Polynomial interpolation in the monomial basis is often considered to be a bad idea in numerical computations. In this paper, we show that this belief is wrong, in the sense that, despite the ill-conditioning of the Vandermonde matrix, polynomial interpolation in the monomial basis is as accurate as polynomial interpolation in a more well-conditioned basis in many cases of interest. Furthermore, we show that the monomial basis is superior to other polynomial bases in a number of applications.

**Keywords**: polynomial interpolation; monomials; Vandermonde matrix; backward error analysis

# Polynomial interpolation in the monomial basis is stable after all

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# 1 Introduction

The approximation of functions is one of the first ideas that appears in any introductory numerical analysis textbook. One of the most efficient methods for approximating a function  $F : [-1,1] \to \mathbb{R}$  is to find an interpolating polynomial  $P_N$  of degree Nwhich satisfies  $P_N(x_j) = F(x_j)$  for a set of (N + 1) collocation points  $\{x_j\}_{j=0,1,\dots,N}$ . In practice, the collocation points are typically chosen to be the Chebyshev points, and the resulting interpolating polynomial, known as the Chebyshev interpolant, is a nearly optimal approximation to F in the space of polynomials of degree N [27]. A common way of evaluating the interpolating polynomial  $P_N$  is to use the Lagrange polynomial basis, which can be done stably using the Barycentric interpolation formula [5, 19]. Alternatively, one can directly compute the monomial coefficients of  $P_N(x) = \sum_{k=0}^N a_k x^k$ , then evaluate  $P_N$  by naive summation. The computation of the monomial coefficient vector  $a := (a_0, a_1, \dots, a_N)^T \in \mathbb{R}^{N+1}$  of the interpolating polynomial  $P_N$  requires the solution to a linear system Va = f, where

$$V := \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^N \\ 1 & x_1 & x_1^2 & \cdots & x_1^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^N \end{pmatrix} \in \mathbb{R}^{(N+1) \times (N+1)}$$
(1)

is a Vandermonde matrix, and  $f := (F(x_0), F(x_1), \dots, F(x_N))^T \in \mathbb{R}^{N+1}$  is a vector of the function values of F at the (N + 1) collocation points on the interval [-1, 1]. It is well known that, when all of the collocation points are real, the condition number of a Vandermonde matrix grows at least exponentially as N increases [4]. It follows that the numerical solution to this linear system is highly inaccurate when N is not small, and, as a result, this algorithm for evaluating  $P_N$  is often considered to be unstable. But, is this really the case? As an example, let  $\{x_j\}_{j=0,1,\dots,N}$  be the set of (N+1) Chebyshev points on the interval [-1, 1], and consider the case where  $F(x) = \cos(2x + 1)$ . We solve the resulting Vandermonde system by LU factorization with partial pivoting. In Figure 1a, we compare the accuracy of the computed monomial expansion with the accuracy of the Chebyshev interpolant evaluated using the Barycentric interpolation formula (which is accurate up to machine precision). One can observe that the computed monomial expansion is, surprisingly, as accurate as the Chebyshev interpolant evaluated using the Barycentric interpolation formula, despite the huge condition number of the Vandermonde matrix reported in Figure 1b.



Figure 1: Polynomial interpolation of  $\cos(2x + 1)$  in the monomial basis. The x-axis label N denotes the order of approximation. The y-axis label "Error" denotes the  $L^{\infty}$  approximation error over [-1, 1], which is estimated by comparing the approximated function values at 10000 equidistant points over [-1, 1] with the true function values.

What happens when the function F becomes more complicated? In Figure 2, we compare the accuracy of the two approximations when  $F(x) = \cos(8x + 1)$  and when  $F(x) = \cos(12x + 1)$ . Initially, the computed monomial expansion is as accurate as the Chebyshev interpolant. However, the convergence of polynomial interpolation in the monomial basis stagnates after reaching a certain error threshold. Furthermore, it appears that, the more complicated a function is, the larger that error threshold becomes. But what does it mean for a function to be complicated in this context? In the next example, we consider functions that require an even higher-order Chebyshev interpolant in order to be approximated to machine precision. In Figure 3, we compare the accuracy of the two approximations when  $F(x) = \frac{1}{x-\sqrt{2}}$  and when  $F(x) = \frac{1}{x-0.5i}$ . These two functions each have a singularity in a neighborhood of the interval [-1, 1], and Chebyshev interpolants of degree  $\geq 40$  are required to approximate them to machine precision. Yet, no stagnation of convergence is observed. Based on all of the previous examples, we conclude that polynomial interpolation in the monomial basis is not as unstable as it appears, and has some subtleties lurking around the corner that are worth further investigation.

These seemingly mysterious experiments can be explained partially from the point of view of backward error analysis. Indeed, the forward error  $||a - \hat{a}||_2$  of the numerical



Figure 2: Polynomial interpolation of more complicated functions in the monomial basis.



Figure 3: Polynomial interpolation of functions with a singularity near the interval [-1,1] in the monomial basis.

solution  $\hat{a}$  to the Vandermonde system Va = f can be huge, but it is the backward error, i.e.,  $\|V\hat{a} - f\|_2$ , that matters for the accuracy of the approximation. This is because a small backward error implies that the difference between the computed monomial expansion, which we denote by  $\hat{P}_N$ , and the exact interpolating polynomial  $P_N$ , is a polynomial that approximately vanishes at all of the collocation points. When the Lebesgue constant associated with the collocation points is small (which is the case for the Chebyshev points), the polynomial  $P_N - \hat{P}_N$  is bounded uniformly by the backward error times a small constant. As a result, we bound the monomial approximation error  $\|F - \hat{P}_N\|_{L^{\infty}([-1,1])}$  by the following inequality:

$$\|F - \widehat{P}_N\|_{L^{\infty}([-1,1])} \le \|F - P_N\|_{L^{\infty}([-1,1])} + \|P_N - \widehat{P}_N\|_{L^{\infty}([-1,1])}.$$
 (2)

We refer to the first and the second term on the right hand side of (2) as the polynomial interpolation error and the backward error, respectively. When the backward error is smaller than the polynomial interpolation error, the monomial approximation error is dominated by the polynomial interpolation error, and the use of a monomial basis does not incur any additional loss of accuracy. Once the polynomial interpolation error becomes smaller than the backward error, the convergence of the approximation stagnates. For example, in Figure 3a, we verify numerically that the backward error is around the size of machine epsilon for all  $N \leq 40$ , so the stagnation is not observed, and polynomial interpolation in the monomial basis (evaluated by naive summation) is as accurate as polynomial interpolation in the Lagrange basis (evaluated by the Barycentric interpolation formula). On the other hand, in Figure 2a, the backward error is around the size of  $10^{-13}$ for  $N \geq 20$ , which leads to stagnation once the polynomial interpolation error is less than  $10^{-13}$ .

The explanation above brings up a new question: when will the backward error be small? When a backward stable linear system solver (e.g., LU factorization with partial pivoting) is used to solve the Vandermonde system Va = f, it is guaranteed that the numerical solution  $\hat{a}$  is the exact solution to the linear system

$$(V + \delta V)\hat{a} = f,\tag{3}$$

for a matrix  $\delta V \in \mathbb{R}^{(N+1)\times(N+1)}$  that satisfies  $\|\delta V\|_2 \leq u \cdot \gamma \|V\|_2$ , where u denotes machine epsilon and  $\gamma$  is a modest constant. It follows that the backward error of the numerical solution is bounded by  $u \cdot \gamma \|V\|_2 \|\hat{a}\|_2$ . We note that  $\|V\|_2$  is small, so the backward error is essentially controlled by the norm of the computed monomial coefficients. In fact, so long as the condition number of the Vandermonde matrix V is not excessively large (i.e.,  $\kappa(V) \leq \frac{1}{2u \cdot \gamma}$ ), one can show that the norm of the monomial coefficients computed by a backward stable solver is around the same size as the norm of the exact monomial coefficients of the interpolating polynomial. Therefore, in this case, the monomial approximation error can be quantified a priori using information about the interpolating polynomial, which implies that a theory of polynomial interpolation in the monomial basis can be developed.

The rest of the paper is organized as follows. In Section 2, we analyze polynomial interpolation in the monomial basis over a smooth simple arc in the complex plane, with the interval as a special case. Based on our results, the monomial approximation error can be accurately estimated a priori, and we show that polynomial interpolation in the monomial basis is as accurate as polynomial interpolation in a more well-conditioned basis in many cases of interst. Furthermore, we show that, so long as the Lebesgue constant associated with the collocation points is small, we require nothing more than a backward stable linear system solver to solve the Vandermonde matrix for the purpose of interpolation. In Section 3, we present applications of polynomial interpolation in the monomial basis. In Section 4, we review related work, and discuss the generalization of our theory to higher dimensions.

# 2 Polynomial interpolation in the monomial basis

Suppose that  $\Gamma$  is a smooth simple arc in  $\mathbb{C}$ . Given a function  $F : \Gamma \to \mathbb{C}$  and a set of collocation points  $Z := \{z_j\}_{j=0,1,\dots,N} \subset \Gamma$ , we denote the Nth order interpolating polynomial of F for the set of collocation points Z by  $P_N$ . In a monomial basis centered at  $\beta \in \mathbb{C}$ , the interpolating polynomial  $P_N(z) = \sum_{k=0}^N a_k (z-\beta)^k$ , and satisfies

$$P_N(z_j) = \sum_{k=0}^N a_k (z_j - \beta)^k = F(z_j),$$
(4)

for j = 0, 1, ..., N. Without loss of generality, we assume the monomial basis center to be the origin (i.e.,  $\beta = 0$ ), and that  $\Gamma$  is inside the unit disk centered at the origin. It is clear that the monomial coefficient vector  $a^{(N)} := (a_0, a_1, ..., a_N)^T$  is the solution to the Vandermonde system

$$\begin{pmatrix} 1 & z_0 & z_0^2 & \cdots & z_0^N \\ 1 & z_1 & z_1^2 & \cdots & z_1^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_N & z_N^2 & \cdots & z_N^N \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{pmatrix} = \begin{pmatrix} F(z_0) \\ F(z_1) \\ \vdots \\ F(z_N) \end{pmatrix}.$$
(5)

We denote the Vandermonde matrix by  $V^{(N)}$ , and the right hand side vector by  $f^{(N)}$ .

