ISDA [22], the outstanding notional of credit derivatives grew from $0.63 Trillion at the beginning of 2001 to $62 Trillion by the end of 2007. Meanwhile, credit derivatives are among the most complex and risky products, often criticized as contributing to the recent credit crisis. Much of the blame for the current credit crisis is being attributed to the mathematical models and quantitative methods associated with credit derivatives. Developing better computational methods for the valuation of credit derivatives is the main focus of this thesis.

Based on the number of underlying assets, credit derivatives are divided into two categories: single-name and multi-name credit derivatives. The default payments for single-name products are based on the default risk of a single reference asset. Such products include credit default swaps (CDS), total return swaps and CDS options. The default payments for multi-name credit derivatives are based on the default combinations of the underlying portfolio. Such derivatives include basket default swaps (BDS), col-

lateralized debt obligations (CDOs) and more exotic time-dependent products, such as forward-starting CDOs (FCDOs), forward-starting BDS (FBDS) and options on tranches. In this thesis, we develop a reduction method for pricing FCDOs, a hybrid method for pricing FBDS, a chaining technique to build a dynamic correlation model for multi-name credit derivatives and a randomization method in the first hitting time approach for modeling the default of a single-name credit derivative[1].

1.1 Mechanics of credit derivatives

In this section, we give a brief review of the credit derivatives covered in this thesis. A more detailed description can be found in the book by Banks, Glantz and Siegel [5] or the product guides by Merrill Lynch [32].

A credit default swap (CDS) is a bilateral contract, in which the seller agrees to provide protection to the buyer against the underlying asset's default in exchange for a series of fixed payments. For example, the seller might provide protection to the buyer against the default of $100 million worth of General Motors (GM) bonds for the next 5 years. If the GM bonds default before the end of the fifth year, the seller pays the loss associated with the GM bonds to the buyer; otherwise, the contract terminates at the end of the fifth year. In return for the protection, the buyer pays a specified rate (known as the premium or spread) to the seller at set dates (e.g., at the end of every three months). Figure 1.1 illustrates a typical CDS structure.

A collateralized debt obligation (CDO) is an agreement to redistribute the credit risk of the collateral pool to priority ordered tranches. Generally, there are five tranches: super-senior, senior, mezzanine, junior and equity, with priority decreasing in that order. All cash collections are distributed to the tranches from super-senior to equity, sequentially; portfolio losses are allocated first to the equity tranche, and then upwards through

[1]The presentation order of these methods is based on their mathematical difficulty of the approaches instead of the complexity of the derivatives.

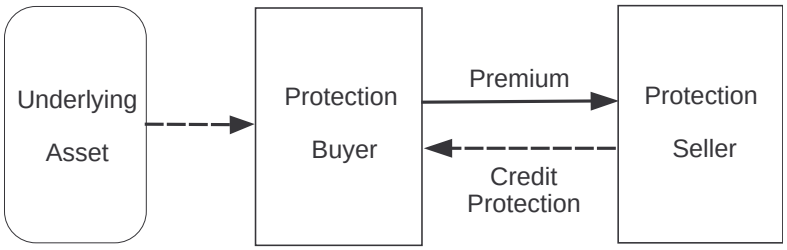


Figure 1.1: CDS structure

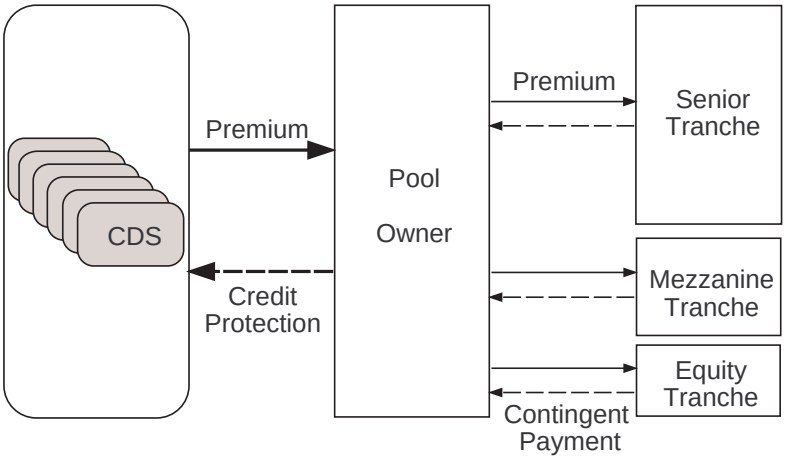


Figure 1.2: CDO structure

the tranches in reverse priority order. Each tranche is specified by an attachment point a and a detachment point b. The buyer of one or more of these tranches sells partial protection to the pool owner by absorbing the pool losses specified by the tranche structure. That is, if the pool losses are less than the tranche attachment point a, the protection seller does not suffer any loss; otherwise, the seller absorbs the losses up to the tranche size $S = b - a$. In return for the protection, the pool owner pays premia to the protection seller at set dates. A CDO is called a synthetic CDO if the risky assets in the underlying pool are CDS. Figure 1.2 depicts a typical synthetic CDO structure[2].

In an mth-to-default basket default swap (BDS), the protection buyer pays premia

[2]In this figure, we have included three tranches only for convenience in drawing the figure, rather than the usual five tranches.

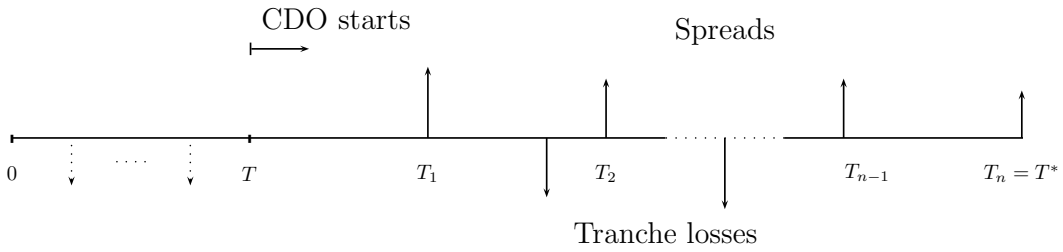


Figure 1.3: Cash flows for a single tranche in a FCDO

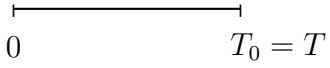
on a specified notional principal at set dates until the mth default occurs among the reference entities or until the maturity of the contract. If the mth default happens before the maturity of the BDS, the protection seller pays the losses caused by the mth default only to the protection buyer.

A forward-starting collateralized debt obligation (FCDO) is a forward contract[3] obligating the holder to buy or sell protection on a specified CDO tranche for a specified spread at a specified future time. For example, a FCDO might obligate the holder to buy protection on a CDO tranche with attachment point a and detachment point b over a future period $[T, T^*]$ for a predetermined spread s. At time T, the contract turns into a single tranche CDO over $[T, T^*]$ with attachment point $(a + L_T)$ and detachment point $(b + L_T)$, where L_T is the aggregate pool losses before T. Figure 1.3 illustrates cash flows for a single tranche in a FCDO. In the figure, the premium dates are T_i, for $i = 1, \ldots, n$. The dotted arrows before T denote the pool losses L_T before the associated CDO starts. Like other forward contracts, the parties associated with FCDOs do not suffer from any loss before T, which makes the contract popular in a volatile market which is expected to stabilize after a short time.

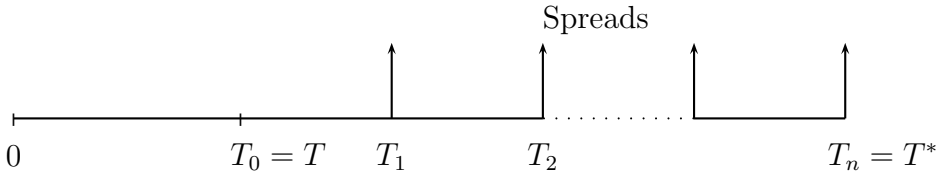
A forward-starting basket default swap (FBDS) is a forward contract obligating the holder to buy or sell a BDS at a specified future time. Figure 1.4 illustrates possible scenarios for an mth-to-default FBDS. In the figure, we denote the maturity date of the

[3] A forward contract is an agreement between two parties to buy or sell an asset at a predetermined future time [17].

(a) Less than m entities survive till T



(b) At least m entities survive till T and the mth default does not occur in $[T, T^*]$



(c) At least m entities survive till T and the mth default occurs in $[T, T^*]$

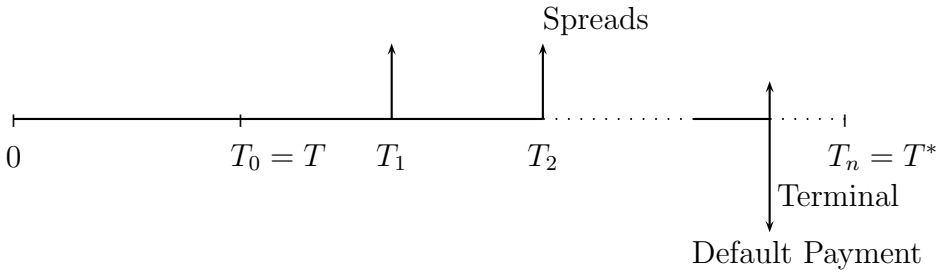


Figure 1.4: Cash flows for an mth-to-default FBDS

forward contract, or equivalently the starting date of the BDS, by T; the maturity date of the BDS by T^*; and the premium dates by T_i, $i = 1, \ldots, n$, where $T = T_0 < T_1 < \ldots < T_n = T^*$. Whether the BDS starts or not is determined by the number of entities left in the basket at T: if less than m names survive till T, the contract terminates without any payments, as shown in case (a); if at least m entities survive till T, the BDS starts, and the cash flows are the same as those in a BDS starting at T, as shown in cases (b) and (c).

1.2 Brief literature review

Credit risk, the risk of an obligor failing any payment obligation, is the cornerstone of credit derivatives. There are two kinds of models for credit risk: structural models and

reduced-form models. In structural models, as pioneered by Merton [43], the default risk is associated with economically meaningful variables. For example, when the asset value of a firm falls below its debt value, the firm defaults. Although appealing, structural models have difficulty fitting market data. Often they can not generate high enough spreads for the short term. Consequently, reduced-form models were developed with the ability to match market data. Here, the default is modeled by a stochastic process not necessarily having a clear economic justification, as describe by Jarrow and Turnbull [31] and Duffie and Singleton [10]. In addition to the calibration advantage, reduced-form models are mathematically simpler than structural models.

Once we have a credit risk model, the valuation of single-name credit derivatives such as CDS becomes straightforward. However for multi-name derivatives such as CDOs, we have to incorporate the default correlations of the underlying portfolio. Due to its computational efficiency, the Gaussian factor copula model [41] has become the market standard model for pricing multi-name credit derivatives. In factor copula models, the default of an obligor is associated with a credit-healthiness variable for the obligor falling below its default boundary. To make the analysis tractable, the credit-healthiness variable is decomposed into a common factor and an idiosyncratic factor, which are assumed to be independent. Then, the default correlation is introduced through the correlation between the credit-healthiness variable and the common factor. Conditional on the common factor, the default of each obligor is independent. Consequently, we can compute or approximate the portfolio loss with efficient numerical methods. This approach is also known as the conditional independence framework [38].

However, the Gaussian factor copula model cannot match market quotes consistently without violating the model assumptions [20]. For example, it has to use different correlation coefficients for different tranches based on the same underlying portfolio. To better match the observable spreads, several modifications have been proposed based on the conditional independence framework. See, for example, [2], [6] and [21]. Most of these

approaches are static one-period models generating a portfolio loss distribution at a fixed maturity. They may not be flexible enough to match market quotes or applicable for new products with strong time-dependent features [1]. To partially overcome this deficiency, Andersen [1] and Sidenius [49] introduced several chaining techniques to build multi-period factor copula models from one-period factor copula models. As these models have to integrate over all the common factors, they require a multi-dimensional integration, which is usually computed by Monte Carlo simulation. This makes the model calibration extremely time consuming. Except for some special cases, for example where the factors are the same for all periods, existing chaining methods cannot avoid multi-dimensional integration. Therefore, current multi-period models are hard to generalize to more than two periods.

Another approach for multi-name credit derivatives is the top-down approach, which models the portfolio loss directly. For example, Bennani [7], Schönbucher [48], and Sidenius, Piterbarg and Andersen [50] propose similar frameworks to model the dynamics of the aggregate portfolio losses by modeling the forward loss rates. With these pool loss dynamics, the pricing of credit derivatives becomes straightforward. However, these models require a large amount of data to calibrate and are currently somewhat speculative [1].

1.3 Main contributions

The purpose of this thesis is to develop dynamic models for multi-name credit derivatives, which are able to calibrate against the market quotes consistently and are applicable to exotic time-dependent credit derivatives, like FCDOs or options on tranches. As computational efficiency is an important concern for practitioners, we employ factor copula models to handle default correlations for multi-name credit derivatives.

Compared with standard credit derivatives, forward-starting products have a random starting portfolio due to defaults that may occur before the maturity of the forward con-

tract (i.e., before T in Figure 1.3). One obvious method is to consider all the possible combinations of the starting pool, then to price the forward-starting products as standard ones [4]. However, as the pool size is often very large, this method may be very computationally expensive because of the associated large combinatorial problem. We propose a reduction method that converts a forward-starting credit derivative to an equivalent standard one, thereby allowing us to use the valuation methods for standard products. In addition to the reduction method, the valuation of FBDS requires the computation of the expectation of the product of two random variables. We use Monte Carlo simulation to estimate the correlation between these two random variables, thereby allowing us to approximate the expectation of the product. We combine this Monte Carlo simulation with an analytical method similar to that described above for FCDOs to obtain an effective hybrid method for the valuation of FBDS. This hybrid method is a novel technique which can be viewed either as a means to accelerate the convergence of Monte Carlo simulation or as a way to estimate parameters in an analytical method that are difficult to compute directly.

Using the method described above, we are able to price FCDOs and FBDS. However, these factor copula models are static models, that are not applicable to more sophisticated time-dependent derivatives. To remedy this deficiency, we develop a novel chaining technique that enables us to build a multi-period factor copula model from several one-period factor copula models. This allows the default correlations to be time-dependent, thereby generating sufficient degrees of freedom to fit market quotes consistently. As discussed above, multi-period factor copula models usually require a multi-dimensional integration, which makes calibration extremely time consuming. However, for our chaining method, the portfolio loss of a completely homogeneous pool possesses the Markov property. Thus, we can compute the portfolio loss distribution analytically without the curse of dimensionality.

The multi-period factor copula model is a discrete-time dynamic model. As a first

step toward a continuous-time dynamic model, we develop a dynamic model for single-name credit derivatives, such as CDS. We develop a first hitting time model, that uses a random initial state. The explicit relation between the default distribution and the initial state distribution is found and the existence conditions of such a relation are also discovered. Extending the single-name model to a multi-name model requires a novel correlation structure to ensure the dynamic, analytical and easy-to-calibrate properties of such a model. We leave this important and challenging problem as future work.

1.4 Outline of thesis

In this thesis, we develop four computational methods for different credit derivatives: a reduction method for FCDOs, a hybrid method for FBDS, a chaining technique to build a dynamic factor copula model for CDOs and a randomization method in the first hitting time setting for CDS.

In Chapter 2, which is based largely on our results in [30], we develop a generic method for pricing a FCDO by converting it to an equivalent synthetic CDO. Then, the value of the FCDO is computed by the well developed methods for pricing the equivalent synthetic one. Numerical results demonstrate that our method is much more accurate and efficient than Monte Carlo simulation, the only other effective numerical method for pricing FCDOs.

In Chapter 3, which is based largely on our results in [27], we propose a fast hybrid method for pricing FBDS. In addition to converting a FBDS to an equivalent BDS, the method combines Monte Carlo simulation with an analytic approach to achieve an effective numerical method. Sensitivity of the hybrid method is also discussed. Numerical results demonstrate the accuracy and efficiency of our proposed hybrid method.

In Chapter 4, which is based largely on our results in [28], we develop a dynamic multi-period factor copula model to overcome the calibration deficiency of factor copula

models by allowing the correlation to be time-dependent. Usually, multi-period factor copula models require multi-dimensional integration. In our model, the portfolio loss of a completely homogeneous pool possesses the Markov property, thus we can compute its distribution across time by a recursive method instead of by Monte Carlo simulation. Numerical results demonstrate the efficiency and flexibility of our model in calibrating against market quotes.

In Chapter 5, which is based largely on our results in [29], we develop a dynamic, analytical and easy-to-calibrate model for CDS based on the first hitting time model. We solve the inverse problem for the first hitting time distribution by randomizing the initial state of the Wiener process. For a wide classes of default distributions, the initial state distribution is available in analytical form. Calibration is also discussed, and numerical results are reported.

Chapter 6 concludes the thesis and discusses future work.

Chapter 2

Reduction method for forward-starting CDOs

The most common approach for pricing FCDOs uses the factor copula model together with Monte Carlo simulation. Such methods are flexible, but are computationally expensive. Therefore, Baheti, Mashal, and Naldi [4] developed a method that considers all the possible pool losses L_T before the maturity of the forward contract. Conditional on a particular L_T, they price single tranches using methods developed for synthetic CDOs. Although their approach is analytical, it is also inefficient due to the large number of default combinations for L_T. For the valuation of FCDOs, we made a fairly simple, but very useful, observation that allows us to convert the FCDO to an equivalent synthetic CDO and then to price the equivalent synthetic CDO by factor copula models. Therefore, our approach avoids having to consider all possible pool losses before T.

In this chapter, we develop our reduction method for FCDOs. Section 2.1 describes the pricing equations for FCDOs. Section 2.2 derives a method to convert a FCDO to an equivalent synthetic CDO. Section 2.3 reviews the Gaussian factor copula model. Section 2.4 introduces a valuation method for synthetic CDOs. Section 2.5 presents two numerical examples.

2.1 Pricing equations

In a FCDO, the protection seller absorbs the pool loss specified by the tranche structure. That is, if the pool loss over $[T, T^*]$ is less than the tranche attachment point a, the seller does not suffer any loss; otherwise, the seller absorbs the loss up to the tranche size $S = b - a$. In return for the protection, the buyer pays periodic premia at specified times $T_1 < T_2 < \ldots < T_n = T^*$, where $T = T_0 < T_1$.

We consider a FCDO containing K instruments with loss-given-default N_k for name k in the original pool. Assume that the recovery rates are constant. Let D_i denote the risk-free discount factors at time T_i, and d_i denote the expected value of D_i in a risk-neutral measure. Denote the original pool loss up to time T_i by L_i, then the effective pool loss over $[T, T_i]$ is $\hat{L}_i = L_i - L_T$. Therefore, the loss absorbed by the specified tranche is

$$\mathcal{L}_i = \min(S, (\hat{L}_i - a)^+), \quad \text{where } x^+ = \max(x, 0) \tag{2.1}$$

We make the standard assumption that the D_i's and L_i's are independent, then D_i and $\mathcal{L}_i$ are also independent.

In general, valuation of a FCDO tranche balances the expectation of the present values of the premium payments (premium leg) against the effective tranche losses (default leg), such that

$$\mathbb{E}\Big[\sum_{i=1}^{n} s(S - \mathcal{L}_i)(T_i - T_{i-1})D_i\Big] = \mathbb{E}\Big[\sum_{i=1}^{n}(\mathcal{L}_i - \mathcal{L}_{i-1})D_i\Big] \tag{2.2}$$

Throughout the thesis, $\mathbb{E}$ denotes the risk-neutral expectation with respect to the risk-neutral probability $\mathbb{P}$. Therefore, the fair spread s is given by

$$s = \frac{\mathbb{E}\big[\sum_{i=1}^{n}(\mathcal{L}_i - \mathcal{L}_{i-1})D_i\big]}{\mathbb{E}\big[\sum_{i=1}^{n}(S - \mathcal{L}_i)(T_i - T_{i-1})D_i\big]} = \frac{\sum_{i=1}^{n}(\mathbb{E}\mathcal{L}_i - \mathbb{E}\mathcal{L}_{i-1})d_i}{\sum_{i=1}^{n}(S - \mathbb{E}\mathcal{L}_i)(T_i - T_{i-1})d_i} \tag{2.3}$$

In the last equality of (2.3), we use the fact that D_i and $\mathcal{L}_i$ ($\mathcal{L}_{i-1}$) are independent[1]. Therefore, the problem is reduced to the computation of the mean tranche losses, $\mathbb{E}\mathcal{L}_i$. To compute this expectation, we have to compute the effective portfolio loss $\hat{L}_i$'s distribution.

[1]Based on the standard assumption of independence between D_i's and L_i's, the stochastic effect of D_i is lost immediately. Therefore, we use d_i directly in the rest of the thesis.

2.2 Reduction method

From (2.1), we know that the expectation of the tranche losses $\mathbb{E}\mathcal{L}_i$ is determined by the distribution of the effective pool losses $\hat{L}_i$. If we denote the default time of name k by τ_k and define the indicator function $\mathbf{1}_{\{\tau_k \le t\}}$ by

$$\mathbf{1}_{\{\tau_k \le t\}} = \begin{cases} 1, & \tau_k \le t \\ 0, & \text{otherwise} \end{cases}$$

then we have

$$\hat{L}_i = L_i - L_T = \sum_{k=1}^{K} N_k \mathbf{1}_{\{\tau_k \le T_i\}} - \sum_{k=1}^{K} N_k \mathbf{1}_{\{\tau_k \le T\}} = \sum_{k=1}^{K} N_k \mathbf{1}_{\{T < \tau_k \le T_i\}} \tag{2.4}$$

This simple, but very useful, observation is key to our approach. The rightmost sum in (2.4) is the expression of the pool losses in a synthetic CDO starting at time T. Therefore, the pool loss in our FCDO is equivalent to the pool loss in this synthetic CDO. The distributions of the effective pool losses $\hat{L}_i$ are determined by whether the underlying names default in $[T, T_i]$, and they can be computed through the equivalent synthetic CDO with modified default probabilities. That is, instead of using the probability that name k defaults before T_i in the synthetic CDO, we use the probability that name k defaults during the period $[T, T_i]$ in the equivalent synthetic CDO.

Remark. According to the argument above, a synthetic CDO can be treated as a special case of a FCDO with $T = 0$.

2.3 Gaussian factor copula model

In this section, we review the market-standard Gaussian factor copula model for pricing synthetic CDOs. However, it is important to note here that our approach is quite general in the sense that any other method based on the conditional independence framework for pricing synthetic CDOs could be used in place of the Gaussian factor copula model in our approach to pricing FCDOs.

Due to their tractability, Gaussian factor copula models are widely used to specify a joint distribution for default times consistent with their marginal distribution. A one-factor model was first introduced by Vasicek [51] to evaluate the loan loss distribution, and the Gaussian copula was first applied to multi-name credit derivatives by Li [41]. After that, the model was generalized by Andersen, Sidenius, and Basu [3], Hull and White [19], and Laurent and Gregory [38], to name just a few. In this section, we review the one-factor Gaussian copula model to illustrate the conditional independence framework and introduce the modified conditional default probability.

2.3.1 One factor copula model

Assume the risk-neutral (cumulative) default probabilities

$$\pi_k(t) = \mathbb{P}(\tau_k \leq t), \quad k = 1, 2, \ldots, K$$

are known. To generate the dependence structure of default times, we introduce random variables U_k, such that

$$U_k = \beta_k X + \sigma_k \varepsilon_k, \quad \text{for } k = 1, 2, \ldots, K \tag{2.5}$$

where X is the systematic risk factor reflecting the health of the macroeconomic environment; ε_k are idiosyncratic risk factors, which are independent of each other and also independent of X; the constants β_k and σ_k, satisfying $\beta_k^2 + \sigma_k^2 = 1$, are assumed known. The random variables X and ε_k follow zero-mean unit-variance distributions, so the correlation between U_i and U_j is $\beta_i \beta_j$.

The default times τ_k and the random variables U_k are connected by a percentile-to-percentile transformation, such that

$$\pi_k(t) = \mathbb{P}(\tau_k \leq t) = \mathbb{P}(U_k \leq b_k(t))$$

where each $b_k(t)$ can be viewed as a default barrier. Thus the dependence among default times is captured by the common factor X.

Models satisfying the assumptions above are said to be based on the conditional independence framework. If we also assume X and ε_k follow standard normal distributions, we get a Gaussian factor copula model. In this case, each U_k also follows a standard normal distribution. Hence we have

$$b_k(t) = \Phi^{-1}(\pi_k(t)). \tag{2.6}$$

where Φ is the standard normal cumulative distribution function.

Conditional on a particular value x of X, the risk-neutral default probabilities are defined as

$$\pi_k(t,x) \equiv \mathbb{P}(\tau_k \leq t \mid X = x) = \mathbb{P}(U_k \leq b_k(t) \mid X = x) \tag{2.7}$$

Substituting (2.5) and (2.6) into (2.7), we have

$$\pi_k(t,x) = \mathbb{P}\big[\beta_k x + \sigma_k \varepsilon_k \leq \Phi^{-1}(\pi_k(t))\big] = \Phi\left[\frac{\Phi^{-1}(\pi_k(t)) - \beta_k x}{\sigma_k}\right] \tag{2.8}$$

In this framework, the default events of the names are assumed to be conditionally independent. Thus, the problem of correlated names is reduced to the problem of independent names. The distribution of the effective pool loss $\hat{L}_i$ satisfies

$$\mathbb{P}(\hat{L}_i = l) = \int_{-\infty}^{\infty} \mathbb{P}_x[\hat{L}_i = l] d\Phi(x) \tag{2.9}$$

where $\mathbb{P}_x[\hat{L}_i = l]$ is the probability of $\hat{L}_i = l$, conditional on a specified value x of X; and $\hat{L}_i = \sum_{k=1}^{K} N_k \mathbf{1}_{\{b_k(T) < U_k \leq b_k(T_i)\}}$, where $\mathbf{1}_{\{b_k(T) < U_k \leq b_k(T_i)\}}$ are mutually independent, conditional on $X = x$. Therefore, if we know the conditional distributions of $\mathbf{1}_{\{b_k(T) < U_k \leq b_k(T_i)\}}$, the conditional distributions of $\hat{L}_i$ can be computed easily, as can $\mathbb{E}\mathcal{L}_i$ in (2.3). To approximate the integral (2.9) we use a quadrature rule such as the Gaussian-Legendre rule. Thus, the integral (2.9) reduces to

$$\mathbb{P}(\hat{L}_i = l) \approx \sum_{m=1}^{M} w_m \mathbb{P}_{x_m}[\hat{L}_i = l]$$

where the w_m and x_m are the quadrature weights and nodes, respectively. Therefore, the main challenge in CDO pricing lies in the evaluation of the distribution of $\hat{L}_i$, conditional on a given value x of X.

2.3.2 Modified conditional default probability

Conditional on a given x, to compute the distribution of $\hat{L}_i$ we need to specify the distribution of $\mathbf{1}_{\{T<\tau_k\leq T_i\}}$, which is equal to the conditional distribution of $\mathbf{1}_{\{b_k(T)<U_k\leq b_k(T_i)\}}$. To this end, we introduce modified conditional default probabilities,

$$\hat{\pi}_k(t,x) = \pi_k(t,x) - \pi_k(T,x), \quad \text{for } t \geq T \tag{2.10}$$

so that the conditional distribution of $\mathbf{1}_{\{T<\tau_k\leq T_i\}}$ satisfies

$$\mathbb{P}_x(\mathbf{1}_{\{T<\tau_k\leq T_i\}} = 1) = \hat{\pi}_k(T_i,x)$$
$$\mathbb{P}_x(\mathbf{1}_{\{T<\tau_k\leq T_i\}} = 0) = 1 - \hat{\pi}_k(T_i,x)$$

where $\mathbb{P}_x$ is the probability conditional on $X = x$. Armed with the modified conditional default probabilities, the conditional distribution of $\hat{L}_i$ for a FCDO can be computed using the methods developed for synthetic CDOs.

2.4 Evaluation of pool loss distribution

Based on the conditionally independent framework, researchers have developed many methods to evaluate the conditional loss distribution for synthetic CDOs. There are generally two kinds of approaches: the first one computes the conditional loss distribution exactly by a recursive relationship or the convolution technique, e.g., Andersen, Sidenius, and Basu [3], Hull and White [19], Laurent and Gregory [38], Jackson, Kreinin, and Ma [26]; the second approach computes the conditional loss distribution approximately by, for example, the normal power or compound Poisson approximations, e.g., De Prisco, Iscoe, and Kreinin [46] and Jackson, Kreinin, and Ma [26]. Here we review one of the exact methods – JKM proposed by Jackson, Kreinin, and Ma [26] – and employ it to solve our numerical examples in the next section. Other methods for pricing synthetic CDOs are equally applicable.

A homogeneous pool has identical loss-given-default, denoted by N_1, but different default probabilities and correlation factors. Hence, conditional on a specified common factor x, the pool losses satisfy

$$\hat{L}_i = \sum_{k=1}^{K} N_k \mathbf{1}_{\{T<\tau_k\leq T_i\}} = N_1 \sum_{k=1}^{K} \mathbf{1}_{\{T<\tau_k\leq T_i\}}$$

Therefore, we can compute the conditional distribution of $\hat{L}_i$ through computing the conditional distribution of the number of defaults $\sum_{k=1}^{K} \mathbf{1}_{\{T<\tau_k\leq T_i\}}$.

Suppose the conditional distribution of the number of defaults over a specified time horizon $[T, T_i]$ in a homogeneous pool with k names is already known. Denote it by $V_k = (p_{k,k}, p_{k,k-1}, \ldots, p_{k,0})^T$, where $p_{k,j} = \mathbb{P}_x(\sum_{l=1}^{k} \mathbf{1}_{\{T<\tau_l\leq T_i\}} = j)$. The conditional distribution of the number of defaults in a homogeneous pool containing these first k names plus the $(k+1)$st name with modified conditional default probability $Q_{k+1} = \hat{\pi}_{k+1}(T_i, x)$ satisfies

$$V_{k+1} = \begin{pmatrix} p_{k+1,k+1} \\ p_{k+1,k} \\ \vdots \\ p_{k+1,1} \\ p_{k+1,0} \end{pmatrix}_{(k+2)\times 1} = \begin{pmatrix} V_k & 0 \\ 0 & V_k \end{pmatrix}_{(k+2)\times 2} \begin{pmatrix} Q_{k+1} \\ 1 - Q_{k+1} \end{pmatrix}_{2\times 1}$$

Using this relationship, V_K can be computed after $K-1$ iterations with initial value $V_1 = (p_{1,1}, p_{1,0})^T = (Q_1, 1-Q_1)^T$. The method has been proved numerically stable by Jackson, Kreinin, and Ma [26].

An inhomogeneous pool, which has different loss-given-default, different default probabilities, and different correlation factors, can be divided into I small homogeneous pools with notionals $N_1, N_2, \ldots, N_I$. The conditional loss distribution for the ith group can be computed using the above method. We denote it by $(p_{i,0}, \ldots, p_{i,d_i})$, where d_i is the maximum number of defaults in group i. Suppose the conditional loss distribution of the first i groups is available. Denote it by $(p_0^{(i)}, \ldots, p_{S_i}^{(i)})$, where $p_s^{(i)}$ is the probability that s

units of the pool default out of the first i groups, for $s = 0, 1, \ldots, S_i = \sum_{j=1}^{i} d_j N_j$. The conditional loss distribution of the pool containing these first i groups plus the $(i+1)$st group satisfies

$$p_s^{(i+1)} = \sum_{\substack{l \in \{0,\ldots,S_i\} \\ (s-l)/N_{i+1} \in \{0,\ldots,d_{i+1}\}}} p_l^{(i)} \cdot p_{i+1,(s-l)/N_{i+1}}, \quad \text{for } s = 0, 1, \ldots, S_{i+1} = S_i + d_{i+1} N_{i+1}$$

To start the iteration, we need to initialize the conditional loss distribution of the first group $(p_0^{(1)}, p_1^{(1)}, \ldots, p_{d_1 N_1}^{(1)})$ by setting possible loss amounts with certain probabilities and impossible loss amounts with probability 0, such that

$$p_s^{(1)} = \begin{cases} p_{1,s/N_1}, & s/N_1 \in \{0, 1, \ldots, d_1\} \\ 0, & \text{otherwise} \end{cases}$$

2.5 Numerical results I

Based on the methods described above, we propose the following steps for pricing FCDOs:

1. Convert $\pi_k(T_i)$ to conditional default probabilities $\pi_k(T_i, x)$ using (2.8) and compute modified conditional default probabilities $\hat{\pi}_k(T_i, x)$ by (2.10);

2. Compute the conditional distribution of $\hat{L}_i$ by the JKM method described in Section 2.4;

3. Approximate $\mathbb{P}(\hat{L}_i = l)$ using a quadrature rule (2.9);

4. Evaluate $\mathbb{E}[\mathcal{L}_i]$ using (2.1);

5. Complete the computation using (2.3).

We compare the results generated by the Monte Carlo method to those obtained by our method. The numerical experiments are based on two FCDOs: one is a homogeneous pool; the other is an inhomogeneous pool. The contracts are 5-year CDOs starting one

Tranche	Attachment	Detachment
Super-senior	12.1%	100%
Senior	6.1%	12.1%
Mezzanine	4%	6.1%
Junior	3%	4%
Equity	0%	3%

Table 2.1: CDO tranche structures

Credit	Time					
rating	1Y	2Y	3Y	4Y	5Y	6Y
Baa2	0.0007	0.0030	0.0068	0.0119	0.0182	0.0223
Baa3	0.0044	0.0102	0.0175	0.0266	0.0372	0.0485

Table 2.2: Risk-neutral cumulative default probabilities

year later with annual premium payments, i.e., $T = T_0 = 1$, $T_1 = 2$, ..., $T_5 = 6 = T^*$. The CDO tranche structures are described in Table 2.1. The continuously compounded interest rates are 4%. The recovery rate of the instruments in the pool is 40%. The risk-neutral cumulative default probabilities for two credit ratings are listed in Table 2.2. The pool structure of the inhomogeneous CDO is defined in Table 2.3, while the homogeneous pool has the same structure except that the notional values are 30 for all names.

We employ Latin hypercube sampling to accelerate the Monte Carlo simulation. Latin hypercube sampling is one of the variance reduction techniques used in high dimensional Monte Carlo simulation. It generates multiple dimensional random variables evenly over the sample space without increasing the sample size. More details can be found in Glasserman [15].