Note that the absolute condition number of the evaluation of  $P_N(z)$ , by naive summation, satisfies

$$\kappa(z) = \sum_{k=0}^{N} |a_k z^k| \le ||a^{(N)}||_2 ||z^{(N)}||_2, \tag{6}$$

where  $z^{(N)} := (1, z, \dots, z^N)^T$ . As a result, large monomial coefficients are detrimental to the accuracy of monomial expansion evaluation. Furthermore, the closer z is to the expansion center, the smaller the condition number becomes.

In order to bound the 2-norm of the computed monomial coefficients, we will require the following lemma, which provides a bound for the 2-norm of the solution to a perturbed linear system.

**Lemma 2.1.** Let N be a positive integer. Suppose that  $A \in \mathbb{C}^{N \times N}$  is invertible,  $b \in \mathbb{C}^N$ , and that  $x \in \mathbb{C}^N$  satisfies Ax = b. Suppose further that  $\hat{x} \in \mathbb{C}^N$  satisfies  $(A + \delta A)\hat{x} = b$ , where  $\delta A \in \mathbb{C}^{N \times N}$ . If there exists an  $\alpha > 1$  such that

$$\frac{\|\delta A\|_2}{\|A\|_2} \le \frac{1}{\alpha \cdot \kappa(A)},\tag{7}$$

then the matrix  $A + \delta A$  is invertible, and  $\hat{x}$  satisfies

$$\frac{\alpha}{\alpha+1} \|x\|_2 \le \|\widehat{x}\|_2 \le \frac{\alpha}{\alpha-1} \|x\|_2.$$

$$\tag{8}$$

**Proof.** By multiplying both sides of  $(A + \delta A)\hat{x} = b$  by  $A^{-1}$ , we have that

$$(I + A^{-1}\delta A)\hat{x} = x,\tag{9}$$

where I denotes the identity matrix. By (7), the term  $A^{-1}\delta A$  satisfies

$$\|A^{-1}\delta A\|_{2} \le \|A^{-1}\|_{2} \|\delta A\|_{2} = \frac{\kappa(A)}{\|A\|_{2}} \|\delta A\|_{2} \le \frac{1}{\alpha} < 1.$$
(10)

Thus, it follows that the matrix  $A + \delta A$  is invertible, and  $\|\hat{x}\|_2$  satisfies

$$\|\widehat{x}\|_{2} \leq \|(I + A^{-1}\delta A)^{-1}\|_{2} \|x\|_{2} \leq \frac{1}{1 - \|A^{-1}\delta A\|_{2}} \|x\|_{2} \leq \frac{\alpha}{\alpha - 1} \|x\|_{2}.$$
 (11)

In addition, by (10),  $||x||_2$  satisfies

$$\|x\|_{2} \leq \|I + A^{-1}\delta A\|_{2} \|\widehat{x}\|_{2} \leq \left(1 + \frac{1}{\alpha}\right) \|\widehat{x}\|_{2}.$$
(12)

The proof is complete by combining (11) and (12).

The following theorem provides a priori bounds for the backward error of the numerical solution to a Vandermonde system, and the monomial approximation error.

**Theorem 2.2.** Let  $\Gamma \subset \mathbb{C}$  be a smooth simple arc. Suppose that  $F : \Gamma \to \mathbb{C}$  is a function. Let  $Z := \{z_j\}_{j=0,1,\dots,N} \subset \Gamma$  be a set of collocation points, and let  $\Lambda_N$  denote associated Lebesgue constant. Suppose that  $P_N$  is the Nth order interpolating polynomial of the function F for the set of collocation points Z. Suppose further that the monomial coefficient vector  $\hat{a}^{(N)} = (\hat{a}_0, \hat{a}_1, \dots, \hat{a}_N)^T$  is the numerical solution to the Vandermonde system  $V^{(N)}a^{(N)} = f^{(N)}$ , computed using a backward stable linear system solver, in the sense that  $\hat{a}^{(N)}$  is the exact solution to

$$(V^{(N)} + \delta V^{(N)})\hat{a}^{(N)} = f^{(N)}, \tag{13}$$

for some  $\delta V^{(N)} \in \mathbb{C}^{N \times N}$  that satisfies

$$\|\delta V^{(N)}\|_{2} \le u \cdot \gamma_{N} \|V^{(N)}\|_{2}, \tag{14}$$

where u denotes machine epsilon and  $\gamma_N$  is a modest factor. Let  $\widehat{P}_N(z) := \sum_{k=0}^N \widehat{a}_k z^k$  be the computed monomial expansion. If the condition number of the Vandermonde matrix satisfies

$$\kappa(V^{(N)}) \le \frac{1}{2u \cdot \gamma_N},\tag{15}$$

then the backward error of the numerical solution  $\hat{a}^{(N)}$  to the Vandermonde system satisfies

$$\|V^{(N)}\widehat{a}^{(N)} - f^{(N)}\|_{2} \le 2u \cdot \gamma_{N} \|V^{(N)}\|_{2} \|a^{(N)}\|_{2},$$
(16)

and the monomial approximation error can be quantified a priori by

$$\|F - \widehat{P}_N\|_{L^{\infty}(\Gamma)} \le 2u \cdot \gamma_N \Lambda_N \|V^{(N)}\|_2 \|a^{(N)}\|_2 + \|F - P_N\|_{L^{\infty}(\Gamma)}$$
(17a)

$$\leq 2u \cdot \gamma_N \Lambda_N \| V^{(N)} \|_2 \| a^{(N)} \|_2 + (\Lambda_N + 1) \| F - P_N^* \|_{L^{\infty}(\Gamma)}, \quad (17b)$$

where  $P_N^*$  denotes the best Nth order polynomial approximation to F in the  $L^{\infty}$  norm over  $\Gamma$ .

**Proof.** When the condition number of  $V^{(N)}$  satisfies

$$\kappa(V^{(N)}) \le \frac{1}{2u \cdot \gamma_N},\tag{18}$$

we have that

$$\frac{\|\delta V^{(N)}\|_2}{\|V^{(N)}\|_2} \le \frac{1}{2 \cdot \kappa(V^{(N)})},\tag{19}$$

by (14). It follows from Lemma 2.1 that the 2-norm of the numerical solution  $\hat{a}^{(N)}$  satisfies

$$\frac{2}{3} \|a^{(N)}\|_2 \le \|\widehat{a}^{(N)}\|_2 \le 2 \|a^{(N)}\|_2.$$
(20)

Then, by the combination of (13), (14) and (20), the backward error of the numerical solution to the Vandermonde system satisfies

$$\|V^{(N)}\widehat{a}^{(N)} - f^{(N)}\|_{2} \le u \cdot \gamma_{N} \|V^{(N)}\|_{2} \|\widehat{a}^{(N)}\|_{2} \le 2u \cdot \gamma_{N} \|V^{(N)}\|_{2} \|\widehat{a}^{(N)}\|_{2}.$$
 (21)

Furthermore, by the triangle inequality, the definition of the Lebesgue constant  $\Lambda_N$ , and (21), the monomial approximation error satisfies

$$\begin{aligned} \|F - \hat{P}_{N}\|_{L^{\infty}(\Gamma)} &\leq \|\hat{P}_{N} - P_{N}\|_{L^{\infty}(\Gamma)} + \|F - P_{N}\|_{L^{\infty}(\Gamma)} \\ &\leq \Lambda_{N} \|V^{(N)} \hat{a}^{(N)} - f^{(N)}\|_{\infty} + \|F - P_{N}\|_{L^{\infty}(\Gamma)} \\ &\leq \Lambda_{N} \|V^{(N)} \hat{a}^{(N)} - f^{(N)}\|_{2} + \|F - P_{N}\|_{L^{\infty}(\Gamma)} \\ &\leq 2u \cdot \gamma_{N} \Lambda_{N} \|V^{(N)}\|_{2} \|a^{(N)}\|_{2} + \|F - P_{N}\|_{L^{\infty}(\Gamma)} \\ &\leq 2u \cdot \gamma_{N} \Lambda_{N} \|V^{(N)}\|_{2} \|a^{(N)}\|_{2} + (\Lambda_{N} + 1) \|F - P_{N}^{*}\|_{L^{\infty}(\Gamma)}. \end{aligned}$$

Observe that the first term on the right hand side of (17a) is an upper bound of the backward error  $||P_N - \hat{P}_N||_{L^{\infty}(\Gamma)}$ , i.e., the extra loss of accuracy caused by the use of a monomial basis. Based on inequality (6), even if the monomial coefficients of  $P_N$  were known analytically, the error caused by the condition number of the evaluation of the interpolating polynomial  $P_N$ , represented in the monomial basis, is around the same size as this upper bound of the backward error, provided that  $\Lambda_N$  is not large. In other words, polynomial interpolation in the monomial basis is stable up to the condition number of evaluation in this basis, so long as  $\Lambda_N$  is not large, a backward stable solver is used to solve the Vandermonde system, and  $\kappa(V^{(N)}) \leq \frac{1}{2u \cdot \gamma_N}$ . We note that the monomial coefficients of a low-degree interpolating polynomial are

We note that the monomial coefficients of a low-degree interpolating polynomial are typically around the same size of the  $L^{\infty}$  norm of the function. Furthermore, the 2-norm of the monomial coefficient vector of an interpolating polynomial generally grows as the order of approximation increases, until the interpolating polynomial converges. Therefore, the upper bound of the backward error is of size around machine epsilon for small N, and, in most cases, grows monotonically with N until the interpolating polynomial converges. The polynomial interpolation error  $||F - P_N||_{L^{\infty}(\Gamma)}$ , on the other hand, generally decays as N increases. As a result, the monomial approximation error is initially dominated by the polynomial interpolation error, and is dominated by the backward error only after the interpolation error becomes sufficiently small, which leads to the stagnation of convergence observed in Figure 2.