Each experiment consists of 100,000 trials, and 100 runs (with different seeds) of each experiment are made. Based on the results of these 100 experiments, we calculate

Notional	Credit Rating	β_k	Quantity
10	Baa2	0.5	5
10	Baa3	0.5	2
10	Baa2	0.6	5
10	Baa3	0.6	5
10	Baa3	0.7	4
10	Baa3	0.8	4
20	Baa3	0.5	7
20	Baa2	0.6	10
20	Baa3	0.6	8
30	Baa2	0.5	15
30	Baa3	0.5	10
60	Baa2	0.4	10
60	Baa2	0.4	8
60	Baa3	0.5	7

Table 2.3: Inhomogeneous pool structure

Pool	Tranche	Monte Carlo	95% CI	Our method
Homogeneous	Equity	1158.19	[1155.11, 1162.18]	1158.25
	Junior	388.83	[386.44, 391.65]	388.80
	Mezzanine	238.37	[236.83, 240.29]	238.27
	Senior	82.88	[81.87, 83.73]	82.89
	Super-Senior	1.29	[1.23, 1.33]	1.29
Inhomogeneous	Equity	1216.41	[1212.70, 1221.00]	1216.35
	Junior	415.54	[412.36, 418.73]	415.46
	Mezzanine	235.04	[232.99, 237.43]	234.89
	Senior	70.26	[69.36, 71.27]	70.21
	Super-Senior	0.80	[0.75, 0.84]	0.79

Table 2.4: Tranche premia (bps)

the mean and the 95% non-parametric confidence interval. Table 2.4 presents the risk premia for these two FCDOs. The results demonstrate that our method is accurate for the valuation of FCDOs.

For the homogeneous FCDO, the running time of one Monte Carlo experiment with 100,000 trials is about 14 times that used by our method; for the inhomogeneous FCDO, the Monte Carlo method uses about 6 times the CPU time used by our method. These comparisons demonstrate that our method is much more efficient than the Monte Carlo method. In addition, it is worth noting that the main reason for this difference in speed-ups between the Monte Carlo method and our method is that our method must compute the actual pool loss notional for the inhomogeneous pool, instead of just the number of defaults for the homogeneous pool. Consequently, for our method, the computation for the inhomogeneous pool is more time consuming than for the homogeneous pool.

Chapter 3

Hybrid method for forward-starting basket default swaps

Another application of the reduction method described in the previous chapter is to price forward-starting basket default swaps (FBDS). FBDS and FCDOs are similar products. However an important difference between them is that the notional of FCDOs at time T equals the tranche size, which is deterministic, while the notional of FBDS at time T depends on the pool losses before T, which is random. This difference makes the valuation of FBDS more complicated. Similar to FCDOs, Monte Carlo simulation and the conditional squared method [52] (which considers all possible combinations of L_T and prices each of them using a standard method) are the only methods previously available for FBDS. Both of these approaches are computationally expensive.

We develop a fast approximation method for the valuation of FBDS. The method converts the FBDS valuation problem to an equivalent BDS valuation problem using the reduction method described in the previous chapter, thereby avoiding the large number of possible default combinations. To complete the pricing, our method combines Monte Carlo simulation with the analytical approach to obtain an accurate and efficient hybrid method. The Monte Carlo method generates a coarse approximation for an important

parameter required by the analytical method. This parameter cannot easily be computed directly. Alternatively, the analytical method can be viewed as a means to accelerate the convergence of Monte Carlo simulation.

This chapter is structured as follows. Section 3.1 derives the pricing equations for FBDS. Section 3.2 describes our hybrid method. Section 3.3 revisits the Gaussian factor copula models. Section 3.4 introduces a valuation method for BDS. Section 3.5 presents two numerical examples. Section 3.6 discusses method sensitivity with respect to the parameters approximated by Monte Carlo simulation.

3.1 Pricing equations

Similar to the assumptions for FCDOs, the underlying pool in a FBDS contains K instruments with loss-given-default N_k for name k. The recovery rates are constant, and the interest rate process is independent of the default process of the basket. Let d_i denote the expected value of the risk-free discount factor corresponding to T_i. Without loss of generality, we assume that the default payment happens at the nearest premium date following (or equal to) the terminal default time, if it occurs before the contract maturity; and no accrued interest is paid out at the terminal default time.

The terminal default time τ, which triggers the default payment, can be expressed as a function of individual default times τ_k. We denote the loss of the FBDS at the terminal default time by

$$L = \begin{cases} g(N_k), & \tau = \tau_k \in (T, T^*] \\ 0, & \text{otherwise} \end{cases}$$

where $g(\cdot)$ is a payoff function.

Let $\mathcal{B}_T$ denote the set of names left in the basket at T. We also denote the number of names in $\mathcal{B}_T$ by $|\mathcal{B}_T|$ and the probability distribution of $\mathcal{B}_T$'s composition by $\mathbb{P}(\mathcal{B}_T)$. The event $m \leq |\mathcal{B}_T| \leq K$ is the event that the BDS associated with the FBDS actually

starts.

In general, the valuation of a FBDS balances the expectation of the present values of the premium payments against the default payments, such that $\mathbb{E}V_{\text{prem}} = \mathbb{E}V_{\text{def}}$. To compute the expectation numerically, we introduce the terminal default probability

$$\Pi_i^{(k)} = \mathbb{P}(\tau = \tau_k \in (T, T_i])$$

We also define the survival indicator function by $\bar{\mathbf{1}}_i = \mathbf{1}_{\{\tau > T_i,\, m \le |\mathcal{B}_T| \le K\}}$. Its probability distribution satisfies

$$\bar{\Pi}_i = \mathbb{P}(\bar{\mathbf{1}}_i = 1) = \mathbb{P}\big(m \le |\mathcal{B}_T| \le K\big) - \sum_{k=1}^{K} \Pi_i^{(k)} \tag{3.1}$$

Under the assumptions above, the value of the default leg satisfies

$$\mathbb{E}V_{\text{def}} = \sum_{k=1}^{K} g(N_k) \sum_{i=1}^{n} d_i \big(\Pi_i^{(k)} - \Pi_{i-1}^{(k)}\big) = \sum_{i=1}^{n} d_i \sum_{k=1}^{K} g(N_k) \big(\Pi_i^{(k)} - \Pi_{i-1}^{(k)}\big) \tag{3.2}$$

Similarly, the value of the premium leg satisfies

$$\mathbb{E}V_{\text{prem}} = \mathbb{E}\Big[s N_T \bar{\mathbf{1}}_i \sum_{j=1}^{i} \Delta T_j \cdot d_j\Big] = s \sum_{i=1}^{n} \Delta T_i \cdot d_i \cdot \mathbb{E}\big[N_T \bar{\mathbf{1}}_i\big] \tag{3.3}$$

where N_T is the sum of the notional values of all names in $\mathcal{B}_T$, and $\Delta T_i = T_i - T_{i-1}$. Therefore, the fair spread can be computed by

$$s = \frac{\mathbb{E}V_{\text{def}}}{\text{DV01}} = \frac{\sum_{i=1}^{n} d_i \sum_{k=1}^{K} g(N_k) \big(\Pi_i^{(k)} - \Pi_{i-1}^{(k)}\big)}{\sum_{i=1}^{n} \Delta T_i \cdot d_i \cdot \mathbb{E}\big[N_T \bar{\mathbf{1}}_i\big]} \tag{3.4}$$

where $\text{DV01} \equiv \sum_{i=1}^{n} \Delta T_i \cdot d_i \cdot \mathbb{E}\big[N_T \bar{\mathbf{1}}_i\big]$ is the dollar value change in the premium leg with respect to 1 bps change in spread.

3.2 Hybrid method

Since $\Pi_i^{(k)}$ is defined in the same form as a similar probability used to value BDS in [24] and [37] with $T = 0$, we can employ the method for BDS to compute the key probability $\Pi_i^{(k)}$ using the reduction method in Chapter 2. That is, instead of using the probability of

name k defaulting before time t, we use the probability of name k defaulting in $(T, t]$. The starting pool of the BDS associated with the FBDS is random in the original formulation of the problem; in our new formulation, after the reduction, the starting pool in the equivalent BDS contains all K names with certainty. Therefore, we avoid the large combinatorial problem due to the consideration of all the possible starting pools in the original formulation of the problem.

Once $\Pi_i^{(k)}$ is known, the computation of $\mathbb{E}V_{\text{def}}$ is straightforward following (3.2). To compute $\mathbb{E}V_{\text{prem}}$ or DV01, we need to compute the expectation $\mathbb{E}\big[N_T\bar{\mathbf{1}}_i\big]$. Suppose we know the correlation ρ_i between N_T and $\bar{\mathbf{1}}_i$, then $\mathbb{E}\big[N_T\bar{\mathbf{1}}_i\big]$ can be computed from

$$\rho_i = \frac{\mathbb{E}\big[N_T\bar{\mathbf{1}}_i\big] - \mathbb{E}\big[N_T\big]\mathbb{E}\big[\bar{\mathbf{1}}_i\big]}{\sqrt{\text{var}\big(N_T\big)\text{var}\big(\bar{\mathbf{1}}_i\big)}} \tag{3.5}$$

where $\mathbb{E}\big[\bar{\mathbf{1}}_i\big] = \bar{\Pi}_i$ and $\text{var}\big(\bar{\mathbf{1}}_i\big) = \bar{\Pi}_i\big(1-\bar{\Pi}_i\big)$. Once $\Pi_i^{(k)}$ is known, the computation of $\bar{\Pi}_i$ defined in (3.1) is straightforward, since the term $\mathbb{P}\big(m \le |\mathcal{B}_T| \le K\big)$ can be computed by the pool loss distribution methods for CDOs, e.g., [3], [38], [19] and [26]. Similarly, the terms $\mathbb{E}\big[N_T\big]$ and $\text{var}\big(N_T\big)$ are easy to compute with known pool loss distribution. Therefore, $\mathbb{E}\big[N_T\bar{\mathbf{1}}_i\big]$ can be computed by

$$\mathbb{E}\big[N_T\bar{\mathbf{1}}_i\big] = \mathbb{E}\big[N_T\big]\bar{\Pi}_i + \rho_i\sqrt{\text{var}\big(N_T\big)\bar{\Pi}_i\big(1-\bar{\Pi}_i\big)} \tag{3.6}$$

Hence, we can compute the premium leg value and complete the valuation.

The only unknown variable in (3.6) is the correlation coefficient ρ_i. We propose to use Monte Carlo simulation to approximate ρ_i. As we show later, the spread is not sensitive to the value of the correlation coefficients. Therefore, only a rough approximation to the ρ_i is needed. This is an important property of this application which contributes to the effectiveness of our hybrid method.

3.3 Gaussian factor copula model revisited

To compute the value of $\Pi_i^{(k)}$ for FBDS, we need to compute the joint distribution of K correlated random variables $\mathbf{1}_{\{T<\tau_k\le T_i\}}$. Using the conditional independence framework, we explain in Section 2.3 how the problem of correlated names is reduced to the problem of independent names. By (3.2) and (3.3), the mean values of the default leg and premium leg for a FBDS can be evaluated as

$$\mathbb{E}V_{\text{def}} = \int_{-\infty}^{\infty} \mathbb{E}_x[V_{\text{def}}]d\Phi(x) = \int_{-\infty}^{\infty} \sum_{i=1}^{n} d_i \Big(\sum_{k=1}^{K} g(N_k)\big(\Pi_i^{(k)}(x) - \Pi_{i-1}^{(k)}(x)\big)\Big)\, d\Phi(x) \quad (3.7)$$

$$\begin{aligned}
\mathbb{E}V_{\text{prem}} &= s\sum_{i=1}^{n} \Delta T_i \cdot d_i \Big(\mathbb{E}\big[N_T\big]\bar{\Pi}_i + \rho_i \sqrt{\text{var}\big(N_T\big)\bar{\Pi}_i\big(1-\bar{\Pi}_i\big)}\Big) \\
&= s\sum_{i=1}^{n} \Delta T_i \cdot d_i \Big(\int_{-\infty}^{\infty} \mathbb{E}_x\big[N_T\big]\bar{\Pi}_i(x)\, d\Phi(x) + \rho_i \int_{-\infty}^{\infty} \sqrt{\text{var}_x(N_T)\bar{\Pi}_i(x)\big(1-\bar{\Pi}_i(x)\big)}\, d\Phi(x)\Big) \\
&= \int_{-\infty}^{\infty} s\sum_{i=1}^{n} \Delta T_i \cdot d_i \Big(\mathbb{E}_x\big[N_T\big]\bar{\Pi}_i(x) + \rho_i \sqrt{\text{var}_x(N_T)\bar{\Pi}_i(x)\big(1-\bar{\Pi}_i(x)\big)}\Big)\, d\Phi(x) \qquad (3.8) \\
&= \int_{-\infty}^{\infty} \mathbb{E}_x[V_{\text{prem}}]d\Phi(x)
\end{aligned}$$

For simplicity, we denote the integrand of (3.8) by $\mathbb{E}_x[V_{\text{prem}}]$. However, it is essential for computational efficiency that we use the unconditional ρ_i in (3.8), rather than the conditional $\rho_i(x)$, as might be expected from the notation of $\mathbb{E}_x[V_{\text{prem}}]$.

As for FCDOs, we use a quadrature rule to approximate the integrals (3.7) and (3.8). Thus, for example, the integral (3.7) is approximated by

$$\mathbb{E}V_{\text{def}} \approx \sum_{m=1}^{M} w_m \mathbb{E}_{x_m}[V_{\text{def}}] \quad (3.9)$$

Therefore, the main challenge in pricing a FBDS lies in computing $\mathbb{P}_x(|\mathcal{B}_T|)$, $\mathbb{P}_x(N_T)$, $\Pi_i^{(k)}(x)$ and $\bar{\Pi}_i(x)$, conditional on a given value x of X.

3.3.1 Modified conditional default intensity

Armed with the conditional default probabilities $\pi_k(T,x)$ defined in (2.7), the conditional distribution of N_T and $|\mathcal{B}_T|$ can also be computed using the methods for CDOs or by brute

force to explore all the possible combinations. Besides the modified conditional default probabilities $\hat{\pi}_k(t,x)$, to compute the conditional probability $\hat{\Pi}_i^{(k)}(x)$ by the methods for BDS, we need to introduce the modified conditional default intensities. Assume the modified conditional default distribution that name k defaults in $(T,t]$ follows the Cox process

$$\mathbb{P}_x(T < \tau_k \leq t) = 1 - \exp\left(-\Lambda_k(t,x)\right) \tag{3.10}$$

where

$$\Lambda_k(t,x) = \int_T^t \lambda_k(u,x)du \tag{3.11}$$

and $\lambda_k(\cdot)$ is the modified conditional default intensity of the kth name. We know

$$\mathbb{P}_x(T < \tau_k \leq t) = \hat{\pi}_k(t,x) \tag{3.12}$$

where $\hat{\pi}_k(t,x)$ is given by (2.10). If we assume $\Lambda_k(t,x)$ is linear between premium dates T_i, then (3.11) implies that $\lambda_k(t,x)$ is a piecewise constant function, satisfying

$$\lambda_k(t,x) = \lambda_k(T_i,x), \quad \text{for } t \in (T_{i-1},T_i]$$

Combining this result with (3.11), we have

$$\Lambda_k(T_i,x) = \Lambda_k(T_{i-1},x) + \lambda_k(T_i,x)\cdot\Delta T_i$$

from which we obtain

$$\lambda_k(T_i,x) = \frac{1}{\Delta T_i}\Big(\Lambda_k(T_i,x) - \Lambda_k(T_{i-1},x)\Big) \tag{3.13}$$

From (3.10) and (3.12), we know

$$\Lambda_k(T_{i-1},x) = -\ln\big(1-\hat{\pi}_k(T_{i-1},x)\big)$$

$$\Lambda_k(T_i,x) = -\ln\big(1-\hat{\pi}_k(T_i,x)\big)$$

Substituting these last two expressions above for $\Lambda_k(T_{i-1},x)$ and $\Lambda_k(T_i,x)$ into (3.13), we obtain

$$\lambda_k(T_i,x) = \frac{1}{\Delta T_i}\ln\Big(\frac{1-\hat{\pi}_k(T_{i-1},x)}{1-\hat{\pi}_k(T_i,x)}\Big), \quad \text{for } i = 1,2,\ldots,n \tag{3.14}$$

3.4 Computation of terminal default probability

Available methods for BDS include the convolution technique by Laurent and Gregory [38] and the recursive method based on the order statistics of individual default times by Iscoe and Kreinin [24]. Here we review the recursive method in [24] and use it in our numerical examples.

In a first-to-default BDS, the conditional probabilities $\Delta\Pi_i^{(k)}(x) = \mathbb{P}_x(\tau = \tau_k \in (T_{i-1}, T_i])$ satisfy

$$\Delta\Pi_i^{(k)}(x) = \Pi_i^{(k)}(x) - \Pi_{i-1}^{(k)}(x) = \frac{\lambda_k(T_i, x)}{\sum_{k=1}^K \lambda_k(T_i, x)}\big(\bar{\Pi}_{i-1}(x) - \bar{\Pi}_i(x)\big) \tag{3.15}$$

where $\lambda_k(\cdot)$ is the modified conditional default intensities defined in (3.14); and $\bar{\Pi}_i(x) = \prod_{k=1}^K \big(1 - \hat{\pi}_k(T_i, x)\big)$.

For the mth-to-default BDS, Iscoe and Kreinin [24] derive the recursive relation between the mth-to-default and the $(m-1)$st-to-default contracts:

$$(m-1)\mathcal{P}_m(\mathcal{B}) = \sum_{j \neq k} \mathcal{P}_{m-1}(\mathcal{B}^{[j]}) - (K - m + 1)\mathcal{P}_{m-1}(\mathcal{B})$$

where $\mathcal{P}_m(\mathcal{B}) = \mathbb{P}(\tau = \tau_k \in (T_{i-1}, T_i])$ for the mth-to-default BDS; and $\mathcal{B}^{[j]}$ is the set of names obtained by excluding name j from $\mathcal{B}$. Iscoe and Kreinin [24] also derive an explicit expression that reduces the mth-to-default case to a set of first-to-default contracts:

$$\mathcal{P}_m(\mathcal{B}) = \sum_{v=0}^{m-1} (-1)^{m-v-1} \binom{K-v-1}{m-v-1} \sum_{\mathcal{J} \subset \mathcal{B}: |\mathcal{J}| = v} \mathcal{P}_1(\mathcal{B}^{[\mathcal{J}]})$$

where $\mathcal{J}$ is a subset of $\mathcal{B}$ and $\mathcal{B}^{[\mathcal{J}]}$ is the set of names obtained by excluding those names belonging to $\mathcal{J}$ from $\mathcal{B}$. Here, for simplicity, we give the recursion for the unconditional probabilities, but a similar recursion is also valid for the conditional probabilities.

3.5 Numerical results II

Based on the methods described above, we propose the following steps for pricing FBDS:

1. Convert $\pi_k(T_i)$ to conditional default probabilities $\pi_k(T_i, x)$ using (2.8) and then compute the modified conditional default probabilities $\hat{\pi}_k(T_i, x)$ using (2.10);

2. Compute the conditional distribution $\Pi_i^{(k)}(x)$ by the recursive method in Section 3.4 and $\mathbb{P}_x(m \le |\mathcal{B}_T| \le K)$ and $\mathbb{P}_x(N_T)$ by the methods for CDOs in Section 2.4 as well as $\bar{\Pi}_i(x)$ using (3.1), $\mathbb{E}_x\big[N_T\big]$ and $\mathrm{var}_x(N_T)$ from $\mathbb{P}_x(N_T)$;

3. Run a Monte Carlo simulation to approximate the ρ_i in (3.5);

4. Evaluate $\mathbb{E}_x[V_{\mathrm{def}}]$ and $\mathbb{E}_x[V_{\mathrm{prem}}]$ by (3.7) and (3.8), respectively;

5. Approximate $\mathbb{E}V_{\mathrm{def}}$ and $\mathbb{E}V_{\mathrm{prem}}$ using a quadrature rule (see (3.9));

6. Complete the computation using the pricing equations (3.4).

The numerical experiments are based on two FBDS: one is a homogeneous pool; the other is an inhomogeneous pool. The contracts are 5-year BDS starting one year later with quarterly premium payments, i.e., $T = T_0 = 1$, $T_i = 1 + 0.25i$, for $i = 1, \ldots, 20$. The continuously compounded interest rates are 4% for each T_i. The recovery rate of the instruments in the pool is 15%. The pool structure of the inhomogeneous FBDS is defined in Table 3.1; the homogeneous pool has the same structure except that the notional values are 100 for all names. The risk-neutral cumulative default probabilities for different credit ratings are listed in Table 3.2.

Table 3.3 lists the premia for the mth-to-default FBDS ($m = 1, \ldots, 4$) computed by the conditional squared method of Zhang [52] ("Analytic" column), and our fast approximation method described above ("Approximation" column) with 10^3 trials in the Monte Carlo simulation to approximate the correlation coefficients ρ_i. The table also lists the 95% confidence interval of the spread computed by a Monte Carlo method ("95% CI" column). The 95% confidence interval is computed as follows: each Monte Carlo experiment consists of 10^6 trials; we repeat each Monte Carlo experiment 500 times; then, we compute the 95% confidence interval from the empirical distribution of those

Name	Notional	Credit Rating	β_k
1	190	C4	0.5
2	80	C6	0.6
3	70	C1	0.9
4	360	C5	0.6
5	100	C2	0.5
6	200	C5	0.4
7	150	C5	0.7
8	123	C2	0.64
9	95	C5	0.55
10	107	C8	0.22

Table 3.1: Inhomogeneous FBDS pool

500 samples. The last column of Table 3.3 lists the relative errors of the spreads computed by our fast approximation method, using the spreads computed by the analytic method for the exact solution. Table 3.4 compares the 95% confidence interval computed by 10^6 trials of Monte Carlo simulation ("95% CI" row) with those computed by our fast approximation method with 100, 1,000 and 10,000 trials ("100", "1,000" and "10,000" rows, respectively). These tables demonstrate that our fast approximation method is accurate for the valuation of FBDS.

For the homogeneous FBDS, the running time of the Monte Carlo simulation with 10^6 trials is about 400 times slower than our fast approximation method; for the inhomogeneous FBDS, the running time of the Monte Carlo simulation is about 20 times slower than our fast approximation method. The main reason for this difference in speed-ups between the Monte Carlo method and our method is that our method must compute the actual pool loss notional for the inhomogeneous pool, instead of just the number of defaults for the homogeneous pool. Consequently, for our method, the computation for

	Time											
Rating	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00
C1	0.000247	0.001578	0.003022	0.004535	0.006115	0.007709	0.009422	0.011203	0.013051	0.014923	0.016902	0.018947
C2	0.000278	0.001776	0.003400	0.005102	0.006880	0.008672	0.010600	0.012604	0.014682	0.016788	0.019015	0.021315
C3	0.000370	0.002368	0.004534	0.006802	0.009173	0.011563	0.014134	0.016805	0.019577	0.022385	0.025353	0.028420
C4	0.000494	0.003156	0.006042	0.009063	0.012219	0.015400	0.018818	0.022369	0.026051	0.029779	0.033718	0.037786
C5	0.000529	0.003381	0.006471	0.009704	0.013080	0.016480	0.020134	0.023926	0.027857	0.031835	0.036035	0.040369
C6	0.000593	0.003787	0.007250	0.010876	0.014663	0.018480	0.022582	0.026843	0.031261	0.035735	0.040462	0.045343
C7	0.000741	0.004733	0.009059	0.013586	0.018312	0.023073	0.028187	0.033497	0.039000	0.044568	0.050449	0.056517
C8	0.000741	0.004734	0.009063	0.013595	0.018329	0.023099	0.028227	0.033553	0.039076	0.044669	0.050577	0.056679
	Time											
Rating	3.25	3.50	3.75	4.00	4.25	4.50	4.75	5.00	5.25	5.50	5.75	6.00
C1	0.021024	0.022951	0.024979	0.027044	0.029147	0.031217	0.033393	0.035605	0.037475	0.040136	0.042934	0.045778
C2	0.023653	0.025819	0.028101	0.030425	0.032791	0.035119	0.037567	0.040056	0.044012	0.047159	0.050472	0.053843
C3	0.031537	0.034426	0.037468	0.040567	0.043721	0.046826	0.050089	0.053408	0.058039	0.062140	0.066449	0.070828
C4	0.041916	0.045742	0.049769	0.053868	0.058039	0.062140	0.066449	0.070828	0.072230	0.077310	0.082643	0.088059
C5	0.044768	0.048840	0.053124	0.057481	0.061911	0.066266	0.070837	0.075479	0.078343	0.085471	0.092967	0.100592
C6	0.050299	0.054891	0.059723	0.064642	0.069646	0.074568	0.079739	0.084993	0.086675	0.092772	0.099172	0.105671
C7	0.062675	0.068377	0.074373	0.080473	0.086675	0.092772	0.099172	0.105671	0.108344	0.115965	0.123965	0.132089
C8	0.062874	0.068614	0.074654	0.080802	0.087058	0.093210	0.099674	0.106241	0.113961	0.123955	0.134122	0.144459

Table 3.2: Risk-neutral cumulative default probabilities

Pool	m	95% CI	Analytic	Approximation	Rel Err
	1	[104.66, 105.35]	105.00	105.07	6.98×10^{-4}
Homogeneous	2	[35.70, 36.08]	35.90	35.92	7.00×10^{-4}
Pool	3	[14.80, 15.08]	14.94	14.94	3.16×10^{-4}
	4	[6.29, 6.46]	6.38	6.38	2.81×10^{-5}
	1	[108.79, 109.75]	109.27	109.24	2.50×10^{-4}
Inhomogeneous	2	[37.23, 37.72]	37.45	37.45	1.38×10^{-4}
Pool	3	[15.18, 15.46]	15.32	15.32	7.52×10^{-5}
	4	[6.35, 6.57]	6.46	6.47	3.65×10^{-4}

Table 3.3: FBDS premia (bps)

Pool	Method	$m = 1$	$m = 2$	$m = 3$	$m = 4$
	95% CI	[104.66, 105.35]	[35.70, 36.08]	[14.80, 15.08]	[6.29, 6.46]
Homogeneous	100	[104.64, 105.42]	[35.82, 35.99]	[14.92, 14.96]	[6.37, 6.38]
Pool	1,000	[104.88, 105.11]	[35.87, 35.93]	[14.93, 14.95]	[6.38, 6.38]
	10,000	[104.96, 105.03]	[35.89, 35.91]	[14.94, 14.94]	[6.38, 6.38]
	95% CI	[108.79, 109.75]	[37.23, 37.72]	[15.18, 15.46]	[6.35, 6.57]
Inhomogeneous	100	[108.89, 109.69]	[37.35, 37.55]	[15.29, 15.34]	[6.46, 6.47]
Pool	1,000	[109.14, 109.42]	[37.42, 37.49]	[15.31, 15.33]	[6.46, 6.47]
	10,000	[109.23, 109.31]	[37.44, 37.47]	[15.32, 15.33]	[6.46, 6.47]

Table 3.4: 95% confidence interval comparison

the inhomogeneous pool is more time consuming than for the homogeneous pool. Our fast approximation method is also faster than the analytic method. For example, for the first-to-default homogeneous FBDS, the running time of the analytic method is about 40 times slower than our fast approximation method. These comparisons demonstrate that our fast approximation method for FBDS outperforms both the Monte Carlo method and the analytic method.

3.6 Sensitivity analysis

Since we use a Monte Carlo method to approximate the correlation coefficients ρ_i, the ρ_i are usually not exact. Therefore, a natural question to ask is: how sensitive is the FBDS spread to small changes in the ρ_i? If the sensitivity is weak, then our approximation method can obtain accurate results with a modest amount of work. Weak sensitivity is a key requirement to ensure that this kind of hybrid method is an effective computational approach.

The sensitivity of the spread with respect to small changes in the ρ_i is determined by

$$\begin{aligned}\frac{\partial s}{\partial \rho_i} &= \frac{\partial s}{\partial \mathrm{DV01}} \frac{\partial \mathrm{DV01}}{\partial \rho_i} \\ &= \frac{\partial(\mathbb{E}[V_{\mathrm{def}}]/\mathrm{DV01})}{\partial \mathrm{DV01}} \Delta T_i \cdot d_i \sqrt{\mathrm{var}(N_T)\bar{\Pi}_i(1-\bar{\Pi}_i)} \\ &= -s \frac{\Delta T_i \cdot d_i \sqrt{\mathrm{var}(N_T)\bar{\Pi}_i(1-\bar{\Pi}_i)}}{\sum_{j=1}^{n} \Delta T_j \cdot d_j (\mathbb{E}[N_T]\bar{\Pi}_j + \rho_j \sqrt{\mathrm{var}(N_T)\bar{\Pi}_j(1-\bar{\Pi}_j)})}\end{aligned} \tag{3.16}$$

The term $\mathrm{var}(N_T)$ in (3.16) is usually much smaller than $\mathbb{E}[N_T]$, as the underlying names usually have credit qualities above the investment grade and the difference in default probabilities between the investment grade level and the best credit level is small. For example, in our numerical experiments, $\sqrt{\mathrm{var}(N_T)}/\mathbb{E}[N_T] \approx 4\%$. Because of the good credit qualities, $\bar{\Pi}_i > 0.5$. Therefore, $\sqrt{\bar{\Pi}_i(1-\bar{\Pi}_i)} < \bar{\Pi}_i$. To obtain an intuitive feeling for the size of $\partial s/\partial \rho_i$, we omit the relatively small terms in the denominator and the

Pool	ρ_i	Spreads (bps)	Rel Err (%)
Homogeneous	-1	[107.37, 36.31, 15.03, 6.40]	[2.26, 1.14, 0.63, 0.35]
	0	[105.36, 35.97, 14.96, 6.38]	[0.34, 0.21, 0.11, 0.05]
	1	[103.42, 35.65, 14.88, 6.36]	[1.50, 0.70, 0.41, 0.25]
Inhomogeneous	-1	[112.03, 37.93, 15.43, 6.49]	[2.56, 1.26, 0.70, 0.39]
	0	[109.66, 37.53, 15.34, 6.47]	[0.36, 0.21, 0.11, 0.05]
	1	[107.39, 37.15, 15.25, 6.45]	[1.72, 0.81, 0.47, 0.29]

Table 3.5: Sensitivity result

minor effects of $\bar{\Pi}_i$ and $\Delta T_i \cdot d_i$, and approximate (3.16) by

$$\frac{\partial s}{\partial \rho_i} \approx -s \frac{\Delta T_i \cdot d_i \sqrt{\text{var}(N_T)} \sqrt{\bar{\Pi}_i(1-\bar{\Pi}_i)}}{\sum_{j=1}^{n} \Delta T_j \cdot d_j \mathbb{E}[N_T] \bar{\Pi}_j} \approx -s \frac{\sqrt{\text{var}(N_T)}}{n\mathbb{E}[N_T]}$$

Therefore, the relative error in the spread due to the error in ρ_i is

$$\frac{|\Delta s|}{s} \approx \frac{|\Delta \rho_i| \sqrt{\text{var}(N_T)}}{n\mathbb{E}[N_T]} \leq \frac{2\sqrt{\text{var}(N_T)}}{n\mathbb{E}[N_T]}$$

as $|\Delta \rho_i| \leq 2$. Furthermore, the relative error due to the errors in all ρ_i is bounded by

$$\sum_{i=1}^{n} \frac{2\sqrt{\text{var}(N_T)}}{n\mathbb{E}[N_T]} = \frac{2\sqrt{\text{var}(N_T)}}{\mathbb{E}[N_T]} \tag{3.17}$$

To further illustrate the weak dependence of the spread s on the correlation coefficients ρ_i, we compute the spreads with all ρ_i set to -1, 0 or 1, respectively. The results are listed in Table 3.5, where the values inside each parenthesis correspond to the spreads and relative errors for $m = 1, 2, 3, 4$, respectively, for the homogeneous and inhomogeneous mth-to-default FBDS considered above. From the table, we see that the maximum relative error for both FBDS is smaller than 3%, which is less than the bound given by (3.17). Moreover, note that taking all the ρ_i to be 0 gives a fairly good rough approximation to the spread for these two examples.

Chapter 4

Dynamic factor copula model

The methods for FCDOs and FBDS discussed in the previous chapters are applicable to any factor copula model based on the conditional independence framework: they are generic methods. However, using the methods based on the Gaussian factor copula model needs considerable care. Hull and White [21] pointed out that the Gaussian factor copula model is static, as the normally distributed common factor determines the default environment for the whole life of the model. A more serious drawback of the factor copula models is the correlation smile problem, by which we mean that the model fails to calibrate to the market spreads for tranches associated with the same underlying pool using the same correlation coefficients. Therefore, we need a dynamic model for both calibration and pricing exotic derivatives with strong time-dependent features.

Most of the existing dynamic models are based on the top-down approach, which are data demanding and currently somewhat speculative [1]. Given this, it is tempting to consider whether we could introduce dynamics into factor copula models to combine their computational efficiency with the ability to calibrate more accurately against market quotes. Extending the idea of chaining one-period factor copula models to obtain a multi-period factor copula model in [1] and [49], we develop a novel chaining technique that avoids the multi-dimensional integration which is usually required in multi-period

factor copula models. In our model, the portfolio loss of a completely homogeneous pool possesses the Markov property, thus we can compute the portfolio loss distribution analytically while avoiding the curse of dimensionality.

The rest of the chapter is organized as follows. Section 4.1 reviews existing chaining methods before introducing our new multi-period model. Section 4.2 discusses calibration. Section 4.3 presents numerical results. Section 4.4 discusses the application of the model for new exotic products.

4.1 Multi-period factor copula models

The pricing equations for synthetic CDOs are a special case of those for FCDOs with $T = 0$, as discussed in Section 2.1. Therefore, the fair spread s of a synthetic CDO is given by

$$s = \frac{\sum_{i=1}^{n} (\mathbb{E}\mathcal{L}_i - \mathbb{E}\mathcal{L}_{i-1}) d_i}{\sum_{i=1}^{n} (S - \mathbb{E}\mathcal{L}_i)(T_i - T_{i-1}) d_i}$$

where the tranche loss $\mathcal{L}_i = \min(S, (L_i - a)^+)$. Therefore, the problem is reduced to the computation of the mean tranche losses, $\mathbb{E}\mathcal{L}_i$.