Figure 4: The level set  $E_{\rho}$  corresponding to  $\Gamma$ , for various values of  $\rho$ . The colorbar indicates the value of  $\rho$ . The smooth simple arc  $\Gamma$  is the white curve in the figure. The plots were made using the source code provided in [22].

The rest of this section is structured as follows. First, we study the polynomial interpolation error  $||F - P_N||_{L^{\infty}(\Gamma)}$  for functions over a smooth simple arc  $\Gamma \subset \mathbb{C}$ . Next, we study the backward error  $||P_N - \hat{P}_N||_{L^{\infty}(\Gamma)}$  by bounding the 2-norm of the monomial coefficients of the interpolating polynomial. Finally, we study the growth rate of the condition number of a Vandermonde matrix, which determines when the backward stability of a linear system solver alone guarantees the a priori error estimates in (17) to hold. Based on these results, we describe a sufficient condition on the function F, such that the monomial approximation error  $||F - \hat{P}_N||_{L^{\infty}(\Gamma)}$  decays to around machine epsilon before the Vandermonde matrix condition number  $\kappa(V^{(N)})$  attains the threshold  $\frac{1}{2u \cdot \gamma_N}$ .

Below, we define a generalization of the Bernstein ellipse for a smooth simple arc in the complex plane.

**Definition 2.1.** Given a smooth simple arc  $\Gamma$  in the complex plane, we define  $E_{\rho}$  to be the level set  $\{x + iy \in \mathbb{C} : G(x, y) = \log \rho\}$ , where  $G : \mathbb{R}^2 \to \mathbb{R}$  is the unique solution to the following exterior Laplace equation

$$\nabla^2 G = 0 \text{ in } \mathbb{R}^2 \setminus \Gamma,$$
  

$$G = 0 \text{ on } \partial \Gamma,$$
  

$$G(x) \sim \log |x| \text{ as } |x| \to \infty.$$
(23)

Furthermore, we let  $E_{\rho}^{o}$  denote the open region bounded by  $E_{\rho}$ .

We note that, when  $\Gamma = [a, b] \subset \mathbb{R}$ , the level set  $E_{\rho}$  is a Bernstein ellipse with foci at a and b, and with parameter  $\rho$ . In Figure 4, we plot examples of level sets  $E_{\rho}$  for an interval and for a sine curve, for various values of  $\rho$ .

The following lemma demonstrates the feasibility of approximating analytic functions over a smooth simple arc  $\Gamma$  in the complex plane by polynomials. We refer the readers to Section 4.5 in [29] for the proof. **Lemma 2.3.** Let  $\Gamma$  be a smooth simple arc in the complex plane. Suppose that the function  $F: \Gamma \to \mathbb{C}$  is analytically continuable to the closure of the region  $E_{\rho}^{o}$  corresponding to  $\Gamma$  for some  $\rho > 1$  (see Definition 2.1). Then, there exists a sequence of polynomials  $\{Q_N\}$  satisfying

$$\|F - Q_N\|_{L^{\infty}(\Gamma)} \le C\rho^{-N},\tag{24}$$

for all  $N \ge 0$ , where  $C \ge 0$  is a constant that is independent of N.

**Remark 2.1.** The magnitude of *C* is generally proportional to  $||F||_{L^{\infty}(E^{o}_{\rho})}$ . See Lemma 2.10 in Section 2.4 for a concrete example in the case where  $\Gamma$  is an interval.

Below, we present two technical lemmas. The following inequality is known as one of Bernstein's inequalities, and bounds the  $L^{\infty}$  norm of a polynomial over the region  $E_{\rho}^{o}$  by the  $L^{\infty}$  norm of the polynomial over  $\Gamma$ .

**Lemma 2.4** (Generalized Bernstein's inequality). Let  $\Gamma$  be a smooth simple arc in the complex plane, and let  $E_{\rho}^{o}$  be the region corresponding to  $\Gamma$  with some parameter  $\rho > 1$  (see Definition 2.1). Then, the  $L^{\infty}$  norm of any polynomial  $P_N$  of degree N over  $E_{\rho}^{o}$  satisfies

$$\|P_N\|_{L^{\infty}(E^o_{\rho})} \le \rho^N \|P_N\|_{L^{\infty}(\Gamma)}.$$
(25)

The following lemma provides a bound for the 2-norm of the monomial coefficients of a polynomial, given the  $L^{\infty}$  norm of the polynomial over the boundary of the unit disk centered at the origin.

**Lemma 2.5.** Let  $P_N : \mathbb{C} \to \mathbb{C}$  be a polynomial of degree N, where  $P_N(z) = \sum_{k=0}^N a_k z^k$  for some  $a_0, a_1, \ldots, a_N \in \mathbb{C}$ . The 2-norm of the coefficient vector  $a^{(N)} := (a_0, a_1, \ldots, a_N)^T$  satisfies

$$\|a^{(N)}\|_{2} \le \|P_{N}\|_{L^{\infty}(\partial D_{1})},\tag{26}$$

where  $D_1$  denotes the open unit disk centered at the origin.

**Proof.** Observe that

$$P_N(e^{i\theta}) = \sum_{k=0}^N a_k e^{ik\theta}.$$
(27)

Thus, by Parseval's identity, we have that

$$\|a^{(N)}\|_{2} = \left(\frac{1}{2\pi} \int_{0}^{2\pi} |P_{N}(e^{i\theta})|^{2} d\theta\right)^{1/2} \le \|P_{N}\|_{L^{\infty}(\partial D_{1})}.$$
(28)

The parameter  $\rho_*$  defined below plays an important role in bounding both the 2-norm of the monomial coefficients of an interpolating polynomial, and the growth rate of the condition number of a Vandermonde matrix. **Definition 2.2.** Let  $E_{\rho}^{o}$  be the region corresponding to a smooth simple arc  $\Gamma \subset \mathbb{C}$  (see Definition 2.1). Define  $\rho_{*} := \inf\{\rho > 1 : D_{1} \subset E_{\rho}^{o}\}$ , where  $D_{1}$  is the open unit disk centered at the origin.

In other words,  $\rho_*$  is the parameter of the smallest  $E_{\rho}^o$  that contains the open unit disk centered at the origin.

The following theorem provides a bound for the 2-norm of the monomial coefficients of an arbitrary interpolating polynomial. We note that Lemma 2.3 provides a sufficient condition for the assumption of this theorem to hold.

**Theorem 2.6.** Let  $\Gamma$  be a smooth simple arc in the complex plane. Suppose that  $F : \Gamma \to \mathbb{C}$  is an analytic function. Suppose further that there exists a finite sequence of polynomials  $\{Q_n\}_{n=0,1,\dots,N}$  that satisfies

$$\|F - Q_n\|_{L^{\infty}(\Gamma)} \le C\rho^{-n}, \quad 0 \le n \le N,$$
(29)

for some constants  $\rho > 1$ ,  $C \ge 0$ . Define  $P_N(z) = \sum_{k=0}^N a_k z^k$  to be the interpolating polynomial of F for the set of collocation points  $Z = \{z_j\}_{j=0,1,\dots,N} \subset \Gamma$ . The 2-norm of the monomial coefficient vector  $a^{(N)} := (a_0, a_1, \dots, a_N)^T$  satisfies

$$\|a^{(N)}\|_{2} \leq \|F\|_{L^{\infty}(\Gamma)} + C\Big(\Lambda_{N}\Big(\frac{\rho_{*}}{\rho}\Big)^{N} + 2\rho_{*}\sum_{j=0}^{N-1}\Big(\frac{\rho_{*}}{\rho}\Big)^{j}\Big),\tag{30}$$

where  $\rho_*$  is defined in Definition 2.2, and  $\Lambda_N$  denotes the Lebesgue constant for Z.

**Proof.** Let  $P_n^* := \operatorname{argmin}_{p \in \mathcal{P}_n} ||F - p||_{L^{\infty}(\Gamma)}$ , where  $\mathcal{P}_n$  denotes the space of polynomials of degree  $\leq n$ . It is well known that  $P_n^*$  exists and is unique. It follows immediately from the definition of  $P_n^*$  and inequality (29) that

$$\|F - P_n^*\|_{L^{\infty}(\Gamma)} \le \|F - Q_n\|_{L^{\infty}(\Gamma)} \le C\rho^{-n},$$
(31)

for all  $0 \le n \le N$ . Let  $E_{\rho_*}^o$  be the region corresponding to  $\Gamma$  with parameter  $\rho_*$  (see Definitions 2.1, 2.2). By the triangle inequality, the polynomial  $P_N^*$  satisfies

$$\begin{aligned} \|P_{N}^{*}\|_{L^{\infty}(E_{\rho_{*}}^{o})} &\leq \|P_{0}^{*}\|_{L^{\infty}(E_{\rho_{*}}^{o})} + \sum_{j=1}^{N} \|P_{j}^{*} - P_{j-1}^{*}\|_{L^{\infty}(E_{\rho_{*}}^{o})} \\ &\leq \|F\|_{L^{\infty}(\Gamma)} + \sum_{j=1}^{N} \rho_{*}^{j} \|P_{j}^{*} - P_{j-1}^{*}\|_{L^{\infty}(\Gamma)} \\ &\leq \|F\|_{L^{\infty}(\Gamma)} + 2C \cdot \rho_{*} \sum_{j=0}^{N-1} \left(\frac{\rho_{*}}{\rho}\right)^{j}, \end{aligned}$$
(32)

where the second inequality follows from Lemma 2.4, and the last inequality comes from (31). It follows that the interpolating polynomial  $P_N$  satisfies

$$\begin{split} \|P_{N}\|_{L^{\infty}(E_{\rho_{*}}^{o})} &\leq \|P_{N} - P_{N}^{*}\|_{L^{\infty}(E_{\rho_{*}}^{o})} + \|P_{N}^{*}\|_{L^{\infty}(E_{\rho_{*}}^{o})} \\ &\leq \rho_{*}^{N}\|P_{N} - P_{N}^{*}\|_{L^{\infty}(\Gamma)} + \|P_{N}^{*}\|_{L^{\infty}(E_{\rho_{*}}^{o})} \\ &\leq \rho_{*}^{N}\Lambda_{N}\|F - P_{N}^{*}\|_{L^{\infty}(\Gamma)} + \|P_{N}^{*}\|_{L^{\infty}(E_{\rho_{*}}^{o})} \\ &\leq \|F\|_{L^{\infty}(\Gamma)} + C\Big(\Lambda_{N}\Big(\frac{\rho_{*}}{\rho}\Big)^{N} + 2\rho_{*}\sum_{j=0}^{N-1}\Big(\frac{\rho_{*}}{\rho}\Big)^{j}\Big), \end{split}$$
(33)

where the second inequality follows from Lemma 2.4, the third inequality comes from the observation that  $P_N - P_N^*$  is the interpolating polynomial of  $F - P_N^*$  for the set of collocation points Z, and the last inequality follows from (31) and (32). Since  $E_{\rho_*}^o$  contains the open unit disk centered at the origin by definition, the 2-norm of the monomial coefficient vector of  $P_N$  satisfies

$$\|a^{(N)}\|_{2} \le \|P_{N}\|_{L^{\infty}(\partial D_{1})} \le \|P_{N}\|_{L^{\infty}(E^{o}_{o_{s}})},$$
(34)

by Lemma 2.5. The proof is complete by combining (33) and (34).