To compute $\mathbb{E}\mathcal{L}_i$, we have to compute the portfolio loss L_i's distribution. Based on the one factor copula model in Section 2.3, the pool loss L_i satisfies

$$\mathbb{P}(L_i = l) = \int_{-\infty}^{\infty} \mathbb{P}_x(L_i = l) d\Phi(x)$$

where $L_i = \sum_{k=1}^{K} N_k \mathbf{1}_{\{U_k \le b_k(T_i)\}}$, and $\mathbf{1}_{\{U_k \le b_k(T_i)\}}$ are mutually independent, conditional on $X = x$. Since we know the conditional distributions of $\mathbf{1}_{\{U_k \le b_k(T_i)\}}$, the conditional distributions of L_i can be computed easily, as can $\mathbb{E}\mathcal{L}_i$.

4.1.1 Drawback of one factor copula model

A significant drawback of the one factor copula model is that it does not allow the β_k's to be time-dependent, which is often required to calibrate the model effectively. If β_k is

a function of time, $\pi_k(t,x)$ defined by (2.7) may be a decreasing function of time, which may lead to an arbitrage opportunity, as explained in the next section. More specifically, for $0 < T_1 < T_2$, to guarantee

$$\pi_k(T_1,x) \leq \pi_k(T_2,x)$$

or equivalently,

$$\Phi\left(\frac{b_k(T_1)-\beta_k(T_1)x}{\sqrt{1-\beta_k(T_1)^2}}\right) \leq \Phi\left(\frac{b_k(T_2)-\beta_k(T_2)x}{\sqrt{1-\beta_k(T_2)^2}}\right)$$

we need

$$\frac{b_k(T_1)-\beta_k(T_1)x}{\sqrt{1-\beta_k(T_1)^2}} \leq \frac{b_k(T_2)-\beta_k(T_2)x}{\sqrt{1-\beta_k(T_2)^2}}$$

As x may be any real value for any fixed $\beta_k(T_1) \neq \beta_k(T_2)$, it is easy to find an x to violate this inequality. For example, if $b_k(T_1) = -2$, $b_k(T_2) = -1.4$, $\beta_k(T_1) = 0.6$ and $\beta_k(T_2) = 0.8$, then

$$\pi_k(T_1,2) = \mathbb{P}(\tau_k \leq T_1 \mid X = 2) = \Phi(-4)$$

$$\pi_k(T_2,2) = \mathbb{P}(\tau_k \leq T_2 \mid X = 2) = \Phi(-5)$$

4.1.2 Existing chaining methods

To overcome this deficiency Andersen [1] and Sidenius [49] pioneered the technique of chaining a series of one-period factor copula models to produce a multi-period factor copula model. However, their approaches must integrate over the multi-dimensional common factors to evaluate the portfolio loss distribution over time, requiring the evaluation of a high-dimensional integral, usually computed by Monte Carlo simulation. Therefore, their models are hard to generalize to more than two periods, except for some special but possibly unrealistic cases such as the common factors remaining the same for all periods. In this section we first review the approaches of Andersen [1] and Sidenius [49]. Then we present our new model, which, for completely homogeneous pools, avoids multi-dimensional integration.

In general the conditional independence framework, including one-period and multi-period factor copula models, has to satisfy two properties: consistency and no arbitrage. By consistency, we mean that the model has to match the marginal default probabilities of the underlyings, i.e.,

$$\mathbb{P}(\tau_k \leq t) = \int_{\mathcal{D}} \mathbb{P}(\tau_k \leq t \mid X^{(t)} = x) dF(x) \tag{4.1}$$

Here $X^{(t)}$ represents the common factors up to time t (it may be a multiple dimensional random variable in the discrete case or a stochastic process in the continuous case); $\mathcal{D}$ is the domain of $X^{(t)}$; and $F(\cdot)$ is the cumulative distribution function of $X^{(t)}$. By no arbitrage, we mean that the cumulative pool loss distribution is a non-decreasing function of time, i.e.,

$$\mathbb{P}(L_i = l) \leq \mathbb{P}(L_j = l), \text{ for } T_i \leq T_j \tag{4.2}$$

To satisfy this constraint in practice, we usually require a stronger condition: the conditional default probability of a single name is non-decreasing over time, i.e.,

$$\mathbb{P}(\tau_k \leq T_1 \mid X^{(T_1)} = x) \leq \mathbb{P}(\tau_k \leq T_2 \mid X^{(T_2)} = y), \text{ for } T_1 \leq T_2 \text{ and } x(t) = y(t), \text{ for } t \leq T_1 \tag{4.3}$$

where $x(t)$ means the value of x at time t. Obviously, if we satisfy condition (4.3), then the cumulative pool loss (4.2) is non-decreasing, which implies no arbitrage. Generally, the consistency property is easy to satisfy, but the no arbitrage property is not, as shown in the previous section.

In this chapter, we extend the factor copula model to a discrete-time dynamic model. For each period $(T_{i-1}, T_i]$ and each name k, we associate a latent random variable

$$Y_{k,i} = \beta_{k,i} X_i + \sqrt{1 - \beta_{k,i}^2}\epsilon_{k,i} \tag{4.4}$$

where X_i is a random variable associated with the common factors for period $(T_{i-1}, T_i]$ and $\epsilon_{k,i}$ are mutually independent random variables associated with idiosyncratic factors for name k and period $(T_{i-1}, T_i]$. To guarantee the no arbitrage property, Andersen [1]

employed a discrete version of the first hitting time model to construct the conditional default probabilities. More specifically, he connected the default time τ_k and the latent random variables by

$$\begin{aligned} \mathbb{P}(\tau_k < t) &= \mathbb{P}(Y_{k,1} \le b_k(T_1)), && t \le T_1 \\ \mathbb{P}(T_{i-1} < \tau_k \le t) &= \mathbb{P}(Y_{k,1} > b_k(T_1), \ldots, Y_{k,i-1} > b_k(T_{i-1}), Y_{k,i} \le b_k(T_i)), && t \in (T_{i-1}, T_i] \end{aligned}$$

Then the conditional default probability for $t \le T_1$ is the same as that in the one-factor copula model. For $t \in (T_{i-1}, T_i]$, the conditional default probability satisfies

$$\mathbb{P}(T_{i-1} < \tau_k \le t \mid X^{(i)} = x^{(i)}) = \mathbb{P}(Y_{k,1} > b_k(T_1), \ldots, Y_{k,i-1} > b_k(T_{i-1}), Y_{k,i} \le b_k(T_i) \mid X^{(i)} = x^{(i)})$$

Here, $X^{(i)}$ is associated with the common factors for the periods up to T_i, or equivalently, $X^{(i)} = \{X_1, X_2, \ldots, X_i\}$.

Similar to the one-factor copula model, we must compute the boundary $b_k(T_i)$ satisfying the consistency property (4.1). For $t \le T_1$, the computation is the same as that for the one factor copula model. However, for $t \in (T_{i-1}, T_i]$, it appears that we must integrate the common factors up to T_i. The complexity of this multi-dimensional integration depends on the assumptions associated with the X_i's. Andersen [1] showed two special cases: (1) X_i are all same and (2) a two-period model, where X is a two-dimensional random variable. In addition to the computation of the default boundary, the multi-dimensional integration also arises when computing the unconditional portfolio loss distribution from the conditional loss distributions.

Sidenius [49] attacked the no arbitrage problem by introducing conditional forward survival probabilities

$$\mathbb{P}(\tau_k > t \mid \tau_k > T_{i-1}, X^{(i)} = x^{(i)}) = \frac{\mathbb{P}(\tau_k > t \mid X^{(i)} = x^{(i)})}{\mathbb{P}(\tau_k > T_{i-1} \mid X^{(i)} = x^{(i)})}, \quad t \in (T_{i-1}, T_i]$$

Using this, he expressed the conditional survival probability for $t \in (T_{i-1}, T_i]$ as

$$\mathbb{P}(\tau_k > t \mid X^{(i)} = x^{(i)}) = \mathbb{P}(\tau_k > t \mid \tau_k > T_{i-1}, X^{(i)} = x^{(i)}) P(\tau_k > T_{i-1} \mid X^{(i-1)} = x^{(i-1)})$$

For $t \leq T_1$, the conditional survival probability is the same as that in the one factor copula model.

The model allows a conditional forward survival probability for each time period $(T_{i-1}, T_i]$ to be associated with each correlation factor, i.e., $\mathbb{P}(\tau_k > t \mid \tau_k > T_{i-1}, X^{(i)} = x^{(i)}) = \mathbb{P}(\tau_k > t \mid \tau_k > T_{i-1}, X_i = x_i)$. For example, if the X_i's associated with the latent random variables $Y_{k,i}$ in (4.4) are independent, then the conditional forward survival probability can be computed by

$$\mathbb{P}(\tau_k > t \mid \tau_k > T_{i-1}, X^{(i)} = x^{(i)}) = \frac{\mathbb{P}\Big(\beta_{k,i}X_i + \sqrt{1-\beta_{k,i}^2}\epsilon_{k,i} > b_k(T_i) \mid X_i = x_i\Big)}{\mathbb{P}\Big(\beta_{k,i}X_i + \sqrt{1-\beta_{k,i}^2}\epsilon_{k,i} > b_k(T_{i-1}) \mid X_i = x_i\Big)}$$

Using the consistency property (4.1), we can calibrate the $b_k(T_i)$ recursively. However, it is impossible to preserve any tractability for general cases. Similarly, the multi-dimensional integration problem cannot be avoided except in some special cases such as all X_i being the same.

Besides the multi-dimensional integration difficulty, another drawback of these two approaches is their lack of economic meaning. For example, in [1], the latent random variables Y_k, which reflect the healthiness of name k, are reset back to zero at the beginning of each period. Therefore, the process forgets its previous position. It is hard to find any economic justification for the approach in [49].

4.1.3 New chaining method

To overcome the high-dimensional integration problem as well as some of the other deficiencies of the methods described above, we use a similar approach based on the same latent random variables (4.4), but we connect $Y_{k,i}$ and τ_k by the forward default probability

$$\mathbb{P}(Y_{k,i} \leq b_k(T_i)) = \mathbb{P}(\tau_k \in (T_{i-1}, T_i] \mid \tau_k > T_{i-1}) = \frac{\mathbb{P}(\tau_k \leq T_i) - \mathbb{P}(\tau_k \leq T_{i-1})}{1 - \mathbb{P}(\tau_k \leq T_{i-1})}$$

If X_i and $\epsilon_{k,i}$ follow standard normal distributions, then each $Y_{k,i}$ also follows a standard normal distribution. Therefore, we can compute the conditional default boundary $b_k(T_i)$ by

$$b_k(T_i) = \Phi^{-1}\big(\mathbb{P}(\tau_k \in (T_{i-1}, T_i] \mid \tau_k > T_{i-1})\big)$$

We can also compute each conditional forward default probability by

$$\mathbb{P}\big(\tau_k \in (T_{i-1}, T_i] \mid \tau_k > T_{i-1}, X_i = x_i\big) = \Phi\left(\frac{b_k(T_i) - \beta_{k,i}x_i}{\sqrt{1-\beta_{k,i}^2}}\right)$$

To compute the conditional pool loss distribution, we need to construct $\mathbb{P}(\tau_k \leq T_i \mid X_1 = x_1, \ldots, X_i = x_i)$ from $\mathbb{P}\big(\tau_k \in (T_{i-1}, T_i] \mid \tau_k > T_{i-1}, X_i = x_i\big)$. Based on the definitions of these terms, we have

$$\begin{aligned}
&\mathbb{P}(\tau_k \leq T_i \mid X_1 = x_1, \ldots, X_i = x_i) \\
&\quad = \mathbb{P}(\tau_k \leq T_{i-1} \mid X_1 = x_1, \ldots, X_{i-1} = x_{i-1}) + \mathbb{P}(\tau_k \in (T_{i-1}, T_i] \mid X_1 = x_1, \ldots, X_i = x_i) \\
&\quad = \mathbb{P}(\tau_k \leq T_{i-1} \mid X_1 = x_1, \ldots, X_{i-1} = x_{i-1}) \\
&\qquad + \mathbb{P}(\tau_k > T_{i-1} \mid X_1 = x_1, \ldots, X_{i-1} = x_{i-1}) \cdot \mathbb{P}\big(\tau_k \in (T_{i-1}, T_i] \mid \tau_k > T_{i-1}, X_i = x_i\big)
\end{aligned}$$

For the rest of the chapter, we denote $P(\tau_k \leq T_{i-1} \mid X_1 = x_1, \ldots, X_{i-1} = x_{i-1})$ by $q_{k,i-1}$ and $\mathbb{P}\big(\tau_k \in (T_{i-1}, T_i] \mid \tau_k > T_{i-1}, X_i = x_i\big)$ by $p_{k,i}$ for simplicity. If $q_{k,i}$ and $p_{k,i}$ are the same for all $k = 1, \ldots, K$, we denote them by q_i and p_i, respectively.

Using the conditional default probabilities $q_{k,i}$, we can efficiently compute the conditional distribution of the pool loss for a completely homogeneous pool, where $\beta_{k,i}$, $\pi_k(t)$ and N_k are the same for $k = 1, \ldots, K$. In this special but important case, the distribution of L_i can be computed from the distribution of number of defaults l_i, as $L_i = N_1 \sum_{k=1}^{K} \mathbf{1}_{\{\tau_k \leq T_i\}} = N_1 l_i$. Therefore, the conditional pool loss distribution of a

completely homogeneous pool satisfies

$$\begin{aligned}
&\mathbb{P}\big(L_i = rN_1 \mid X_1 = x_1, \ldots, X_i = x_i\big) = \mathbb{P}\big(l_i = r \mid X_1 = x_1, \ldots, X_i = x_i\big) \\
&\quad = \binom{K}{r}\big(q_{i-1} + (1-q_{i-1})p_i\big)^r\big((1-q_{i-1})(1-p_i)\big)^{K-r} \\
&\quad = \binom{K}{r}\left(\sum_{m=0}^{r}\binom{r}{m}q_{i-1}^m(1-q_{i-1})^{r-m}p_i^{r-m}\right)(1-q_{i-1})^{K-r}(1-p_i)^{K-r} \\
&\quad = \sum_{m=0}^{r}\binom{K}{m}q_{i-1}^m(1-q_{i-1})^{K-m}\cdot\binom{K-m}{r-m}p_i^{r-m}(1-p_i)^{K-m-(r-m)} \\
&\quad = \sum_{m=0}^{r}\mathbb{P}\big(l_{i-1} = m \mid X_1 = x_1, \ldots, X_{i-1} = X_{i-1}\big)\mathbb{P}\big(\hat{l}_{(i-1,i]}^{K-m} = r-m \mid X_i = x_i\big) \qquad (4.5)
\end{aligned}$$

where $\hat{l}_{(i-1,i]}^{K-m}$ is the number of defaults during $(T_{i-1}, T_i]$ with the pool size $K-m$, and its distribution is computed using the conditional forward default probability p_i.

To compute the tranche loss, we need to compute the unconditional pool loss distribution from the conditional ones, i.e., we need to integrate over the common factors X_i. Generally, this requires a multi-dimensional integration, for which Monte Carlo simulation is usually used. However, we can avoid the multi-dimensional integration in this special case by exploiting the independence of the X_i's:

$$\begin{aligned}
\mathbb{P}\big(l_i = r\big) &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sum_{m=0}^{r}\mathbb{P}\big(l_{i-1} = m \mid X_1 = x_1, \ldots, X_{i-1} = X_{i-1}\big) \\
&\qquad\qquad \cdot\,\mathbb{P}\big(\hat{l}_{(i-1,i]}^{K-m} = r-m \mid X_i = x_i\big)d\Phi(X_1)\ldots d\Phi(X_i) \\
&= \sum_{m=0}^{r}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\mathbb{P}\big(l_{i-1} = m \mid X_1 = x_1, \ldots, X_{i-1} = X_{i-1}\big)d\Phi(X_1)\ldots d\Phi(X_{i-1}) \\
&\qquad\qquad \cdot\int_{-\infty}^{\infty}\mathbb{P}\big(\hat{l}_{(i-1,i]}^{K-m} = r-m \mid X_i = x_i\big)d\Phi(X_i) \\
&= \sum_{m=0}^{r}\mathbb{P}\big(l_{i-1} = m\big)\mathbb{P}\big(\hat{l}_{(i-1,i]}^{K-m} = r-m\big) \qquad (4.6)
\end{aligned}$$

Therefore, the unconditional pool loss distribution possesses the Markov property and can be computed recursively.

Compared with the approach by Andersen [1], the latent process of our model is also reset to zero at the beginning of each period. However, in our model it describes the

healthiness of the forward default probability. The process for the default probability actually remembers its position at the end of the previous period: how the process evolves for the new period depends on the latent process of the forward default probability. Therefore, our model is meaningful from the economical perspective.