When the condition number of the Vandermonde matrix is not excessively large (i.e., when  $\kappa(V^{(N)}) \leq \frac{1}{2u \cdot \gamma_N}$ ), Lemma 2.3 and Theorem 2.6 can be combined with the a priori monomial approximation error estimates in (17). The following theorem characterizes the growth rate of the condition number of a Vandermonde matrix a priori.

**Theorem 2.7.** Suppose that  $V^{(N)}$  is a Vandermonde matrix with (N + 1) collocation points  $Z = \{z_j\}_{j=0,1,...,N} \subset \mathbb{C}$ . Suppose further that  $\Gamma \subset \mathbb{C}$  is a smooth simple arc such that  $Z \subset \Gamma$ . Let  $\Lambda_N$  denote the Lebesgue constant for the set of collocation points Z over  $\Gamma$ . The condition number of  $V^{(N)}$  satisfies

$$\kappa(V^{(N)}) \le \rho_*^N \Lambda_N \|V^{(N)}\|_2, \tag{35}$$

where  $\rho_*$  is defined in Definition 2.2. When  $N \leq \log(1/u)/\log(\rho_*)$ , where u denotes machine epsilon, the condition number of  $V^{(N)}$  is bounded by

$$\kappa(V^{(N)}) \le \frac{1}{u} \Lambda_N \|V^{(N)}\|_2.$$
(36)

**Proof.** Let  $f^{(N)} = (f_0, f_1, \ldots, f_N)^T \in \mathbb{C}^{N+1}$  be an arbitrary vector. Suppose that  $P_N$  is an interpolating polynomial of degree N for the set of points  $\{(z_j, f_j)\}_{j=0,1,\ldots,N}$ . Suppose further that  $E^o_{\rho_*}$  denotes the region corresponding to  $\Gamma$  defined in Definition 2.1. The  $L^\infty$ norm of  $P_N$  over the boundary of  $D_1$  (i.e., the unit disk centered at the origin) satisfies

$$\|P_N\|_{L^{\infty}(\partial D_1)} \le \|P_N\|_{L^{\infty}(E^o_{\rho_*})} \le \rho^N_* \|P_N\|_{L^{\infty}(\Gamma)} \le \rho^N_* \Lambda_N \|f^{(N)}\|_{\infty},$$
(37)

where the second inequality is a direct application of Lemma 2.4, and the last inequality follows from the definition of the Lebesgue constant. Suppose that  $a^{(N)} = (a_0, a_1, \ldots, a_N)^T$ is the solution to the Vandermonde system  $V^{(N)}a^{(N)} = f^{(N)}$ . It follows immediately that  $P_N(z) = \sum_{j=0}^N a_j z^j$ . Therefore, by Lemma 2.5, the 2-norm of  $a^{(N)}$  satisfies

$$\|a^{(N)}\|_{2} \leq \|P_{N}\|_{L^{\infty}(\partial D_{1})} \leq \rho_{*}^{N} \Lambda_{N} \|f^{(N)}\|_{\infty} \leq \rho_{*}^{N} \Lambda_{N} \|f^{(N)}\|_{2},$$
(38)

from which it follows that

$$\|(V^{(N)})^{-1}\|_{2} = \sup_{f^{(N)} \neq 0} \left\{ \frac{\|(V^{(N)})^{-1} f^{(N)}\|_{2}}{\|f^{(N)}\|_{2}} \right\} = \sup_{f^{(N)} \neq 0} \left\{ \frac{\|a^{(N)}\|_{2}}{\|f^{(N)}\|_{2}} \right\} \le \rho_{*}^{N} \Lambda_{N}.$$
(39)

Therefore, the condition number of  $V^{(N)}$  is bounded by

$$\kappa(V^{(N)}) = \|V^{(N)}\|_2 \|(V^{(N)})^{-1}\|_2 \le \rho_*^N \Lambda_N \|V^{(N)}\|_2.$$
(40)

When  $N \leq \log(1/u)/\log(\rho_*)$ , we have  $\rho_*^N \leq \frac{1}{u}$ , from which it follows that (36) holds.

**Definition 2.3.** Given a smooth simple arc  $\Gamma$  in the complex plane, we define  $N^* := \log(1/u)/\log(\rho_*)$ , where u denotes machine epsilon, and  $\rho_*$  is defined in Definition 2.2.

Without loss of generality, suppose that  $\Gamma$  is a smooth simple arc inside the unit disk centered at the origin. When the set of collocation points  $Z \subset \Gamma$  are chosen such that the associated Lebesgue constant is small, we find that the condition  $\kappa(V^{(N)}) \leq \frac{1}{u}$ is generally satisfied for  $N \leq N^*$ , where  $N^*$  is defined below. We refer the readers to Figures 5 and 10a for numerical evidence. Since  $\gamma_N$  is typically very small, it is safe to assume that  $\kappa(V^{(N)}) \lesssim \frac{1}{2u \cdot \gamma_N}$  when  $N \leq N^*$ .



Figure 5: The condition number of a Vandermonde matrix with Chebyshev collocation points over an interval  $\Gamma$ , and its upper bound, for different orders of approximation. It is easy to verify that  $\rho_* = 1 + \sqrt{2}$ ,  $N^* \approx 40.8$  when  $\Gamma = [-1, 1]$ , and  $\rho_* = 3 + 2\sqrt{2}$ ,  $N^* \approx 20.4$  when  $\Gamma = [0, 1]$ . One can observe that  $\kappa(V^{(N)}) \leq \frac{1}{u}$  when  $N \leq N^*$ .

The following corollary shows that, when the term  $C\rho^{-N}$  appearing in Lemma 2.3 is approximately equal to Cu, for some  $N \leq N^*$ , the 2-norm of the monomial coefficient vector of the interpolating polynomial is bounded by approximately  $C(N + \Lambda_N)$ .

**Corollary 2.8.** Without loss of generality, let  $\Gamma$  be a smooth simple arc inside the unit disk. Suppose that  $F : \Gamma \to \mathbb{C}$  is analytically continuable to the closure of the region  $E_{\rho}^{o}$  corresponding to  $\Gamma$  for some  $\rho > 1$ . Let  $Z := \{z_{j}\}_{j=0,1,\ldots,N} \subset \Gamma$  be a set of collocation points, and let  $\Lambda_{N}$  denote the associated Lebesgue constant. Suppose that  $a^{(N)} = (a_{0}, a_{1}, \ldots, a_{N})^{T}$  is the monomial coefficient vector of the Nth degree interpolating polynomial of F for the set of collocation points Z. If  $\rho^{-N} \leq u$  for some  $N \leq N^{*}$  (see Definition 2.3), the 2-norm of  $a^{(N)}$  satisfies

$$\|a^{(N)}\|_{2} \leq \|F\|_{L^{\infty}(\Gamma)} + C\left(\Lambda_{N} + 2\rho_{*}N\right),\tag{41}$$

where C is the same constant appearing in Lemma 2.3, and  $\rho_*$  is defined in Definition 2.2.

**Proof.** By Lemma 2.3, there exists a sequence of polynomials  $\{Q_n\}$  that satisfies

$$\|F - Q_n\|_{L^{\infty}(\Gamma)} \le C\rho^{-n},\tag{42}$$

for all  $n \ge 0$ , where  $C \ge 0$  is a constant. Suppose that  $\rho^{-N} \le u$  for some  $N \le N^*$ . By the definition of  $N^*$ , we have that  $\rho^N \ge \frac{1}{u} = \rho_*^{N^*} \ge \rho_*^N$ , which implies that  $\frac{\rho_*}{\rho} \le 1$ . Since the precondition of Theorem 2.6 is satisfied by (42), it follows from the inequality  $\frac{\rho_*}{\rho} \le 1$  that (41) holds.

Recall that, when  $N \leq N^*$ , it is generally safe to assume that  $\kappa(V^{(N)}) \lesssim \frac{1}{2u \cdot \gamma_N}$ . As a result, when a backward stable linear system solver is used and when the assumptions made in Corollary 2.8 are satisfied, the monomial approximation error is bounded by

$$\begin{aligned} \left\| F - \widehat{P}_N \right\|_{L^{\infty}(\Gamma)} &\leq 2u \cdot \gamma_N \Lambda_N \| V^{(N)} \|_2 \| a^{(N)} \|_2 + Cu \cdot (\Lambda_N + 1) \\ &\lesssim Cu \cdot \Lambda_N (N^2 + \Lambda_N N + 1), \end{aligned}$$
(43)

based on the a priori error estimate (17b), where the last inequality follows from the combination of (41) and the naive estimate  $\|V^{(N)}\|_2 \leq \|V^{(N)}\|_F \leq N+1$ . In practice, we find that the bound  $\|F - \hat{P}_N\|_{L^{\infty}(\Gamma)} \leq Cu$  generally holds under those same assumptions. We also note that the value of C is generally proportional to  $\|F\|_{L^{\infty}(E_{0}^{o})}$  (see Remark 2.1).