For a more general pool, it still holds that the event that r defaults occur before T_i is equivalent to the event that m defaults occur before T_{i-1} and $r-m$ defaults occur during $(T_{i-1}, T_i]$, for $m = 0, \dots r$. That is,

$$\mathbb{P}(l_i = r) = \sum_{m=0}^{r} \mathbb{P}(l_{i-1} = m, l_{(i-1,i]} = r-m) = \sum_{m=0}^{r} P(l_{i-1} = m) \cdot \mathbb{P}(l_{(i-1,i]} = r-m \mid l_{i-1} = m)$$

Moreover, this relationship extends to the conditional probabilities:

$$\mathbb{P}(l_i = r \mid X_1 = x_1, \dots, X_i = x_i) = \sum_{m=0}^{r} \mathbb{P}(l_{i-1} = m \mid X_1 = x_1, \dots, X_{i-1} = x_{i-1}) \cdot \mathbb{P}(l_{(i-1,i]} = r - m \mid l_{i-1} = m, X_1 = x_1, \dots, X_i = x_i)$$

Under the assumptions of our model, we can simplify the above expression using

$$\mathbb{P}(l_{(i-1,i]} = r-m \mid l_{i-1} = m, X_1 = x_1, \dots, X_i = x_i) = \mathbb{P}(l_{(i-1,i]} = r-m \mid l_{i-1} = m, X_i = x_i)$$

Therefore,

$$\mathbb{P}(l_i = r \mid X_1 = x_1, \dots, X_i = x_i) = \sum_{m=0}^{r} \mathbb{P}(l_{i-1} = m \mid X_1 = x_1, \dots, X_{i-1} = x_{i-1}) \cdot \mathbb{P}(l_{(i-1,i]} = r - m \mid l_{i-1} = m, X_i = x_i)$$

To obtain the unconditional pool loss distribution, we need to integrate over the common factors, as we did in (4.6). Therefore, in our model, the Markov property holds for a general pool:

$$\mathbb{P}\big(l_i = r\big) = \sum_{m=0}^{r} \mathbb{P}\big(l_{i-1} = m\big) \cdot \mathbb{P}\big(l_{(i-1,i]}^{K-m} = r - m \mid l_{i-1} = m\big)$$

However, as the default probability of each name may be different in a general pool, we end up with another combinatorial problem: we need to consider all possible combinations of $l_{i-1} = m$ defaults.

Obviously, the completely homogeneous pool is a special case. However, it is of considerable practical importance, since such pools often arise in practice. Moreover, the pool loss of a general pool is generally approximated by the pool loss of a completely homogeneous one for computational efficiency in calibration and in the valuation of bespoke contracts.

Remark. For simplicity, we used the Gaussian factor copula model to illustrate our new discrete dynamical multi-period factor copula model. However, it is important to note that our approach can be applied to build a multi-period factor copula model from any one factor copula model based on the conditional independence framework.

4.2 Calibration

Our goal is to calibrate our model against the market tranche quotes on the same underlying pool. To illustrate our approach, we use the tranche quotes of the credit indexes, CDX and ITRAXX. As our model allows the correlation factor loadings to be time-dependent, we can introduce dynamics into the model by letting the correlation factor loadings follow particular dynamic processes. This added flexibility gives our dynamic model enough degrees of freedom to calibrate consistently against market quotes.

We obtain the spread quotes for the indexes and tranches on CDX and ITRAXX from the Thomson Datastream. We approximate the default probabilities of a single name using the index spreads, which are the average spreads of the 125 names in CDX or ITRAXX. Due to the data availability and popularity, we calibrate our model against the four mezzanine tranches with maturities 5 years, 7 years and 10 years. Therefore, we have to fit 12 market tranche quotes on the same underlying pool.

To fit these 12 tranche quotes, we must incorporate sufficient degrees of freedom into our model. As the correlation factor loadings are time-dependent in our model, they can be any dynamic process within the range $[0, 1]$. Therefore, we can obtain

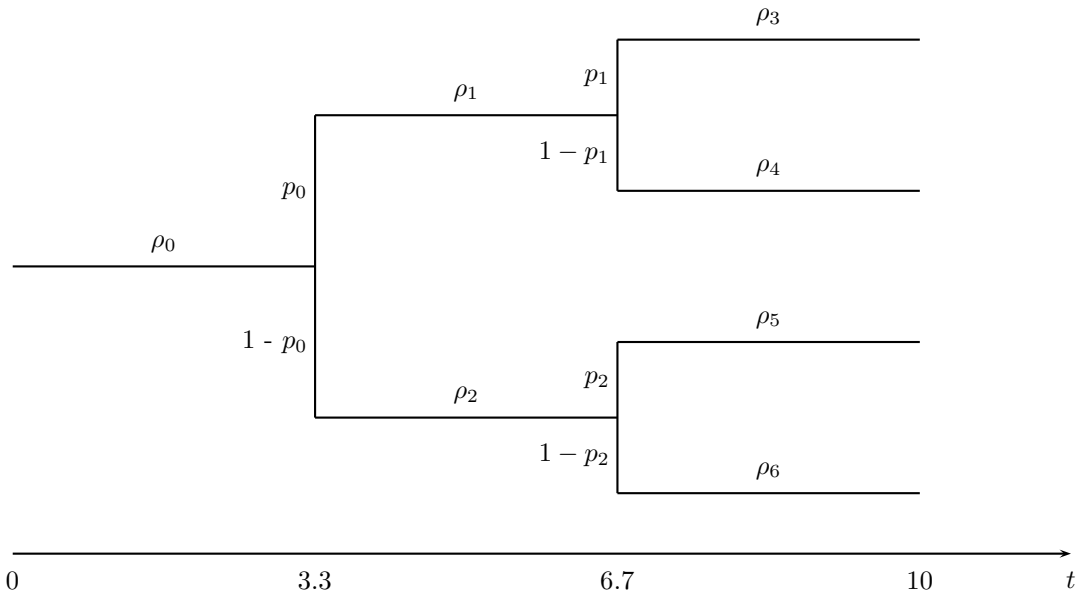


Figure 4.1: A dynamic tree structure

sufficient degrees of freedom by building a suitable dynamic process for the correlation factor loadings. To illustrate our approach, we employ a binomial tree structure for the correlation factor loadings in our numerical examples. We assume that the correlation factor loading process is a piecewise constant function over time and each branch of the tree describes one possible path of the factor loading process. To compute the tranche prices, we only need to take the expectation of the tranche prices on each branch. Figure 4.1 illustrates an equally-spaced three-period[1] tree, where ρ_j is the value of the correlation factor loading and p_j is the probability of the process taking the upper branch. With this tree structure, the correlation factor loading process has four possible paths for a 10-year maturity contract. For example, for an annual payment tranche contract, one possible path for the $\beta_{k,i}$'s is $(\rho_0, \rho_0, \rho_0, \rho_1, \rho_1, \rho_1, \rho_3, \rho_3, \rho_3, \rho_3)$ with probability $p_0 p_1$. We can increase or decrease the degrees of freedom of the tree by adjusting the number of periods or the tree structure, e.g., constraining the general tree to be a binomial tree.

[1] As illustrated in this example, the number of periods for the tree may be different from the number of periods of our model, which equals the number of premium payments.

Time	1Y	2Y	3Y	4Y	5Y
Probability	0.0041	0.0052	0.0069	0.0217	0.0288

Table 4.1: Risk-neutral cumulative default probabilities

4.3 Numerical results III

We begin by comparing the results generated by the Monte Carlo method to those obtained by the recursion (4.6) on an example with arbitrarily chosen parameters. The numerical experiments are based on 5-year CDOs with 100 underlying names and annual premium payments. The tranche structure is the same as those of CDX, i.e., six tranches with attachment and detachment points, 0%–3%, 3%–7%, 7%–10%, 10%–15%, 15%–30% and 30%–100%. We assume a constant interest rate of 4% and a constant recovery rate of 40%. For simplicity, we assume that all $\beta_{k,i} = 0.6$. The risk-neutral cumulative default probabilities are listed in the Table 4.1.

Each Monte Carlo simulation consists of 100,000 trials, and 100 runs (with different seeds) for each experiment are made. Based on the results of these 100 experiments, we calculate the mean and the 95% non-parametric confidence interval. Table 4.2 presents the risk premia for the CDOs. For our example, the running time of one Monte Carlo experiment with 100,000 trials is about 14 times that used by our recursive method. These results demonstrate that the recursive relationship (4.6) is accurate and efficient.

To calibrate against the market quotes, we employ the tree structure for the correlation factor loadings discussed in the previous section. In particular, we use an equally-spaced four-period tree. However, we add constraints by using the same growth rate μ_j and probability p_j for period j, as shown in the tree in Figure 4.2. To guarantee that $\beta_{k,i} \in [0, 1]$, we constrain $\rho_0 > 0$, $\mu_k > 0$ and $\beta_{k,i} = \min(1, \rho_{J(i)})$, where $J(i)$ is the index of ρ associated with $\beta_{k,i}$. Therefore, we have 7 parameters in total to calibrate against 12 tranche quotes. We compute the parameters by solving an associated optimization problem. For the objective function of the optimization problem, we could use either the

Tranche	Monte Carlo	95% CI	Recursion
0% – 3%	953.40	[946.71, 960.62]	951.60
3% – 7%	182.09	[179.51, 184.81]	181.59
7% – 10%	58.95	[57.26, 60.33]	58.77
10% – 15%	22.21	[21.01, 23.39]	22.09
15% – 30%	3.47	[3.03, 3.78]	3.44
30% – 100%	0.07	[0.03, 0.09]	0.07

Table 4.2: Tranche premia (bps)

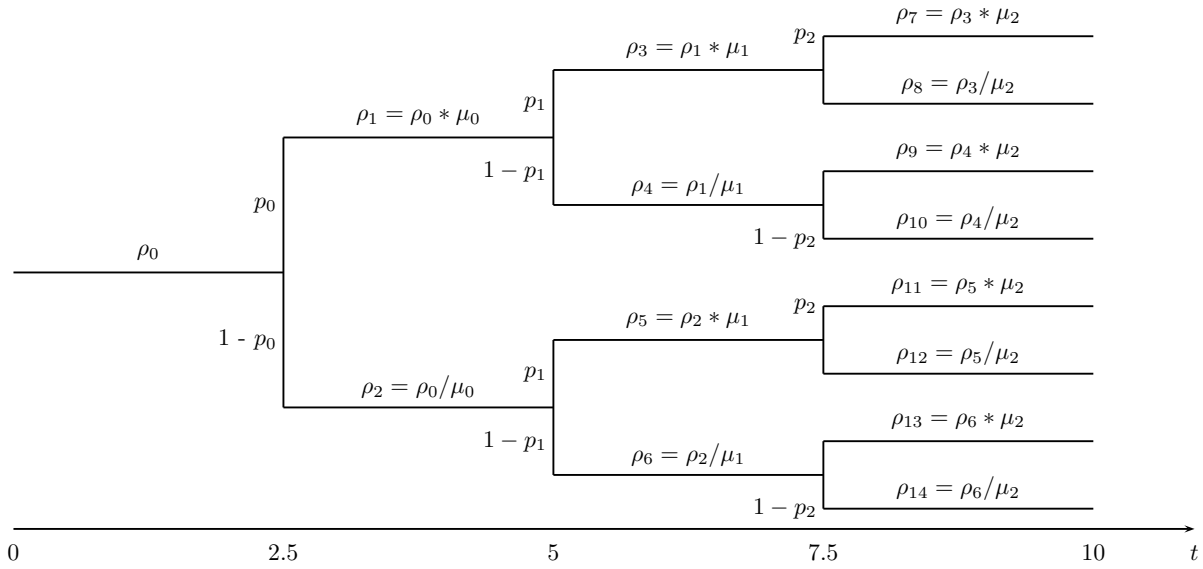


Figure 4.2: A particular dynamic tree example

absolute error in the spreads

$$f_{\text{abs}} = \sum (m_i - s_i)^2, \text{ for } i = 1, \ldots, 12$$

or the relative error in the spreads

$$f_{\text{rel}} = \sum (m_i - s_i)^2 / m_i^2, \text{ for } i = 1, \ldots, 12$$

where m_i is the market spread quote for tranche i and s_i is the model spread for tranche i.

Table 4.3 lists the calibration result for the tranche quotes of CDX series 8 on April 4, 2007. The upper half of the table uses the absolute spread error as the objective function, while the lower half of the table uses the relative spread error as the objective function. In both cases, the rows "Parameter" display the values of the parameters in our model, in the order $\rho_0, \mu_0, p_0, \mu_1, p_1, \mu_2, p_2$.

Table 4.4 lists the calibration results for the same data using the Gaussian factor copula model and the normal inverse Gaussian factor copula model [34]. In the table, "NIG(1)" means the normal inverse Gaussian factor copula model with one extra parameter for fat-tailness, and "NIG(2)" means the normal inverse Gaussian factor copula model with two extra parameters for skewness and fat-tailedness. Our results in Table 4.3 are far superior to the results of the three models in Table 4.4.

In additional to the market data on a single day, we calibrate our model against market spreads of CDX series 8 on each Wednesday from March 23, 2007 to July 4, 2007. Figure 4.3 plots the absolute errors and relative errors of the 12 tranches using the four-period tree structure with 7 parameters. The unit of the absolute error is basis points and the unit of the relative error is percentage. For market data before the credit crunch (July, 2007), our model is able to match the data quite well with 7 parameters. For market data after the credit crunch, the calibration error increases dramatically. We believe this is because the market quotes exhibit arbitrage due to the large demand and

Maturity	5 yr			7 yr			10 yr		
Tranche	Market	Model	Abs Err	Market	Model	Abs Err	Market	Model	Abs Err
3 – 7	111.81	110.13	1.68	251.44	254.65	3.21	528.31	528.37	0.06
7 – 10	22.31	20.90	1.41	54.69	59.51	4.82	134.00	134.21	0.21
10 – 15	10.42	7.99	2.43	26.47	28.45	1.98	63.30	61.38	1.92
15 – 30	4.34	1.97	2.37	9.50	12.08	2.58	20.46	23.36	2.90
Parameter	0.73	0.43	0.98	0.32	0.57	0.11	0.63	$f_{\text{abs}} = 8.52$	
Tranche	Market	Model	Rel Err	Market	Model	Rel Err	Market	Model	Rel Err
3 – 7	111.81	109.88	1.73%	251.44	300.00	19.31%	528.31	560.57	6.11%
7 – 10	22.31	21.37	4.23%	54.69	54.00	1.26%	134.00	141.36	5.49%
10 – 15	10.42	10.79	3.57%	26.47	25.01	5.52%	63.30	60.20	4.90%
15 – 30	4.34	4.36	0.37%	9.50	9.86	3.79%	20.46	22.30	8.99%
Parameter	0.55	0.65	0.80	0.42	0.71	0.15	0.57	$f_{\text{rel}} = 25.01\%$	

Table 4.3: Calibration result of CDX 8 on April 4, 2007

Maturity	5 yr				7 yr				10 yr			
Tranche	Market	Gaussian	NIG(1)	NIG(2)	Market	Gaussian	NIG(1)	NIG(2)	Market	Gaussian	NIG(1)	NIG(2)
3 – 7	111.81	149.77	84.48	92.12	251.44	379.65	240.59	240.36	528.31	653.48	537.32	536.43
7 – 10	22.31	14.61	32.42	33.21	54.69	80.52	62.03	64.61	134.00	248.90	154.68	148.07
10 – 15	10.42	1.51	21.42	19.71	26.47	14.80	36.18	35.30	63.30	77.84	66.95	65.44
15 – 30	4.34	0.02	12.28	9.36	9.50	0.49	19.02	16.18	20.46	5.49	29.00	26.38
Abs err		39.98	32.14	24.86		131.62	18.88	18.54		171.19	24.39	17.42
Parameter	Gaussian: 0.30				NIG(1): 0.46, 0.37				NIG(2): 0.44, 0.99, -0.61			
Tranche	Market	Gaussian	NIG(1)	NIG(2)	Market	Gaussian	NIG(1)	NIG(2)	Market	Gaussian	NIG(1)	NIG(2)
3 – 7	111.81	164.22	89.70	86.76	251.44	383.20	289.34	265.15	528.31	635.06	642.09	616.51
7 – 10	22.31	21.07	23.40	24.17	54.69	94.04	53.01	53.28	134.00	255.83	173.08	151.31
10 – 15	10.42	2.88	12.52	12.50	26.47	20.92	24.02	25.14	63.30	89.10	54.25	53.37
15 – 30	4.34	0.07	4.96	4.46	9.50	0.98	8.77	9.23	20.46	8.06	15.40	16.95
Rel err		130.96%	31.95%	31.26%		128.06%	19.53%	8.35%		118.37%	46.17%	31.39%
Parameter	Gaussian: 0.33				NIG(1): 0.34, 0.44				NIG(2): 0.35, 0.99, -0.63			

Table 4.4: Comparison of calibration results by different models

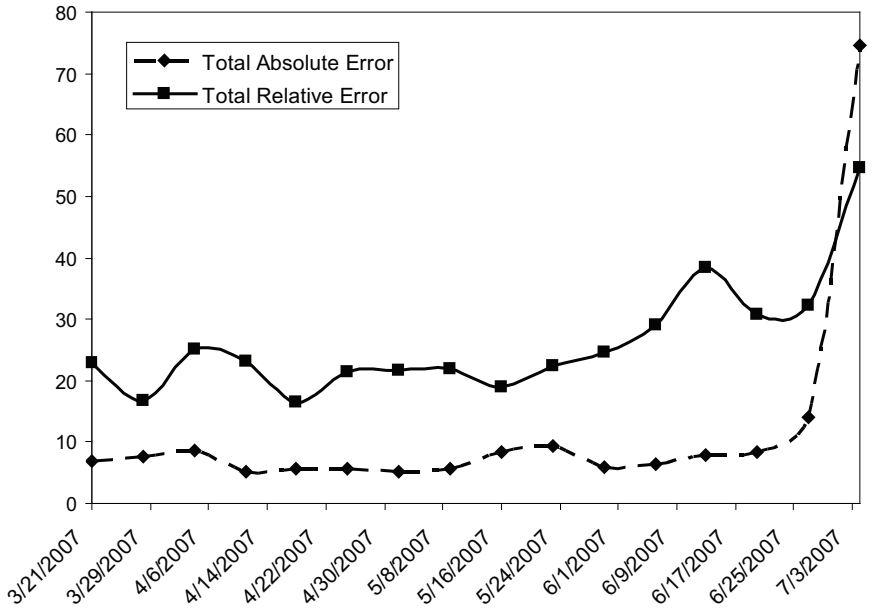


Figure 4.3: Weekly calibration result of CDX 8

supply gap. As the financial crisis developed, traders tried to sell the credit derivatives they were holding, but no one wanted to buy them.

In addition to our approaches, Eckner [12] calibrated against market tranche quotes successfully using an affine-type reduced form model and based on the conditional independence framework. Metzler [44] proposed a different approach to introduce correlations into a multi-name first hitting time model, as explained in the next chapter. He also successfully calibrated against market quotes using his approach. Both of them achieved almost the same results as our model[2].

4.4 Application to forward products

After calibrating against market quotes consistently, our multi-period factor copula model can be applied to price exotic time-dependent credit derivatives such as FCDOs. By the

[2]Note that Eckner [12], Metzler [44] and our model use different data sets.

reduction method in Chapter 2, we need to compute the conditional distribution of the effective pool losses $\hat{L}_i$ using the modified conditional default probability

$$\mathbb{P}(T < \tau_k \leq T_i \mid X^{(i)}) = \mathbb{P}(\tau_k \leq T_i \mid X^{(i)}) - \mathbb{P}(\tau_k \leq T \mid X^{(0)})$$

where $X^{(0)}$ means the common factor before T. In our multi-period factor copula model, the modified conditional default probability satisfies

$$\begin{aligned}
&\mathbb{P}(T < \tau_k \leq T_i \mid X^{(i)}) \\
=&\mathbb{P}(T < \tau_k \leq T_{i-1} \mid X^{(i-1)}) + \mathbb{P}(\tau_k > T_{i-1} \mid X^{(i-1)}) \cdot \mathbb{P}(\tau_k \in (T_{i-1}, T_i] \mid \tau_k > T_{i-1}, X_i)
\end{aligned}$$

For $i = 1$, we have

$$\mathbb{P}(T < \tau_k \leq T_1 \mid X^{(1)}) = \big(1 - \mathbb{P}(\tau_k \leq T \mid X^{(0)})\big) \cdot \mathbb{P}(\tau_k \in (T, T_1] \mid \tau_k > T, X_1)$$

This term is used to compute the distribution of the effective pool loss over $(T, T_1]$. Even though the distribution of number of defaults l_i in the effective pool still follows

$$\mathbb{P}(l_i = r \mid X^{(i)}) = \sum_{m=0}^{r} \mathbb{P}(l_{i-1} = m \mid X^{(i-1)}) \cdot \mathbb{P}(l_{(i-1,i]} = r - m \mid l_{i-1} = m, X^{(i)})$$

we cannot omit the common factors before T, as they occur in the base case of the recursion, i.e., the distribution of l_1. That makes our dynamic factor copula model computationally inefficient for pricing FCDOs, as we need to integrate all the possible scenarios of the common factors before T.

We plan to continue to work on developing a fast dynamic factor copula model, but, for now, we suggest to price these time-dependent products using Monte Carlo simulation and based on models that calibrate consistently against market quotes, such as our dynamic factor copula models. An effective method for forward starting products and options based on such models is an interesting and challenging future research topic.

Chapter 5

Randomized first hitting time model

The multi-period factor copula model described in the previous chapter is a discrete-time dynamic model. To develop a continuous-time dynamic model based on the conditional independence framework, we need a continuous-time credit risk model for a single name, as such a model is the fundamental building block for a multi-name model. As structural models are employed in factor copula models, we concentrate in this chapter on first hitting time models, one particular class of structural models.

Structural models were introduced by Merton [43] to model credit risk. In his original model, the default time of a firm is set to its debt maturity, if the firm's asset value falls below its debt value at maturity. It is well known that credit spreads for short maturities generated by Merton's model are too low [14]. Therefore, Black and Cox [8] employed a first hitting time model, in which the firm's default time is set to the first time when its asset value falls below its debt value. However the Black and Cox model and its extensions such as [42], [39] and [13], cannot generate high enough short spreads, either.

Duffie and Lando [37] introduced an incomplete information approach, which adds random noise into the asset process[1] to better match the observed market spreads. Similarly, Giesekce [14] assumed a stochastic default barrier in the first hitting time model.

[1]Equivalently, the underlying asset process starts from a random initial position.

Both approaches assume a particular distribution for the noise, e.g., a normal distribution, then calibrate the distribution parameters against market data.

In this chapter, we develop a randomized first hitting model to better fit the market data. Our approach introduces extra degrees of freedom by randomizing the initial state of the Wiener process in the first hitting time setting. In our approach, we do not assume a particular distribution function, such as a normal distribution, for the noise. That is, instead of calibrating parameters in a pre-assumed distribution function, we directly calibrate the distribution function itself. Therefore, our approach is a generalization of the incomplete information approach.

The rest of this chapter is organized as follows. Section 5.1 presents the mathematical derivation of our approach and studies several important properties of the model. Section 5.2 describes calibration. Section 5.3 presents calibration results. Section 5.4 discusses the difficulty of extending our single-name model to a multi-name one.

5.1 Randomization in the first hitting time problem

Let X_t be an arbitrary stochastic process with continuous sample paths and let $b(t)$ be a continuous absorbing boundary, satisfying the relation $X_0 \geq b(0)$. The random variable

$$\tau = \begin{cases} \inf\{t \geq 0 : \ X_t < b(t)\}, & \text{if there exists } t \text{ such that } X_t < b(t) \\ \infty, & \text{otherwise} \end{cases}$$

is called the *first hitting time* for the process X_t.

If X_t is a diffusion process, the problem of finding the distribution of τ is a classical one; it was solved by Khintchine [36] in 1933 for sufficiently smooth boundaries. Since then many books and research papers have been published in this area. Lerche [40] summarized known analytical results obtained by the mid-1980s, Karatzas and Shreve [35] considered the link between analytical methods and the martingale approach, Durbin [11]

discussed computational aspects of the problem, and Hobson, Williams, and Wood [16] analyzed Taylor expansions of the probability distribution of τ.

Calibration of the default model in the credit risk context leads to the following inverse problem:

Given a stochastic process X_t and a distribution $F(t)$,

find a boundary $b(t)$ such that $\mathbb{P}(\tau \leq t) = F(t)$.

This problem was considered in [25] for the case in which X_t is a Brownian random walk[2]. A detailed analysis of the inverse problem in the discrete time setting is given in [23] as well as a solution based on the Monte Carlo method.

Existence of the solution to the continuous-time inverse problem is analyzed in [9]. In [45], an integral equation for the boundary is derived when X_t is a Wiener process. Unfortunately, this problem appears to be very difficult: an analytical solution is known in a few cases only.

In this chapter, we consider a modified inverse problem. Let $X_t = a + W_t$ with a random initial position, $X_0 = a$, where a is a non-negative random variable, and W_t is a standard Wiener process. We fix the boundary $b(t) = \mu t$, where $\mu \geq 0$ is a fixed constant. For a fixed μ, let the first hitting time for X_t and $b(t) = \mu t$ be denoted by $\tau^{(\mu)}$. The modified inverse problem is formulated as follows:

Given a stochastic process X_t and a distribution $F(t)$,

find a distribution of the random variable a such that $\mathbb{P}\left(\tau^{(\mu)} \leq t\right) = F(t)$. (5.1)

An equivalent form of this first hitting time problem that we often use in this chapter is to let $X_t = W_t - \mu t$ and $b(t) = -a$.

The randomization of the initial state of the process X_t allows us to find an analytically tractable solution for a large class of distributions $F(t)$, without changing the covariance

[2] X_t is a discrete time process with Gaussian increments.

structure of the process. We begin by considering the inverse problem (5.1) for $F(t)$ restricted to the class of Gamma distributions with the probability density function

$$p_\gamma(t) = \lambda \cdot \frac{(\lambda t)^{\gamma-1}}{\Gamma(\gamma)} \cdot \exp(-\lambda t), \quad \gamma > 0,\ \lambda > 0$$

The idea behind our solution is as follows: given $X_0 = a$, the conditional density function $f_a(t)$ of $\tau^{(\mu)}$ and the conditional distribution function

$$F_a(t) = \mathbb{P}(\tau \leq t \,|\, X_0 = a)$$

are available in closed form. Let $g(a)$ be the density function of the initial state $X_0 = a$. Then, the distribution function of the first hitting time satisfies

$$F(t) = \int_0^\infty F_a(t) g(a)\, \mathrm{d}a. \tag{5.2}$$

Therefore, the random variable $\tau^{(\mu)}$ is the mixture of the random variables $\tau_a^{(\mu)}$, representing the first hitting time for the process $X_t = W_t - \mu t$ and the constant boundary $b(t) = -a$. The density function of $\tau_a^{(\mu)}$ has a simple analytical form. From (5.2), we find the Laplace transform $\hat{g}(s)$ of the density function g, and derive conditions on the parameter μ providing existence of the solution.

The rest of this section is organized as follows. Subsection 5.1.1 reviews some of the existing results on the first hitting time distribution of a Wiener process. Subsection 5.1.2 presents our solution to Problem (5.1). Subsection 5.1.3 discusses existence of the density function of the initial state.

5.1.1 First hitting time for a Wiener process

We start with a brief review of the properties of the first hitting time of a Wiener process W_t with constant and linear boundaries [35]. Consider the constant boundary $b(t) = -a$, $a > 0$. In this case, $\mu = 0$. Using the reflection principle, we have

$$\mathbb{P}\{\tau_a^{(0)} \leq t\} = 2\mathbb{P}\{W_t \leq -a\} = 2\Phi\left(\frac{-a}{\sqrt{t}}\right)$$

where $\Phi(x)$ is the standard normal cumulative distribution function. The probability density function of the first hitting time satisfies the relation

$$f_a(t) = \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{a^2}{2t}\right) \tag{5.3}$$

In the case of a constant boundary (i.e., $\mu = 0$), the random variable $\tau_a^{(0)}$ is finite almost surely but $\mathbb{E}\tau_a^{(0)} = \infty$. If we take a mixture of random variables $\tau_a^{(0)}$ with a mixing density g, the resulting random variable $\tau^{(0)}$ will also have infinite expectation. In this case it is not possible to match a class of distributions $F(t)$ with a finite first moment. Subsection 5.1.3 provides the existence condition of g for this case.

Now consider the first hitting time $\tau_a^{(\mu)}$ of a Wiener process with drift $X_t = W_t - \mu t$ and a constant boundary $b(t) = -a$. Obviously, the first hitting time for X_t coincides with the first hitting time of W_t and the boundary $b_\mu(t) = \mu t - a$. Using the Girsanov theorem, we find[3]

$$\mathbb{P}\left(\tau_a^{(\mu)} \leq t\right) = \int_0^t \frac{a}{\sqrt{2\pi s^3}} \exp\left(-\frac{(a-\mu s)^2}{2s}\right) \mathrm{d}s \tag{5.4}$$

Therefore, given a value of a, the conditional density function $f_a(t)$ for the first hitting time of $X_t = W_t - \mu t$ with the boundary $b(t) = -a$ and the associated conditional distribution function $F_a(t)$ are given by

$$f_a(t) = \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{(a-\mu t)^2}{2t}\right) \tag{5.5}$$

$$F_a(t) = \Phi\left(\frac{-a+\mu t}{\sqrt{t}}\right) + e^{2\mu a}\Phi\left(\frac{-a-\mu t}{\sqrt{t}}\right) \tag{5.6}$$

From (5.6), it follows that

$$\mathbb{P}\left(\tau_a^{(\mu)} < \infty\right) = \begin{cases} 1, & \text{if } \mu \geq 0 \\ e^{2\mu a}, & \text{otherwise} \end{cases} \tag{5.7}$$

[3]The details of the derivation can be found in [35].

Thus, if $\mu > 0$, then the first hitting time in our problem is finite almost surely. The moment generating function, $G_a(s) = \mathbb{E}[e^{s\tau_a^{(\mu)}}]$, is

$$G_a(s) = \int_0^\infty e^{st} f_a(t)\,\mathrm{d}t = \exp\left[(\mu - \sqrt{\mu^2 - 2s})a\right] \tag{5.8}$$

If $\mu > 0$, the expected value of the first hitting time $\tau_a^{(\mu)}$ is finite:

$$\mathbb{E}[\tau_a^{(\mu)}] = \frac{a}{\mu}$$

Therefore

$$\mathbb{E}[\tau^{(\mu)}] = \frac{\mathbb{E}[a]}{\mu}$$

5.1.2 Density function of the initial state

Now we can find the relation between the distribution function $F(t)$ and the density function $g(\cdot)$ of the initial distribution $X_0 = a$ of the process $X_t = W_t + a$.

Proposition 1. *Suppose that the random variable $\tau^{(\mu)}$ has an absolutely continuous distribution function $F(t)$ with the density function $f(t)$. Denote the Laplace transform of f by*

$$\hat{f}(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t, \quad s \geq 0$$

If a density function $g(t)$ of the initial state $X_0 = a$ exists, then its Laplace transform

$$\hat{g}(s) = \int_0^\infty e^{-st} g(t)\,\mathrm{d}t$$

satisfies the equation

$$\hat{g}(s) = \hat{f}\left(\frac{s(s+2\mu)}{2}\right) \tag{5.9}$$

REMARK. The function $\hat{g}(s)$ defined by (5.9) may not be a Laplace transform of a density function of a random variable. In this case, a solution to Problem (5.1) does not exist. For this reason, we formulate Proposition 1 in a conditional form.

Proof. The density function $f(t)$ of the random variable $\tau^{(\mu)}$ satisfies the equation

$$f(t) = \int_0^\infty f_a(t) \cdot g(a)\,\mathrm{d}a$$

Applying the Laplace transform to this equation, we find

$$\hat{f}(s) = \int_0^\infty G_a(-s) g(a)\,\mathrm{d}a \tag{5.10}$$

where $G_a(s)$ is the moment generation function (5.8) for the first hitting time problem. Since $G_a(-s) = \exp\left(a\mu - a\sqrt{\mu^2 + 2s}\right)$, we obtain

$$\hat{f}(s) = \hat{g}\left(\sqrt{\mu^2 + 2s} - \mu\right)$$

which is equivalent to (5.9).

Let us now find the density $g(a)$ of the initial distribution $X_0 = a$ of the process $X_t = W_t + a$, if the first hitting time distribution is the Gamma distribution. In this case, $f(t) = p_\gamma(t)$, for some $\gamma > 0$. Then we have

$$\hat{f}(s) = \frac{\lambda^\gamma}{(\lambda + s)^\gamma}$$

From (5.9), we obtain

$$\hat{g}(s) = \frac{(2\lambda)^\gamma}{((s+\mu)^2 - (\mu^2 - 2\lambda))^\gamma} \tag{5.11}$$

Rearranging (5.11), we find

$$\hat{g}(s) = \frac{(\mu + \sqrt{\mu^2 - 2\lambda})^\gamma}{\left(s + \mu + \sqrt{\mu^2 - 2\lambda}\right)^\gamma} \cdot \frac{(\mu - \sqrt{\mu^2 - 2\lambda})^\gamma}{\left(s + \mu - \sqrt{\mu^2 - 2\lambda}\right)^\gamma} \tag{5.12}$$

The latter representation of the Laplace transform $\hat{g}(s)$ corresponds to the sum of two independent random variables $\xi_1 + \xi_2$ having Gamma distributions with a common shape parameter γ: $\xi_1 \sim \Gamma(\gamma, \lambda_1)$ and $\xi_2 \sim \Gamma(\gamma, \lambda_2)$, where

$$\lambda_1 = \mu - \sqrt{\mu^2 - 2\lambda}, \quad \lambda_2 = \mu + \sqrt{\mu^2 - 2\lambda}$$

From (5.11) it follows that μ must satisfy the inequality

$$\mu \geq \sqrt{2\lambda} \tag{5.13}$$

Otherwise, the solution of Problem (5.1) does not exist. Indeed, if $\mu < \sqrt{2\lambda}$, the function $g(a)$, corresponding to $\hat{g}(s)$, takes negative values. Thus, we proved

Proposition 2. *Consider the first hitting time of the process $X_t = a + W_t$ and the linear boundary $b(t) = \mu t$. If the distribution $F(t)$ belongs to the class of Gamma distributions with the rate parameter λ satisfying* (5.13)*, then Problem (5.1) has a solution and the Laplace transform of the distribution of the initial state $X_0 = a$ is given by* (5.11)*. If* (5.13) *is not satisfied, then the solution does not exist.*

5.1.3 Existence of a solution

In the previous section, we reduced the inverse problem to the solution of the integral equation for the density function $g(\cdot)$,

$$\hat{f}(s) = \int_0^\infty G_a(-s) \cdot g(a)\,\mathrm{d}a, \quad s > 0 \tag{5.14}$$

where $G_a(s)$ is defined by (5.8) and $\hat{f}(s)$ is the Laplace transform of a given density function $f(t)$. In the case $f(t) = p_\gamma(t)$, we found the solution to the problem defined by (5.1), if the parameter μ satisfies (5.13).