Based on Lemma 2.3, Remark 2.1, Corollary 2.8 and the discussion above, a sufficient condition on the function F for the monomial approximation error to be around the size of machine epsilon is that F is analytically continuable to the closure of a region  $E_{\rho}^{o}$ , where  $\rho^{-N} \leq u$ , on which it is bounded by a modest constant. We note that this sufficient condition is generally satisfied by non-oscillatory functions that can be resolved by an Nth order interpolating polynomial to machine precision, where  $N \leq N^*$ . It follows that, for this class of functions, polynomial interpolation in the monomial basis is as accurate as polynomial interpolation in a more well-conditioned basis, when the order of approximation is  $\leq N^*$ . In the following section, we deal with the case where this sufficient condition is not satisfied by F.

## 2.1 Practical a priori backward error estimates

When a function is not sufficiently smooth, stagnation of convergence is observed when it is interpolated in the monomial basis, due to the backward error of the approximation. In this section, we show that this stagnation of convergence can be accurately estimated a priori in many cases.

Without loss of generality, let  $\Gamma$  be a smooth simple arc inside the unit disk centered at the origin, and suppose that  $F : \Gamma \to \mathbb{C}$  is a function that satisfies  $||F||_{L^{\infty}(\Gamma)} \leq 1$ . Suppose further that  $\{P_N\}_{N=0,1,\dots}$  is a sequence of interpolating polynomials of F, and that the Lebesgue constants for the associated collocation points are slowly growing. We define  $\widetilde{N} := \min\{N : ||F - P_N||_{L^{\infty}([-1,1])} \approx u\}$  to be the degree of the first interpolating polynomial in the sequence that approximates F to machine precision, and we assume that  $\widetilde{N} < \infty$ . We divide our discussion into two cases: the case where  $\widetilde{N} \leq N^*$ , and the case where  $\widetilde{N} > N^*$  (see Definition 2.3).

First, we present a practical a priori estimate for the 2-norm of the monomial coefficient vector  $a^{(\tilde{N})}$  of the  $\tilde{N}$ th order interpolating polynomial  $P_{\tilde{N}}$ , in the case where  $\tilde{N} \leq N^*$ . Define  $\rho_{\tilde{N}} := (\frac{1}{u})^{1/\tilde{N}}$ . Unless F is pathological, F is analytically continuable to the closure of the region  $E^o_{\rho_{\tilde{N}}}$  corresponding to  $\Gamma$  (see Definition 2.1), and the extrapolation

of F by  $P_{\widetilde{N}}$  satisfies

$$\left\|F - P_{\widetilde{N}}\right\|_{L^{\infty}(E^{o}_{\rho_{\widetilde{N}}})} \le 1,\tag{44}$$

(see, for example, [28]). Note that  $\widetilde{N} \leq N^*$  is equivalent to the condition  $\rho_* \leq \rho_{\widetilde{N}}$ , due to the inequality  $\rho_*^{\widetilde{N}} \leq \rho_*^{N^*} = \frac{1}{u} = (\rho_{\widetilde{N}})^{\widetilde{N}}$ . Thus, the open unit disk  $D_1$  centered at the origin is a subset of  $E_{\rho_{\widetilde{N}}}^o$ , from which it follows that the  $L^{\infty}$  norm of  $P_{\widetilde{N}} - F$  over  $\partial D_1$  satisfies

$$\|F - P_{\widetilde{N}}\|_{L^{\infty}(\partial D_{1})} \le \|F - P_{\widetilde{N}}\|_{L^{\infty}(E^{o}_{\rho_{*}})} \le \|F - P_{\widetilde{N}}\|_{L^{\infty}(E^{o}_{\rho_{\widetilde{N}}})} \le 1, \qquad (45)$$

by (44), and the 2-norm of the monomial coefficient vector  $a^{(\tilde{N})}$  of  $P_{\tilde{N}}$  satisfies

$$\|a^{(\widetilde{N})}\|_{2} \leq \|P_{\widetilde{N}}\|_{L^{\infty}(\partial D_{1})} \leq \|P_{\widetilde{N}} - F\|_{L^{\infty}(\partial D_{1})} + \|F\|_{L^{\infty}(\partial D_{1})} \leq \|F\|_{L^{\infty}(\partial D_{1})} + 1, \quad (46)$$

by Lemma 2.5. Therefore, when  $\widetilde{N} \leq N^*$ , the additional error caused by the use of a monomial basis can be predicted using (46), provided that the analytic continuation of F is known, and that the Vandermonde system is solved using a backward stable solver. In this case, polynomial interpolation in the monomial basis is as accurate as polynomial interpolation in a more well-conditioned basis, until the polynomial interpolation error decays to around  $u \|F\|_{L^{\infty}(\partial D_1)}$ .

Next, we show that, in the case where  $\tilde{N} > N^*$ , the monomial approximation error  $||F - \hat{P}_N||_{L^{\infty}(\Gamma)}$  is generally dominated by the polynomial interpolation error  $||F - P_N||_{L^{\infty}(\Gamma)}$  for all  $N \leq N^*$ . In this case, it is generally true that the polynomial interpolation error  $||F - P_N||_{L^{\infty}(\Gamma)}$  decays more slowly than the condition number  $\kappa(V^{(N)})$  grows. Thus, we make the assumption that  $||F - P_N||_{L^{\infty}(\Gamma)} \approx C\rho^{-N}$  for some  $\rho < \rho_*$ . It follows from Theorem 2.6 that

$$u \|a^{(N)}\|_{2} \leq u \Big( \|F\|_{L^{\infty}(\Gamma)} + C\Big(\Lambda_{N}\Big(\frac{\rho_{*}}{\rho}\Big)^{N} + 2\rho_{*}\sum_{j=0}^{N-1}\Big(\frac{\rho_{*}}{\rho}\Big)^{j}\Big)\Big)$$

$$\leq u \Big( \|F\|_{L^{\infty}(\Gamma)} + \frac{C(\Lambda_{N} + 2\rho_{*})}{\rho_{*}/\rho - 1}\Big(\frac{\rho_{*}}{\rho}\Big)^{N+1}\Big)$$

$$\leq u \cdot C\Big(\frac{\rho_{*}}{\rho}\Big)^{N+1} = C\rho_{*}^{N+1-N^{*}} \cdot \rho^{-(N+1)},$$

$$(47)$$

for all  $N \ge 0$ , where the last equality follows from the identity  $\rho_*^{N^*} = u$ . In particular, when  $N \le N^*$ , the inequality (47) becomes

$$u\|a^{(N)}\|_2 \lesssim C\rho_* \cdot \rho^{-(N+1)} \lesssim C\rho^{-N} \approx \|F - P_N\|_{L^{\infty}(\Gamma)}.$$
(48)

Thus, the backward error is generally smaller than the polynomial interpolation error when  $N \leq N^*$ , in which case the monomial approximation error is dominated by the polynomial interpolation error. In other words, in the case where  $\tilde{N} > N^*$ , polynomial interpolation in the monomial basis is as accurate as polynomial interpolation in a more well-conditioned basis, so long as the order of approximation is  $\leq N^*$ .

We validate the claims made in this section by extensive numerical experiments presented in Sections 2.4 and 2.5.

#### 2.2 Worst-case backward error analysis

Recall from Theorem 2.2 that the backward error of polynomial interpolation in the monomial basis, i.e.,  $\|\hat{P}_N - P_N\|_{L^{\infty}(\Gamma)}$ , is bounded by approximately  $u \cdot \Lambda_N \|a^{(N)}\|_2$ . When this upper bound of the backward error is less than the error tolerance of the computation, polynomial interpolation in the monomial basis is as accurate as polynomial interpolation in a more well conditioned basis, up to the error tolerance. In this section, we provide the largest value that this upper bound of the backward error can attain.

Suppose that  $F : \Gamma \to \mathbb{C}$  is a function. Without loss of generality, we assume that  $||F||_{L^{\infty}(\Gamma)} \leq 1$ . Suppose further that  $P_N$  is the Nth order interpolating polynomial of F, for a set of (N + 1) collocation points over  $\Gamma$ . Let  $a^{(N)} \in \mathbb{C}^{N+1}$  be the monomial coefficient vector of  $P_N$ , and let  $f^{(N)} \in \mathbb{C}^{N+1}$  be the vector of function values of F at the collocation points. By inequality (39) in the proof of Theorem 2.7, the 2-norm of  $a^{(N)}$ satisfies

$$\|a^{(N)}\|_{2} = \|(V^{(N)})^{-1}f^{(N)}\|_{2} \le \|(V^{(N)})^{-1}\|_{2}\|f^{(N)}\|_{2} \le \rho_{*}^{N}\Lambda_{N},$$
(49)

where  $\rho_*$  is defined in Definition 2.2, and  $\Lambda_N$  is the Lebesgue constant for the collocation points. In the case where  $\Lambda_N$  is small, it follows that the maximal backward error for polynomial interpolation in the monomial basis is of around size  $u \cdot \rho_*^N$ , where u denotes machine epsilon. Therefore, so long as the order of approximation  $N \leq \log(\varepsilon/u)/\log(\rho_*)$ , where  $\varepsilon$  is the error tolerance of the computation, polynomial interpolation in the monomial basis is as accurate as polynomial interpolation in a more well conditioned basis, up to the error tolerance, for any function F.

### 2.3 A backward stable linear system solver is all you need

If a Vandermonde system is solved using regularization, e.g., the truncated singular value decomposition (TSVD), then the 2-norm of the numerical solution is bounded by approximately the 2-norm of the exact solution, regardless of the condition number of the matrix [7]. Furthermore, it is possible to show that, in this case, the a priori error estimates in (17) are valid for arbitrarily high-order approximation. In this section, we show that, in almost all cases, the use of a regularization method is meaningless for polynomial interpolation in the monomial basis, and a backward stable solver is all that is required.

Let  $\Gamma$ , F,  $\{P_N\}_{N=0,1,\ldots}$  and  $\widetilde{N}$  be the same as introduced in the previous section. In the case where  $\widetilde{N} \leq N^*$  (see Definition 2.3), there is no need to set the order of the approximation N to be larger than  $N^*$ , because, once  $P_N$  approximates F to machine precision, the 2-norm of the monomial coefficient vector of  $P_N$ , which determines the additional error caused by the use of a monomial basis, will not become any smaller as Nincreases. In the case where  $\widetilde{N} > N^*$ , we make the same assumption as in the previous section that  $\|F - P_N\|_{L^{\infty}(\Gamma)} \approx C\rho^{-N}$  for some  $\rho < \rho_*$  (see Definition 2.2). Based on the inequality (47), we have that the upper bound of  $u\|a^{(N)}\|_2$ , i.e.,  $C\rho_*^{N+1-N^*} \cdot \rho^{-(N+1)}$ , becomes larger than the polynomial interpolation error  $\|F - P_N\|_{L^{\infty}(\Gamma)} \approx C\rho^{-N}$  when  $N > N^*$ . In most situations, the exact value of  $u\|a^{(N)}\|_2$  is close to this upper bound, from which it follows that the smallest a priori bound for the monomial approximation error  $\|F - \widehat{P}_N\|_{L^{\infty}(\Gamma)}$  is attained when  $N \approx N^*$ . In practice, we observe the same phenomenon, and find that there is generally no benefit in setting the order of the approximation N to be larger than  $N^*$ . As a result, we conclude that, in almost all cases, polynomial interpolation in the monomial basis is of interest only when the order of approximation is  $\leq N^*$ , from which it follows that a backward stable solver is all that is required for solving a Vandermonde system in this context.

**Remark 2.2.** What are the situations when it is meaningful to solve a large Vandermonde system using regularization? Consider the following. Suppose that the function F is analytically continuable to the unit disk  $D_1$  centered at the origin, over which it is bounded by a modest constant. Suppose further that F is badly behaved outside  $D_1$ . It is possible to show that the first assumption guarantees that the 2-norm of the monomial coefficient vector of  $P_{\tilde{N}}$  is of modest size, and the second assumption ensures that  $\tilde{N} > N^*$ . An example of such a function is  $F(z) = z^{100}$ .

**Remark 2.3.** It takes  $\mathcal{O}(N^3)$  operations to solve a Vandermonde system of size  $N \times N$ by a standard backward stable solver, e.g., LU factorization with partial pivoting. Since  $N^*$  is smaller than 50 in most situations, the Vandermonde matrix, used for polynomial interpolation in the monomial basis, is typically not large, from which it follows that its solution can be computed rapidly by LAPACK, for example. There also exist some specialized algorithms that solve Vandermonde systems in  $\mathcal{O}(N^2)$  operations, e.g., the Björck-Pereyra algorithm [6]. The backward stability of these algorithms in the most general case remains unknown [18].

#### 2.4 Interpolation over an interval

In this section, we consider polynomial interpolation in the monomial basis over an interval  $\Gamma = [a, b] \subset \mathbb{R}$ . We suggest the use of the Chebyshev points on the interval [a, b] as the collocation points, because of the following two well-known lemmas related to Chebyshev approximation.

The lemma below, originally proved in [9], bounds the growth rate of the Lebesgue constant for the Chebyshev points.

**Lemma 2.9.** Let  $\Lambda_N$  be the Lebesgue constant for the (N + 1) Chebyshev points on an interval [a,b]. For any nonnegative integer N, the Lebesgue constant  $\Lambda_N$  satisfies

$$\Lambda_N \le \frac{2}{\pi} \log(N+1) + 1. \tag{50}$$

The following lemma provides a sufficient condition for the Chebyshev interpolant of a function to converge geometrically. The proof can be found in, for example, Theorem 8.2 in [27]. Recall that the level set  $E_{\rho}$  for an interval [a, b] is a Bernstein ellipse with foci at a and b, and with parameter  $\rho$  (see Figure 4a).

**Lemma 2.10.** Suppose that  $F : [a, b] \to \mathbb{C}$  is analytically continuable to the region  $E_{\rho}^{o}$ (see Definition 2.1), and satisfies  $||F||_{L^{\infty}(E_{\rho}^{o})} \leq M$  for some  $M \geq 0$ . The Nth order Chebyshev interpolant  $P_{N}$  of F satisfies

$$\|F - P_N\|_{L^{\infty}([a,b])} \le \frac{4M}{\rho - 1} \rho^{-N},\tag{51}$$

for all  $N \geq 0$ .

We note that the lemma above is a stronger version of Lemma 2.3, as it specifies the constant factor C. As a result, the constant C in Corollary 2.8 is known in this case.

**Remark 2.4.** As is noted in Figure 5, we have that  $\rho_* = 1 + \sqrt{2}$ ,  $N^* \approx 40.8$  when  $\Gamma = [-1, 1]$ , and  $\rho_* = 3 + 2\sqrt{2}$ ,  $N^* \approx 20.4$  when  $\Gamma = [0, 1]$  (see Definitions 2.2 and 2.3).

In the rest of this section, we provide several numerical experiments. In Figure 6, we compute monomial expansions for functions which can be resolved by a Chebyshev interpolant of degree  $N \leq N^*$  (see Definition 2.3) over  $\Gamma = [-1, 1]$ . In addition to the estimated values of  $||F - P_N||_{L^{\infty}([a,b])}$  and  $||F - \hat{P}_N||_{L^{\infty}([a,b])}$ , we plot three extra curves in each figure: an a priori approximation of the upper bound of the backward error (see the legend of Figure 6 for a precise definition); an a priori approximation of the upper bound of the error caused by the condition number of the monomial expansion evaluation, i.e.,  $u \cdot ||F||_{L^{\infty}(\partial D_1)}$ ; and the exact upper bound of the error caused by the condition number of the monomial expansion evaluation, i.e.,  $u \cdot ||a^{(N)}||_2$ . In Figure 7, we provide similar experiments for the case where  $\Gamma = [0, 1]$ . In Figure 8, we consider polynomial interpolation, in the monomial basis, of functions that cannot be resolved to machine precision by a Chebyshev interpolant of degree  $N \leq N^*$ . Based on these experimental results, we make the following observations:

- 1. In many cases of interest, polynomial interpolation in the monomial basis (evaluated by naive summation) is as accurate as polynomial interpolation in the Lagrange polynomial basis (evaluated by the Barycentric interpolation formula), when the order of approximation is  $\leq N^*$ .
- 2. Provided that the function is resolved by an interpolating polynomial of degree  $N \leq N^*$  to machine precision, the norm of the expansion coefficients  $||a^{(N)}||_2$  is around the same size as  $||F||_{L^{\infty}(\partial D_1)}$ . This confirms the effectiveness of our practical estimate  $||a^{(N)}||_2 \approx ||F||_{L^{\infty}(\partial D_1)}$  proposed in Section 2.1.
- 3. The monomial approximation error is correctly captured by our upper bound of the backward error, and this bound is reasonably tight. Furthermore, the convergence generally stagnates after the monomial approximation error decays to  $u \cdot ||a^{(N)}||_2$ , which implies that the error caused by the condition number of the monomial expansion evaluation is a tight lower bound for the monomial approximation error.

## 2.5 Interpolation over a smooth simple arc in the complex plane

In this section, we consider polynomial interpolation in the monomial basis over a smooth simple arc  $\Gamma \subset \mathbb{C}$ . Without loss of generality, we assume that  $\Gamma$  is inside the unit disk centered at the origin.

In this more general setting, similar to the special case where  $\Gamma$  is an interval, there exists a class of collocation points, known as adjusted Fejér points, whose associated Lebesgue constant also grows logarithmically [30]. However, these points are extremely costly to construct numerically, which makes them unsuitable for practical applications. Fortunately, in the context of polynomial interpolation in the monomial basis, it is sufficient for the Lebesgue constant  $\Lambda_N$  to be small when N satisfies  $\kappa(V^{(N)}) \leq \frac{1}{2u \cdot \gamma_N}$ ,



Figure 6: Polynomial interpolation in the monomial basis over  $\Gamma = [-1, 1]$ . The label "Barycentric" denotes  $||F - P_N||_{L^{\infty}([-1,1])}$ , estimated using the Barycentric interpolation formula. The label "Vandermonde" denotes the estimated value of  $||F - \hat{P}_N||_{L^{\infty}([-1,1])}$ . The label "Back. err. bound" denotes the estimated backward error bound  $u \cdot \Lambda_{\widetilde{N}} ||V^{(\widetilde{N})}|| ||F||_{L^{\infty}(\partial D_1)}$ , where  $\widetilde{N} = \min\{N : ||F - P_N||_{L^{\infty}([-1,1])} \approx u\}$ . Recall that  $N^* \approx 40.8$  when  $\Gamma = [-1, 1]$ , and that the functions F in Figures 6a-6d are the same functions that appear in Section 1.



Figure 7: Polynomial interpolation in the monomial basis over  $\Gamma = [0, 1]$ . See the caption of Figure 6 for the definition of the labels "Barycentric", "Vandermonde" and "Back. err. bound". Recall that  $N^* \approx 20.4$  when  $\Gamma = [0, 1]$ .



Figure 8: Polynomial interpolation, in the monomial basis, of functions that are more difficult to resolve. See the caption of Figure 6 for the definition of the labels "Barycentric" and "Vandermonde".

where  $u \cdot \gamma_N$  is the upper bound of the relative backward error of the linear system solver (see Section 2.3). Suppose that  $\gamma : [-1,1] \to \mathbb{C}$  is a parametrization of  $\Gamma$ . When the Jacobian  $\gamma'(t)$  does not have large variations, and when N is not large, we find that the Lebesgue constant for the set of nodes  $Z = \{\gamma(t_j)\}_{j=0,1,\dots,N}$ , where  $\{t_j\}_{j=0,1,\dots,N}$  is the set of (N+1) Chebyshev points on the interval [-1,1], is generally small. This guarantees that the set of collocation points Z is nearly optimal for polynomial interpolation, and that the backward stability of the linear system solver alone is sufficient for the a priori error estimates in (17) to hold for  $N \leq N^*$  (see Definition 2.3).