Now we would like to find general conditions on the Laplace transform $\hat{f}(s)$ providing existence of the density function $g(a)$, satisfying the integral equation (5.14). The following proposition gives a sufficient condition for existence of the density $g(a)$.

Proposition 3. *Fix the boundary $b(t) = \mu t$. Consider a non-negative random variable τ. Let $\mathfrak{R}$ denote the class of functions $\hat{f}(s) = \mathbb{E}[e^{-s\tau}]$ of the argument s represented in the form*

$$\hat{f}(s) = \sum_{k=1}^{K} \frac{\alpha_k}{(s + \lambda_k)^{l_k}}, \quad l_k \in \mathbb{R}_+, \quad \lambda_k, \alpha_k > 0 \tag{5.15}$$

Then there exists $\mu_* > 0$ such that for all $\mu \geq \mu_*$ the integral equation (5.14) *has a solution $g(a)$ satisfying the conditions*

$$g(a) \geq 0, \quad \int_0^\infty g(a)\,\mathrm{d}a = 1$$

PROOF. We start the proof of Proposition 3 with the following remark.

Let $\vec{p} = (p_1, p_2, \ldots, p_K)$ be a finite probability distribution. Therefore

$$\sum_{k=1}^{K} p_k = 1, \quad p_k \geq 0 \quad \text{for } k = 1, 2, \ldots, K$$

If each density function $g_k(a)$ solves Problem (5.1) for each associated density $f_k(t)$ and the linear boundary $b(t) = \mu t$, then the convex combination

$$g(a) = \sum_{k=1}^{K} p_k g_k(a)$$

solves Problem (5.1) for the density

$$f(t) = \sum_{k=1}^{K} p_k f_k(t)$$

Now consider a density function $f(t)$ such that $\hat{f}(s) \in \mathfrak{R}$. Since $\hat{f}(0) = 1$ and $\alpha_k > 0$, $\lambda_k > 0$, we obtain

$$\sum_{k=1}^{K} \frac{\alpha_k}{\lambda_k^{l_k}} = 1 \tag{5.16}$$

Let $p_k = \frac{\alpha_k}{\lambda_k^{l_k}}$. Obviously, $p_k > 0$ and $\sum_{k=1}^{K} p_k = 1$. Then the function $\hat{f}(s)$ defined by (5.15) can be represented as a convex combination

$$\hat{f}(s) = \sum_{k=1}^{K} p_k \hat{f}_k(s)$$

where

$$\hat{f}_k(s) = \frac{\lambda_k^{l_k}}{(s + \lambda_k)^{l_k}}$$

The Laplace transform $\hat{f}_k(s)$ corresponds to a random variable $\tau_k \sim \Gamma(l_k, \lambda_k)$ with a Gamma distribution. Let

$$\mu_* = \max_{1 \leq k \leq K} \sqrt{2\lambda_k}.$$

Then, by Proposition 2, for $\mu \geq \mu_*$ and for each $k = 1, 2, \ldots, K$, there exists $g_k(a)$ solving Problem (5.1) for the density $f_k(t)$. Consequently, the density

$$g(a) = \sum_{k=1}^{K} p_k g_k(a)$$

solves Problem (5.1) for the probability density function

$$f(t) = \sum_{k=1}^{K} p_k \lambda_k \exp(-\lambda_k t) \frac{(\lambda_k t)^{l_k}}{\Gamma(l_k)}$$

Thus, the Proposition is proved.

Consider now the class of hyper-Erlang distributions, $\mathcal{HE}$, that contains all non-negative random variables ξ satisfying the following relation for the Laplace transform:

$$\xi \in \mathcal{HE} \Leftrightarrow \mathbb{E}[e^{-s\xi}] = \sum_{j=1}^{k} p_j \frac{\lambda_j^{n_j}}{(\lambda_j + s)^{n_j}}$$

where $\sum_{j=1}^{k} p_j = 1$, $p_j > 0$, $\lambda_j > 0$ and n_j are positive integers.

It is very well known that the class of hyper-Erlang distributions, $\mathcal{HE}$, is dense in the space of all non-negative random variables [33]. That is, for any random variable $\eta \geq 0$, there exists a sequence of random variables $\xi_n \in \mathcal{HE}$, such that

$$\xi_n \stackrel{w}{\Longrightarrow} \eta$$

where $\stackrel{w}{\Longrightarrow}$ denotes the weak convergence of the corresponding probability measures. Then

$$\lim_{n\to\infty} \mathbb{E}[\exp(-s\xi_n)] = \mathbb{E}[\exp(-s\eta)], \quad \forall s \geq 0$$

and $\mathbb{E}[\exp(-s\xi_n)] \in \mathfrak{R}$. However, this property does not allow us to conclude that Problem (5.1) can be solved for an arbitrary distribution $F(t)$ using our randomization construction. For instance, consider the degenerate random variable τ, such that $\mathbb{P}(\tau = c) = 1$, where $c > 0$ is a constant. In this case

$$\mathbb{E}[\exp(-s\tau)] = e^{-sc}$$

Let $\lambda_k = k \cdot c^{-1}$ and $l_k = k$. Consider a sequence of random variables τ_k with Gamma distributions, i.e., $\tau_k \sim \Gamma(\lambda_k, l_k)$. Then we have

$$\mathbb{E}[\exp(-s\tau_k)] = \frac{\lambda_k^{l_k}}{(\lambda_k + s)^{l_k}} = \frac{1}{\left(1 + \frac{sc}{k}\right)^k}$$

and

$$\lim_{k\to\infty} \mathbb{E}[\exp(-s\tau_k)] = e^{-sc}, \quad s \geq 0$$

but there is no finite μ, satisfying the inequality $\mu \geq \sqrt{2\lambda_k}$ for all k. Thus Problem (5.1) does not have a solution for any μ in this case.

For the same reason, there is no solution to Problem (5.1) if the function $F(t)$ is discontinuous. If we impose the condition: there exists $\lambda_* > 0$ such that

$$\lambda_k \leq \lambda_*, \quad \text{for all } k = 1, 2, \ldots \tag{5.17}$$

Then for any $\mu \geq \sqrt{2\lambda_*}$, there exists a solution $g_k(a)$ of Problem (5.1), given the density $f_k(t)$ and the mixture

$$g(a) = \sum_{k\geq 0} p_k g_k(a)$$

solves Problem (5.1) for

$$f(t) = \sum_{k\geq 0} p_k f_k(t)$$

The class of distributions satisfying condition (5.17) has a positive density $f(t)$ for all $t > 0$.

Proposition 3 defines a class $\mathfrak{R}$ for which a density $g(a)$ exists for $\mu > 0$. As mentioned above, if $\mu = 0$, it is impossible to match a class of distributions with a finite first moment, such as the class $\mathfrak{R}$. The following proposition gives a sufficient condition for the existence of the density for the case $\mu = 0$.

Proposition 4. *Fix the boundary $b(t) = 0$. Consider a non-negative random variable τ. Let $\mathfrak{R}_0$ denote the class of functions $\hat{f}(s) = \mathbb{E}[e^{-s\tau}]$ of the argument s represented in the form*

$$\hat{f}(s) = \sum_{k=1}^{K} \frac{\alpha_k}{(\sqrt{2s} + \lambda_k)^{l_k}}, \quad l_k \in \mathbb{R}_+, \quad \lambda_k, \alpha_k > 0 \tag{5.18}$$

Then the integral equation (5.14) *has a solution $g(a)$ satisfying the conditions*

$$g(a) \geq 0, \quad \int_0^\infty g(a)\,\mathrm{d}a = 1$$

PROOF. Similar to the proof of Proposition 3, the function $\hat{f}(s)$ defined in (5.18) can be represented as a convex combination

$$\hat{f}(s) = \sum_{k=1}^{K} p_k \hat{f}_k(s)$$

where

$$p_k = \frac{\alpha_k}{\lambda_k^{l_k}} \quad \text{and} \quad \hat{f}_k(s) = \frac{\lambda_k^{l_k}}{(\sqrt{2s} + \lambda_k)^{l_k}}$$

From (5.9) of Proposition 1, we have

$$\hat{g}_k(s) = \hat{f}_k\left(\frac{s^2}{2}\right) = \frac{\lambda_k^{l_k}}{(s + \lambda_k)^{l_k}}$$

which corresponds to the Gamma distribution

$$g_k(a) = \lambda_k \exp(-\lambda_k a) \frac{(\lambda_k a)^{l_k}}{\Gamma(l_k)}$$

Then the density

$$g(a) = \sum_{k=1}^{K} p_k g_k(a)$$

solves the Problem (5.1) with $\mu = 0$ for the density function $f(t)$ with Laplace transform $\hat{f}(s) \in \mathfrak{R}_0$.

5.2 Calibration

The randomized first hitting time model has analytical solutions for a large class of default distributions. To calibrate our model against market CDS spreads, we need to assume one particular default distribution, e.g., linear combination of gamma distributions[4]. Given the default distribution, the CDS spreads are computed as follows.

Suppose the protection buyer pays premia at $t_1 < \ldots < t_n = T$, where $0 = t_0 < t_1$ and T is the contract maturity. We consider a constant recovery rate R. If the underlying

[4]As shown in the calibration results, one gamma distribution with two parameters usually matches 10 market spreads quite well.

defaults during $(t_{i-1}, t_i]$, the protection seller pays the loss to the buyer at t_i. Then, the fair spread of the CDS satisfies

$$s = \frac{\sum_{i=1}^{n}(1-R)\big(\mathbb{P}(\tau \le t_i) - \mathbb{P}(\tau \le t_{i-1})\big)d_i}{\sum_{i=1}^{n}\big(1-\mathbb{P}(\tau \le t_i)\big)d_i}$$

where d_i is the expected value of discount factor for t_i in a risk-neutral measure.

Generally, the market CDS spreads are available at several maturities for a particular underlying asset. Therefore, we have to calibrate all of these spreads consistently using the same default distribution. As we model the default using our randomized first hitting time model, we have to find the distribution of the initial state. To obtain it, we need only to employ the relation between initial state and default distribution in Subsection 5.1.2.

5.3 Numerical results IV

For simplicity, we assume that the default time follows a gamma distribution with parameters γ and λ. In addition, we assume $\mu = \sqrt{2\lambda}$. Then by (5.12), the Laplace transformation of the initial state $X_0 = a$ for the process $X_t = W_t + a$ satisfies

$$\hat{g}(s) = \frac{(\sqrt{2\lambda})^{2\gamma}}{\left(s + \sqrt{2\lambda}\right)^{2\gamma}}.$$

Therefore, the initial state a follows a gamma distribution with parameters 2γ and $\sqrt{2\lambda}$. To verify that a single gamma distribution suffices to match market quotes, we collect daily quotes for eight CDS with ratings from AAA to BBB$-$. The market quotes of each CDS contain 392 daily spreads from 2 January, 2006 to 29 June, 2007.

Figure 5.1 plots representative calibrated results. The left panel plots the parameter values versus time for a CDS rating A$-$; the right panel is the spread calibration results. In the left panel, the "Alpha" curve plots the value of 10γ, and the "Beta" curve plots the value of $1/\lambda$. In the right panel, two curves marked with ▲ and • represent the average

spread fit; the other two curves represent the worst fit out of these 392 days. Figure 5.2 plots the calibrated default probabilities and the initial state distributions for different ratings. From these plots, we can conclude that the gamma distribution captures the default distribution fairly well and that the parameters of our randomized first hitting time model are relatively stable over time.

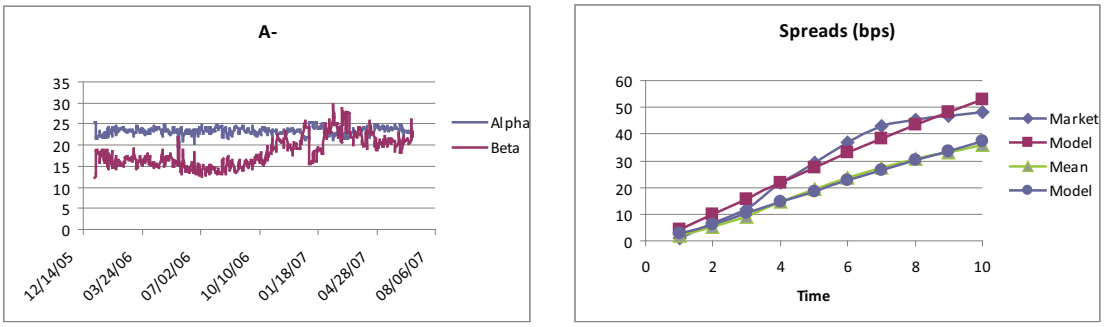


Figure 5.1: Typical calibration results

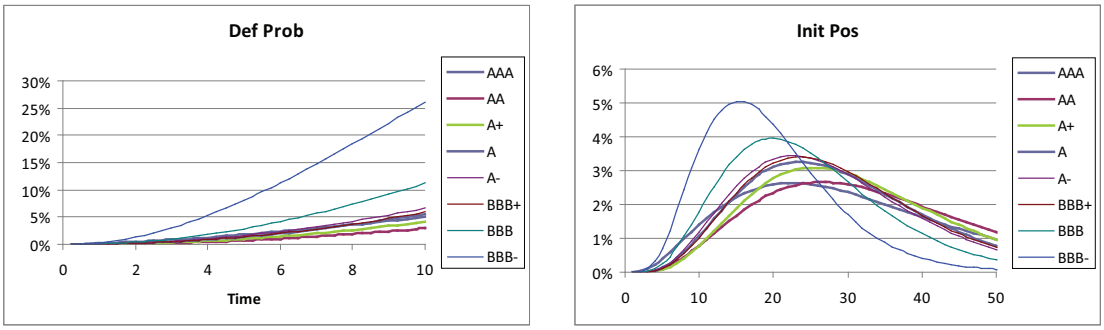


Figure 5.2: Default probability and initial state distributions

5.4 Challenges in extending to multi-name models

A natural way to extend our single-name model to a multi-name model is by decomposing the Wiener process of the first hitting time model into two parts, such that

$$W(t) = \rho W_X(t) + \sqrt{1-\rho^2}\widetilde{W}(t)$$

where $W_X(t)$ represents the common factor; $\widetilde{W}(t)$ represents the idiosyncratic factor associated with the particular name only; and ρ is the correlation coefficient between the name and common factor. A similar approach has been employed by Hull, Predescu and White [18] and Iscoe, Kreinin and Rosen [25]. This approach is intuitive, but it ends up either the same as the Gaussian factor copula model [18], which is unable to match market quotes effectively, or computationally inefficient without any analytical or semi-analytical solution [25]. Therefore, it is an important, but challenging, problem to extend the single-name model to a multi-name model while keeping the dynamic, analytical and easy-to-calibrate properties of the single-name model in the continuous-time setting.

A different and novel correlation structure is required to extend our single-name model to a multi-name first-hitting time model. Recently, a new correlation framework for first hitting time models has been introduced by Metzler [44]. The approach introduces the correlation of defaults by decomposing the drift and diffusion coefficients of a diffusion process in the first hitting time setting. Further study of such approaches and how to apply them to our single-name model deserves more study in the future.

Chapter 6

Conclusions and future work

In this thesis, we have studied four computational methods for the valuation of credit derivatives. First, we developed a generic method for FCDOs based on the conditional independence framework. We avoided the large combinatorial problem associated with pricing FCDOs by transforming the computation of the effective pool loss distribution of a FCDO to the computation of the pool loss distribution of an equivalent CDO. We also extended the reduction method to the valuation of FBDS. In particular, we proposed a hybrid method combining Monte Carlo simulation with an analytic approach to achieve an effective method.

Furthermore, we proposed a dynamic multi-period factor copula model, which can be calibrated fairly easily and matches the market quotes quite well. Using the independence of the common factors and the conditional forward default probability, we showed that the loss of a completely homogeneous pool possesses the Markov property. Therefore, we avoided the computationally expensive multi-dimensional integration that must be computed in previously proposed multi-period factor copula models.

Finally, we introduced a randomized first hitting time model for credit risk. We obtained extra degrees of freedom by randomizing the initial state of a standard Wiener process in the first hitting time setting. This initial state was traditionally a deterministic

variable in most credit risk models. For models requiring many degrees of freedom to match market quotes, randomizing the deterministic parameters may generate enough freedom to provide an elegant solution.

In the future, we would like to apply the reduction technique and the hybrid method to other forward-starting credit derivatives or more general credit options. For the multi-period factor model, we have developed an efficient method for completely homogenous pools only using an independent latent process across time. Therefore, key open questions include how to extend the model to a general pool and a general latent process. We would also like to study promising correlation structures to achieve a multi-name first hitting time model, which is dynamic, analytical and easy-to-calibrate.

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