The calculation of  $N^*$  requires the computation of the level set  $E_{\rho}$  of a general smooth simple arc (see Definition 2.1). This level set is costly to compute, since it requires the solution to the Laplace equation (23), which is generally not available in a closed-form expression (an exception is when  $\Gamma$  is a line segment). As a result, it is often more convenient to estimate  $N^*$  directly, by estimating  $\kappa(V^{(N)})$ . We also note that an analogue of Lemma 2.10 is unavailable, so the magnitude of the constant factor C in Lemma 2.3 is unknown. However, as is noted in Remark 2.1, the factor C is generally proportional to  $\|F\|_{L^{\infty}(E_{\rho}^o)}$ , from which it follows that C is small when F is bounded by a modest constant over  $E_{\rho}^o$ .

In the rest of the section, we provide several numerical experiments. Consider the case where  $\Gamma$  is a parabola, and is parametrized by  $\gamma : [-1,1] \to \mathbb{C}$ ,  $\gamma(t) := t + i\alpha(t^2 - 1)$ , for  $\alpha = 0.2, 0.4, 0.6$ . In Figure 9, we plot these parabolas, including their associated level sets  $E_{\rho}$ , for various values of  $\rho$ . The value of  $\rho_*$  for each parabola is estimated from the plots. In Figure 10, we estimate the condition number of the Vandermonde matrix and the Lebesgue constant for the collocation points, for different values of  $\alpha$ . One can observe that the condition number  $\kappa(V^{(N)})$  is approximately below the threshold  $\frac{1}{2u \cdot \gamma_N}$  for  $N \leq N^*$ , and the Lebesgue constant is of approximately size one. In Figure 11, we report the monomial approximation error  $\|F - \hat{P}_N\|_{L^{\infty}(\Gamma)}$ , and an a priori approximation of the upper bound of the error caused by the condition number of the monomial expansion evaluation (i.e.,  $u \cdot \|F\|_{L^{\infty}(\partial D_1)}$ ), and the exact upper bound of the error caused by the condition  $(i.e., u \cdot \|a^{(N)}\|_2)$ , for various functions F over  $\Gamma$ . Based on the experimental results, it is clear that the observations made at the end of Section 2.4 are also applicable to the case where  $\Gamma$  is a parabola. In fact, these observations apply to any simple arc that is sufficiently smooth.

**Remark 2.5.** In certain applications, the function  $F : \Gamma \to \mathbb{C}$  is defined by the formula  $F(z) := \sigma(\gamma^{-1}(z))$ , where  $\gamma : [-1,1] \to \mathbb{C}$  is an analytic function that parametrizes the curve  $\Gamma$ , and  $\sigma : [-1,1] \to \mathbb{C}$  is analytic. In this case, the analytic continuation of function F(z) can have a singularity close to  $\Gamma$  even when  $\sigma$  is entire, because the inverse of the parametrization (i.e.,  $\gamma^{-1}$ ) will have so-called Schwarz singularities at  $z = \gamma(t^*)$ , where  $\gamma'(t^*) = 0$ . In [22], the authors show that, the higher the curvature of the arc  $\Gamma$  is, the closer the singularity induced by  $\gamma^{-1}$  is to  $\Gamma$ . As a result, the approximation of such a function F by polynomials is efficient only when the curvature of  $\Gamma$  is small (see Section 2.3 in [22] and Section 4.6 in [29] for a detailed statement and a proof).

# 3 Applications

After justifying the feasibility of polynomial interpolation in the monomial basis, a natural question to ask is: why would one want to do it in the first place? First of all, the



Figure 9: The level set  $E_{\rho}$  of a parabola, for various values of  $\rho$ . The colorbar indicates the value of  $\rho$ . The smooth simple arc  $\Gamma$  is the white curve in the figure. The value of  $\rho_*$  (see Definition 2.2) is estimated for each arc  $\Gamma$ .



Figure 10: The Lebesgue constant for collocation points over a parabola, and the condition number of the corresponding Vandermonde matrix. The collocation points are chosen to be  $\{\gamma(t_j)\}$ , where  $\{t_j\}$  is a set of Chebyshev points over [-1,1], and  $\gamma : [-1,1] \to \mathbb{C}$  is the parametrization of the parabola defined in Section 2.5. The *x*-axis label *N* denotes the order of approximation. The value of  $\rho_*$  is set to be 2.6, based on the estimate in Figure 9. It follows that  $N^* \approx 37.7$ . One can observe that  $\kappa(V^{(N)}) \leq \frac{1}{n}$  when  $N \leq N^*$ .

monomial basis is the simplest polynomial basis to manipulate. The evaluation of an Nth order monomial expansion requires N multiplications with the use of Horner's rule. Monomial expansions for the derivative and anti-derivative of a monomial expansion can also be computed with N multiplications. Besides these obvious advantages, we present several applications that demonstrate the unique merits of function representation in a monomial basis.

## 3.1 Oscillatory integrals and singular integrals

Given an oscillatory (or singular) function  $\Psi : \Gamma \to \mathbb{C}$  and a smooth function  $F : \Gamma \to \mathbb{C}$ over a smooth simple arc  $\Gamma \subset \mathbb{C}$ , the calculation of

$$\int_{\Gamma} \Psi(z) F(z) \,\mathrm{d}z \tag{52}$$

by standard quadrature rules is extremely expensive and inaccurate due to the oscillations (or the singularity) of  $\Psi$ . However, when F is a monomial, there exists a wide class of integrals in the form (52) that can be efficiently computed to high accuracy by, for example, recurrence relations that are derived using integration by parts. Therefore, when the smooth function F is accurately approximated by a monomial expansion of order N, such integrals can be efficiently evaluated by the formula

$$\sum_{k=0}^{N} a_k \Big( \int_{\Gamma} \Psi(z) z^k \, \mathrm{d}z \Big), \tag{53}$$

where  $\{a_k\}_{k=0,1,\dots,N}$  denotes the coefficients of the monomial expansion.



Figure 11: Polynomial interpolation in the monomial basis over a parabola. The interpolation is performed on the parabolas shown in Figure 9a. The *x*-axis label N denotes the order of approximation. The label "Vandermonde" denotes the estimated value of  $||F - \hat{P}_N||_{L^{\infty}([-1,1])}$ .

In the rest of this section, we present examples of oscillatory integrals and singular integrals of this kind.

**Remark 3.1.** When the function F is too complicated to be approximated to high accuracy over  $\Gamma$  by a single monomial expansion, one can subdivide the domain  $\Gamma$ , and approximate F by monomial expansions over each subpanel.

**Remark 3.2.** When one needs to compute the oscillatory (or singular) integral (52) for multiple smooth functions F, it is unnecessary to compute a monomial expansion for each F. Instead, the adjoint method can be used to compute a quadrature rule  $\{(z_i, w_i)\}_{i=0,1,\dots,N}$ , such that

$$\int_{\Gamma} \Psi(z) F(z) \, \mathrm{d}z \approx \sum_{i=0}^{N} w_i \Psi(z_i) F(z_i), \tag{54}$$

for any function F that can be accurately approximated by a monomial expansion of order N. We refer the readers to Section 2.2.2 in [22] for a detailed overview of the method.

#### 3.1.1 Fourier integrals

Given a smooth function  $G : [a, b] \to \mathbb{C}$  and a real number c, it takes  $\mathcal{O}(c)$  operations to compute the Fourier integral

$$\int_{a}^{b} e^{icx} G(x) \,\mathrm{d}x \tag{55}$$

by a standard quadrature rule for smooth functions, as the number of points required to resolve the integrand is proportional to the value of c. Consequently, the evaluation of such an integral is prohibitively expensive when c is large. By a change of variables, the integral (55) can be decomposed into the sum of the integral of a smooth function, and an oscillatory integral of the form

$$\int_{-1}^{1} e^{i\omega x} F(x) \,\mathrm{d}x,\tag{56}$$

where  $\omega \in \mathbb{R}$ , and  $F : [-1,1] \to \mathbb{R}$  is smooth. Thus, without loss of generality, it is sufficient to consider the numerical evaluation of (56) alone. Observe that (56) can be efficiently evaluated to high accuracy with a cost independent of  $\omega$  when the function Fis a monomial, using the following recurrence relations:

$$\int_{-1}^{1} e^{i\omega x} dx = \frac{1}{i\omega} (e^{i\omega} - e^{-i\omega}),$$
(57a)

$$\int_{-1}^{1} e^{i\omega x} x^{k+1} \, \mathrm{d}x = \frac{1}{i\omega} \Big( e^{i\omega} + (-1)^{k} e^{-i\omega} - (k+1) \int_{-1}^{1} e^{i\omega x} x^{k} \, \mathrm{d}x \Big), \quad (57b)$$

for all  $k \ge 0$ . The use of this recurrence relation for computing Fourier integrals was first proposed in [11], and the resulting algorithm is known as the Filon-type method. When this method is used, the smooth function F is typically approximated by piecewise polynomials of low degrees [11, 20], in part due to the belief that higher-order polynomial interpolation in the monomial basis is unstable. By approximating F by a higher-order monomial expansion as described in this paper, the Filon-type method is made substantially more accurate.

We note that this technique also generalizes to higher dimensions and to more complicated oscillatory functions  $\Psi$ , and we refer the readers to [21] for an overview.

#### 3.1.2 Layer potentials

Given a smooth simple arc  $\Gamma \subset \mathbb{C}$  and a target point  $\xi \in \mathbb{C}$ , the evaluation of the layer potentials

$$\int_{\Gamma} \log(z-\xi)F(z) \,\mathrm{d}z \quad \text{and} \quad \int_{\Gamma} \frac{F(z)}{z-\xi} \,\mathrm{d}z, \tag{58}$$

is of great importance in the integral equation method for the numerical solution of partial differential equations [23]. When  $\xi$  is close to  $\Gamma$ , the integrands become nearly singular and, as a result, standard quadrature rules cannot be used to compute the integrals efficiently. In [14, 15], the authors observe that the integrals (58) can be computed efficiently to high accuracy when F(z) is a monomial, with the use of the following recurrence relations:

$$\int_{\Gamma} \frac{1}{z - \xi} \, \mathrm{d}z = \log(1 - \xi) - \log(-1 - \xi) \pm 2\pi i \mathcal{N}_{\xi},\tag{59a}$$

$$\int_{\Gamma} \frac{z^{k+1}}{z-\xi} \,\mathrm{d}z = \xi \int_{\Gamma} \frac{z^k}{z-\xi} \,\mathrm{d}z + \frac{1+(-1)^k}{k+1},\tag{59b}$$

$$\int_{\Gamma} \log(z-\xi) z^k \, \mathrm{d}z = \frac{1}{k+1} \Big( \log(1-\xi) + (-1)^k \log(-1-\xi) - \int_{\Gamma} \frac{z^{k+1}}{z-\xi} \, \mathrm{d}z \Big), \quad (59c)$$

for all  $k \geq 0$ , where  $\mathcal{N}_{\xi} \in \mathbb{Z}$  denotes a winding number whose value depends on the relative location of  $\xi$  with respect to  $\Gamma$ . Therefore, the layer potentials (58) can be efficiently evaluated once F is approximated by monomials. We refer the readers to Section 2.2.1 in [22] for a detailed overview of this method.

#### 3.1.3 Hadamard finite-part integrals

Integrals of the form

$$\int_{a}^{b} (x-a)^{\nu} \log^{m} (x-a) G(x) \, \mathrm{d}x,$$
(60)

where  $G: [a, b] \to \mathbb{C}$  is smooth,  $\nu \in \mathbb{R}$ , and  $m \ge 0$  is an integer, appear often in applications. By a change of variables, the integral (60) can be written as a combination of integrals of the form

$$\int_0^1 x^\nu \log^m(x) F(x) \,\mathrm{d}x,\tag{61}$$

where  $F: [0,1] \to \mathbb{C}$  is smooth. When  $\nu \leq -1$ , this integral is divergent, in which case we can consider only its "finite part" (see, for example, [10]). Let  $\epsilon > 0$ , and write

$$\int_{\epsilon}^{1} x^{\nu} \log^{m}(x) F(x) \, \mathrm{d}x = F_{0}(\epsilon) + F_{1}(\epsilon), \tag{62}$$

where

$$F_1(\epsilon) = a_1 \Psi_1(\epsilon) + a_2 \Psi_2(\epsilon) + \dots + a_n \Psi_n(\epsilon)$$
(63)

is a combination of given functions  $\Psi_1, \Psi_2, \ldots, \Psi_n$  which become infinite as  $\epsilon \to 0$ . Discarding the "infinite part"  $F_1(\epsilon)$ , we define the Hadamard finite part of (61) by

f.p. 
$$\int_0^1 x^{\nu} \log^m(x) F(x) \, \mathrm{d}x = \lim_{\epsilon \to 0} F_0(\epsilon).$$
(64)

It is possible to show that the finite part of (61) is equal to its meromorphic continuation in  $\nu$  to (a subset of) the region  $\operatorname{Re}(\nu) \leq -1$ .

When F(x) is a monomial, we can evaluate the finite part explicitly using the formula

f.p. 
$$\int_0^1 x^{\nu} \log^m(x) \cdot x^k \, \mathrm{d}x = \frac{(-1)^m m!}{(\nu + k + 1)^{m+1}}.$$
 (65)

Therefore, the finite part integral (61) can be accurately and efficiently evaluated for all  $\nu \notin \{-1, -2, \ldots\}$ , once F(x) is approximated by monomials.

#### 3.2 Root finding

One method for computing the roots of a smooth function  $F : [-1, 1] \to \mathbb{C}$  is to first approximate it by a polynomial  $P_N(x) = \sum_{j=0}^N a_j x^j$  to high accuracy, and then to compute the roots of  $P_N$  as the eigenvalues of the companion matrix

$$C(P_N) := \begin{pmatrix} 0 & 0 & \dots & 0 & -\frac{a_0}{a_N} \\ 1 & 0 & \dots & 0 & -\frac{a_1}{a_N} \\ 0 & 1 & \dots & 0 & -\frac{a_2}{a_N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -\frac{a_{N-1}}{a_N} \end{pmatrix}.$$
(66)

Recently, a backward stable algorithm that computes the eigenvalues of  $C(P_N)$  in  $\mathcal{O}(N^2)$  operations with  $\mathcal{O}(N)$  storage has been proposed in [2]. This algorithm is backward stable in the sense that the computed roots are the exact roots of a perturbed polynomial  $\hat{P}_N(x) = \sum_{j=0}^N (a_j + \delta a_j) x^j$ , so that the backward error satisfies  $\|\delta a^{(N)}\|_2 \leq u \|a^{(N)}\|_2$ , where u denotes machine epsilon,  $\delta a^{(N)} := (\delta a_0, \delta a_1, \dots, \delta a_N)^T$  and  $a^{(N)} := (a_0, a_1, \dots, a_N)^T$ . It follows that

$$\|P_N - \widehat{P}_N\|_{L^{\infty}([-1,1])} \le u \|\delta a^{(N)}\|_1 \lesssim u\sqrt{N+1} \|a^{(N)}\|_2.$$
(67)

Since  $||a^{(N)}||_2$  can potentially be much larger than  $||P_N||_{L^{\infty}([-1,1])}$ , the companion matrixbased root finding algorithm is not, in general, backward stable in the function  $P_N$ , and thus not backward stable in the function F. In this paper, however, we show that, when F is sufficiently smooth, F can be approximated uniformly by an interpolating polynomial  $P_N(x) = \sum_{j=0}^N a_j x^j$ , so that  $\|a^{(N)}\|_2 \approx \|F\|_{L^{\infty}([-1,1])}$ . If F is an analytic on a neighborhood of [-1,1], then there exists some partition of [-1,1] such that F is sufficiently smooth on each subinterval, and this partition can be determined by adaptive subdivision. Therefore, polynomial interpolation in the monomial basis, as described in this paper, can be combined with the algorithm of [2] to yield a backward stable rootfinder in the function F.

# 4 Discussion

Since the invention of digital computers, most research on the topic of polynomial interpolation in the monomial basis focuses on showing that this is a bad idea. For example, the condition number of Vandermonde matrices has been studied extensively in recent decades [12, 4], and it is known that its growth rate is at least exponential, unless the collocation nodes are distributed uniformly on the unit circle centered at the origin [24]. As a result, the computed monomial coefficients are generally highly inaccurate when the dimensionality of the Vandermonde matrix is not small. For this reason, other more well-conditioned bases are often used for function representations [27, 8]. However, the fact that the monomial coefficients are computed inaccurately does not imply that polynomial interpolation in the monomial basis is unstable, since it is the backward error  $\|V\hat{a} - f\|_2$  of the numerical solution  $\hat{a}$  to the Vandermonde system Va = f that determines the accuracy of the approximation (a similar situation also occurs in the method of fundamental solutions [3]). As we show in this paper, so long as a backward stable linear system solver is used and  $\kappa(V)$  is not excessively large, i.e.,  $\kappa(V) \lesssim \frac{1}{u}$ , where u denotes machine epsilon, this backward error is bounded by approximately  $u ||a||_2$ . In fact, if the Vandermonde system is solved using regularization, then this backward error bound holds regardless of the magnitude of  $\kappa(V)$ . If, in addition, the Lebesgue constant associated with the collocation points is small, then the approximation error of the resulting monomial expansion is essentially bounded by  $u \|a\|_2$  plus the approximation error of the exact interpolating polynomial. Therefore, it is numerically stable to solve an ill-conditioned Vandermonde matrix for the purpose of interpolation, in the sense that the extra error  $u \|a\|_2$  is approximately equal to the condition number of the evaluation of the exact interpolating polynomial represented in the monomial basis. In other words, even if the monomial coefficients were known analytically, the resulting error would be the same.

The observation that monomials can approximate sufficiently well-behaved functions uniformly to high accuracy is not new (see, for example, [13, 14]), and many interesting applications have appeared [16, 25]. Yet, the general attitude towards polynomial interpolation in the monomial basis has remained skeptical, in part because a complete theory has been unavailable until now. We show that the monomial approximation error can be accurately estimated a priori, and that polynomial interpolation in the monomial basis is as accurate as polynomial interpolation in a more well-conditioned basis in many cases of interst.

In the literature, it is also observed that the monomial coefficients can sometimes be computed to relative accuracy by the Björck-Pereyra algorithm [6, 17]. We note that this phenomenon occurs only for a very limited set of functions, so the observation itself does not yield a general algorithm for accurate polynomial interpolation in the monomial basis. Another line of research that shares the same flavor is function approximation by frames. A frame is a complete system of functions that is redundant, but provides infinite representations with bounded coefficients [1] (we note that the monomial basis is not a frame for the space of smooth functions). As a result, function approximation by a frame also involves solving an ill-conditioned linear system. The main difference between our work and frame approximation is that, in almost all cases, polynomial interpolation in the monomial basis is of interest only when  $\kappa(V) \lesssim \frac{1}{u}$  (see Section 2.3), from which it follows that we require nothing more than a backward stable solver to guarantee the stability of the algorithm. On the other hand, when a function is approximated by a frame, the condition number of the matrix involved generally has to be  $\gtrsim \frac{1}{u}$  for the approximation to be accurate, and the use of a regularization method for solving the linear system is a must.

While not discussed in this paper, an analogue of Theorem 2.2 can be shown to hold in higher dimensions, from which it follows that higher-dimensional polynomial interpolation in the monomial basis is also stable under similar conditions. The generalization of the rest of the theorems in this paper to the multivariate case is, however, not so immediate. Nevertheless, we find that, in practice, higher-dimensional polynomial interpolation in the monomial basis is as accurate as higher-dimensional polynomial interpolation in a more well-conditioned basis, in many cases of interest. We provide heuristics for accurate and stable 2-D polynomial interpolation in the monomial basis in our recent work on Newtonian potential evaluation [26], where the monomial basis is used for the approximation of the anti-Laplacian of a 2-D function.

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